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Electrostatic Stability of a Collisionless Plane Discharge.

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Summary. — In a collisionless plane discharge the stream of ions falling to the walls is found to be stable against ion wave instabilities for the case in which the ionization function is constant. If ionization in the sheath region is suppressed, however, ion oscillations may be excited in the sheath. Such oscillations would resemble those recently observed by others.

1. – Introduction.

In a previous article (1) the author pointed out the possibility that ion waves can be excited by the streaming of ions into a sheath, but that it is in general extremely difficult to predict whether or not the stability criterion is satisfied because the ion velocity distribution is ordinarily unknown. It is the purpose of this article to investigate the stability of the ion stream in and near a sheath in a particularly simple case in which the ion velocity distribution is known. This is the case of a low-pressure discharge between infinite parallel plane electrodes with a spacing much smaller than the ion-neutral mean free path, so that the ions make no collisions before reaching the wall. This theoretically simple example has previously been analysed by several authors (2-4) with the result that the ion distribution is independent of the manner in which the ions are created.

(1) F. F. CHEN: *Phys. Fluids*, **4**, 1448 (1961).
(2) L. TONKS and I. LANGMUIR: *Phys. Rev.*, **34**, 876 (1929).
(3) E. R. HARRISON and W. B. THOMPSON: *Proc. Phys. Soc.*, **74**, 2, 145 (1959).
(4) P. L. AUER: *Nuovo Cimento*, **22**, 3, 548 (1961).

This distribution is shown in Fig. 2 and will be derived in Section **3**. Since the distribution is sharply peaked (in fact, it has an integrable singularity at the maximum velocity) and the mean velocity is greater than $(kT_e/M)^{\frac{1}{2}}$ even outside the sheath, there is a possibility that the ion stream is unstable even in the plasma region; this possibility will be discussed in Section **4**.

Even if the ion stream were stable in the plasma, it may become unstable in the sheath, where it undergoes further acceleration. If the growth rate were large enough ($\mathrm{Im}\,\omega \approx \mathrm{Re}\,\omega$), an initial perturbation can grow perhaps 200 times in the sheath, since the latter is of the order of 5 Debye lengths in thickness in Hg [5]. This possibility is discussed in Section **5**. Of course in this case the oscillations will not propagate backward into the plasma as was suggested in ref. [1], since the walls are assumed to be cold and will not emit primary electrons.

A recent experiment by OTT *et al.* [5] under conditions approaching those considered here revealed low frequency oscillations in the sheath on an electrically floating flat metal plate inserted into a low-pressure mercury discharge. These oscillations were detected by a thin electron beam traversing the sheath in a direction parallel to the plate. A weak magnetic field was used to compensate for the curvature of the beam caused by the electric field of the sheath. Although the observations indicated that the oscillations were connected with this magnetic field, it is interesting to see whether such oscillations would in principle be expected in the absence of such a magnetic field.

2. – Statement of the problem.

Suppose that a plasma is created by an unspecified mechanism between two infinitely large flat plates placed at $z = \pm L$. In practice, the ionization is usually accomplished by a stream of fast electrons moving parallel to the plates; but since we shall neglect both the density of the fast electrons and the magnetic field of their current, one can imagine that ions are formed by ultraviolet radiation.

Poisson's equation is, in the usual notation,

$$\frac{\mathrm{d}^2 V}{\mathrm{d}z^2} = -4\pi e(n_i - n_e)\,. \tag{1}$$

It will be assumed that the potential falls monotonically from 0 at the origin $z = 0$.

[5] G. v. GIERKE, W. OTT and F. SCHWIRZKE: *Proc. Fifth Intl. Conf. on Ionization Phenomena in Gases, Munich, 1961* (Amsterdam); W. OTT: *Thesis* (Munich, 1961).

If the electrons are nearly in thermal equilibrium, their density is given by

$$n_e = \tfrac{1}{2}\, n_0 \exp\left[eV/kT\right]\left[1 + \operatorname{erf} \frac{e}{kT}\,(V - V_w)\right], \tag{2}$$

where n_0 is the density at $z = 0$, kT is the electron temperature, and V_w is the potential of the walls. The term containing the error function takes into account electrons lost to the walls under the assumptions that the electrons have an exactly Maxwellian distribution at $z = 0$ and that they make no collisions in a length L. The requirement on the mean free path is easily satisfied, but one would expect that the high-energy tail of the distribution would be replenished by collisions, so that the two assumptions are not consistent. However, the work of GABOR *et al.* [6] has shown that perhaps high frequency oscillations in the wall sheath populate the tail of the distribution much faster than collisions, thus accounting for the observed lack of depletion of fast electrons. Thus the use of eq. (2) is experimentally justified, although it is unreasonable theoretically. In any case, the computation of the rate of replenishment of the high-energy tail and of its shape, either by collisions or by oscillations, is beyond the scope of this paper.

If the assumptions leading to eq. (2) are valid, the equality of ion and electron fluxes to the walls requires that $-eV_w/kT$ be of the order of 5 in mercury, so that we may safely replace the error function by unity in the plasma region. In the immediate neighborhood of the wall, the electron density will depend sensitively on the mechanism of population of the high-energy tail; for lack of a better approximation, we shall assume eq. (2) is valid.

As for the ions, if $q(z)$ is the number of ions created per cm³ per s and if the ions are created at rest and fall unhindered toward the wall under the force of the electric field in the plasma, the ion density at any point is given by

$$n_i(z) = \int_0^z \frac{q(z')\,\mathrm{d}z'}{v(z, z')}, \tag{3}$$

where

$$v(z, z') = \left[\frac{2e}{M}\,\bigl(V(z') - V(z)\bigr)\right]^{\frac{1}{2}}. \tag{4}$$

We shall now employ the following dimensionless variables:

$$\left\{\begin{aligned} \eta &= -\,eV/kT\,, & & \\ u &= v/v_s\,, & v_s &= (2kT/M)^{\frac{1}{2}},\\ \xi &= z/h\,, & h &= (kT/4\pi n_0 e^2)^{\frac{1}{2}}\\ g &= qh/n_0 v_s & & \end{aligned}\right. \tag{5}$$

[6] D. GABOR, E. A. ASH and D. DRACOTT: *Nature*, **176**, 916 (1955).

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In these units, velocity is directly related to potential difference: $u^2=\eta$. If we combine eqs. (1)–(5), Poisson's equation becomes

$$\text{(6)}\qquad \frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2}=\int\limits_0^{\eta}\frac{g(\eta')(\mathrm{d}\xi/\mathrm{d}\eta')\,\mathrm{d}\eta'}{(\eta-\eta')^{\frac{1}{2}}}-\frac{1}{2}\exp[-\eta][1+\operatorname{erf}(\eta_w-\eta)]\,.$$

This is a singular integral equation which can be solved numerically to give $\eta(\xi)$ everywhere in the region $|z|\leqslant L$. The boundary conditions are that $\mathrm{d}\eta/\mathrm{d}\xi=0$ at $z=0$ and that the ion and electron currents to the wall be equal. The ion current is merely $\int q(z)\,\mathrm{d}z$; *i.e.*, the total number of ions created per second. The electron current is given, as a function of η_w, by a knowledge of the shape of the high-energy tail of the distribution function. If the latter is known, η_w is determined by the equality of currents, and this boundary condition assures a unique solution of (6). Since the electron current is not known, however, there is not much to gain by an exact numerical computation. We shall, instead, take advantage of the known analytic solution ([3]) for the quasineutral region and use an approximate equation for the sheath region, which, as indicated above, cannot be treated exactly in any case. In this manner, we hope to gain more physical insight into the problem than by a numerical computation.

Our procedure, then, is as follows. To determine whether an ion wave instability is possible, we need to know the derivative of the ion velocity distribution. In the plasma region, where quasineutrality obtains, an expedient way to find this is to invert the equation

$$\text{(6a)}\qquad \frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2}=\int\limits_0^{\eta}\frac{g(\eta')(\mathrm{d}\xi/\mathrm{d}\eta')\,\mathrm{d}\eta'}{(\eta-\eta')^{\frac{1}{2}}}-\exp[-\eta]\,.$$

To do this, we shall follow the procedure of Harrison and Thompson ([3]), but shall retain the correction term $\mathrm{d}^2\eta/\mathrm{d}\xi^2$, which measures the deviation from neutrality. This correction term will be evaluated by using the quasineutral solution. The latter is known to be inaccurate beyond $\eta=0.854$ and may not be sufficiently accurate even at lower values of η. Consideration of the correction term will show that indeed as far as the derivative of the ion distribution is concerned, the quasineutral solution is insufficiently accurate beyond $\eta=0.6$. It will then be assumed that the quasineutral solution can be trusted up to $\eta=0.6$, and it will be shown that the plasma is stable up to this point. Beyond $\eta=0.6$, we shall use an approximate sheath equation, in which the electron density is assumed to be given by eq. (2), and the ions are assumed to be monoenergetic. This equation should be fairly accurate except in the part of the sheath nearest the wall, where ion waves do not have time to grow anyway, even if they could be excited.

3. – The distribution functions.

As shown in Appendix I, eq. (6*a*) can be inverted to give

$$\pi g(\eta)\,\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = G(\eta) + \eta^{-\frac{1}{2}}\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2}\right)_0 + I_2(\eta)\,, \tag{7}$$

where

$$G(\eta) = \eta^{-\frac{1}{2}} - 2F(\eta^{\frac{1}{2}})\,, \tag{8}$$

$$F(x) = \exp[-x^2]\int_0^x \exp[t^2]\,\mathrm{d}t\,, \tag{9}$$

and

$$I_2(\eta) = \int_0^{\eta^{\frac{1}{2}}} \left(\frac{\mathrm{d}^3\eta}{\mathrm{d}\xi^3}\right)'\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)'\left(\frac{\mathrm{d}\eta'}{(\eta-\eta')^{\frac{1}{2}}}\right). \tag{10}$$

Here $G(\eta)$ is the usual plasma solution which assumes strict charge neutrality, and the other two terms in (7) are corrections due to the non-vanishing second derivative of η.

The solution of the approximate equation

$$g(\eta)\,\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \pi^{-1}\,G(\eta)\,, \tag{11}$$

for the case $g = \text{constant} = 2/\pi$ is shown in Fig. 1. The curve for the case when g is proportional to electron density is quite similar [3]. At the point $\eta = \eta_0 = 0.854$ the function $G(\eta)$ vanishes, and hence, by (11), $\mathrm{d}\xi/\mathrm{d}\eta$ vanishes and the curve turns around. At this point the quasi-neutral approximation has broken down, and eq. (6) must be solved exactly. The sheath solution presumably joins on to the plasma solution in some manner such as that depicted by the dotted line in Fig. 1.

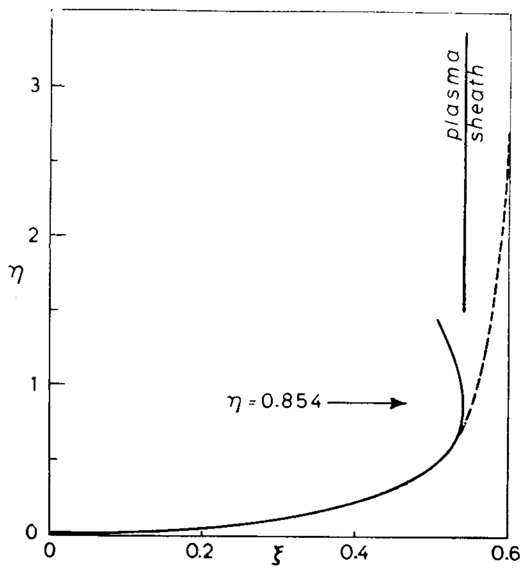

Fig. 1. – The plasma solution for normalized potential η as a function of normalized distance ξ from the mid-plane, for the case $g = 2/\pi$. The dashed curve shows schematically the sheath solution.

The value of η_0 is independent of $g(\eta)$, although the point ξ_0 where $\eta=\eta_0$ may vary with $g(\eta)$; and it is convenient to think of $\eta=.854$ as the point of transition between the plasma and the sheath (4).

We wish to note in passing that although the function $q(z)$ can be prescribed by the experimentalist (by, say, shaping the beam of ionizing electrons), the function $g(\eta)$ cannot be arbitrarily chosen, since eq. (11) must be satisfied. In particular, it is not possible to establish a cut-off $\eta_c<\eta_0$ such that $g(\eta)=0$ for $\eta>\eta_c$. From (11) we see that if $g(\eta)=0$ when $G(\eta)\neq 0$, $\mathrm{d}\eta/\mathrm{d}\xi$ must vanish. The potential will then remain at η_c until the sheath region near the wall is reached, where eq. (11) can be violated. But in this case it will be impossible to set up a sheath because the ions will not have enough energy to satisfy the sheath criterion (7). This is *not* to say that there is no solution to the problem. What will happen when one tries to establish such a cut-off by concentrating the ionizing electron stream into a narrow beam at $z=0$ is that the potential will rise from 0 to essentially η_0 *within the beam*, so that $g(\eta)$ does not have a cut-off below η_0, and the potential will be nearly η_0 from the edge of the beam up to the sheath at the wall. In this case the ion stream will satisfy the sheath criterion. In this paper we shall be concerned not with such pathological functions $q(z)$ but with the physically interesting cases $g=\text{const.}$ and $g(\eta)\sim e^{-\eta}$ or cases in between; we may then assume that η and all its derivatives increase monotonically with ξ.

If $n_0 f_i(\eta, u)$ is the number of ions per cm³ with velocity between u and $u+\mathrm{d}u$ at a point where the potential is η, then we have

(12) $$n_i = n_0\int_0^\infty f_i(\eta, u)\,\mathrm{d}u\,.$$

In terms of the dimensionless variables (5), eqs. (3) and (4) become

(13) $$u(\xi, \xi') = (\eta-\eta')^{\frac{1}{2}}\,,$$

(14) $$n_i = n_0\int_0^\eta \frac{g(\eta')(\mathrm{d}\xi/\mathrm{d}\eta)'\,\mathrm{d}\eta'}{(\eta-\eta')^{\frac{1}{2}}}\,.$$

Since $\mathrm{d}u=-\frac{1}{2}(\eta-\eta')^{-\frac{1}{2}}\mathrm{d}\eta'$, eq. (14) becomes

(15) $$n_i = n_0\int_0^{\eta^{\frac{1}{2}}} 2g(\eta')\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)'\mathrm{d}u\,.$$

(7) F. F. CHEN: to be published; see also D. BOHM: *Characteristics of Electrical Discharges in Magnetic Fields*, ed. by A. GUTHRIE and R. K. WAKERLING (New York, 1949), chap. 3.

Comparison with (12) shows that

$$f_1(\eta, u) = 2g(\eta')\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)', \tag{16}$$

where $\eta' = \eta - u^2$, and the primed quantities are evaluated at the point at which the ions were created in order to have velocity u at the point where the potential is η.

When the plasma solution is valid, eqs. (11) and (8) give

$$f_i^{(0)}(\eta, u) = \frac{2}{\pi}\left[\frac{1}{x} - 2F(x)\right], \tag{17}$$

where

$$x^2 = \eta - u^2 . \tag{18}$$

Thus the ion velocity distribution is uniquely defined in terms of the potential everywhere in the plasma and is independent of the ionization function $q(z)$. This gives some hope of determining the stability of the ion stream without knowing the exact experimental details.

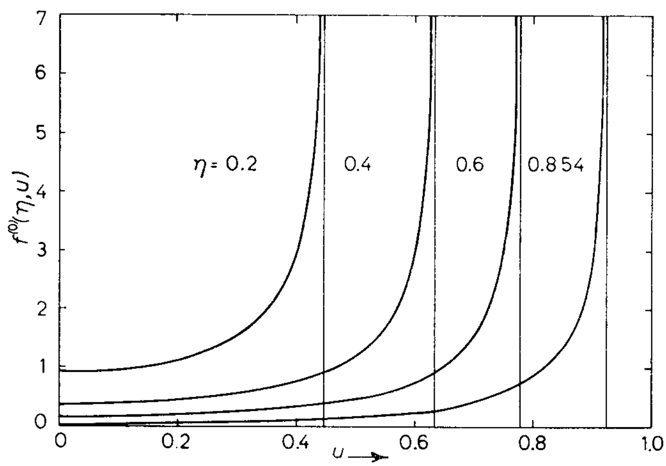

Fig. 2. – The ion velocity distribution $f_i^{(0)}(\eta, u)$, corresponding to the plasma solution, for various values of η.

The function $f_i^{(0)}(\eta, u)$ is shown in Fig. 2 for various values of η. This distribution is peaked at high velocities, since it is apparent from Fig. 1 that most of the ionization occurs at small values of η. The integrable singularity

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at $u=\eta^{\frac{1}{2}}$ arises because $\mathrm{d}\eta/\mathrm{d}\xi=0$ at $\eta=0$ for reasons of symmetry (cf. eq. (16)). One notes also that $\mathrm{d}f_i^{(0)}/\mathrm{d}u$ vanishes at $u=0$ but that $f_i^{(0)}$ vanishes at $u=0$ only for $\eta=\eta_0$.

The correction to $f_i^{(0)}$ due to charge unbalance is defined as $f_i^{(1)}$, so that $f_i=f^{(0)}+f_i^{(1)}$, and, by (7),

$$f_i^{(1)}(\eta,\,u)=\frac{2}{\pi}\left[\frac{1}{x}\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2}\right)_0+I_2(x^2)\right]. \tag{19}$$

If the electrons are assumed to be in thermal equilibrium, their velocity distribution is given by

$$f_e(\eta,\,u)=\frac{\varepsilon}{\sqrt{\pi}}\exp[-\eta]\exp[-\varepsilon^2u^2], \tag{20}$$

where

$$\varepsilon=(m/M)^{\frac{1}{2}}. \tag{21}$$

By using Appendix II, it can be verified that

$$\int f_i^{(0)}\,\mathrm{d}u=\int f_e\,\mathrm{d}u=\exp[-\eta].$$

4. – Stability in the plasma.

A necessary condition ([8]) for the onset of an electrostatic instability is that the total distribution function, weighted by the masses, have a minimum, or

$$\frac{\mathrm{d}f_e}{\mathrm{d}u}+\varepsilon^2\frac{\mathrm{d}f_i}{\mathrm{d}u}=0. \tag{22}$$

Let us first consider the lowest order distribution function $f_i^{(0)}$. Differentiating (17) with the help of Appendix II, we have

$$f_{i1}^{(0)}\equiv\frac{\mathrm{d}}{\mathrm{d}u}f_i^{(0)}(\eta,\,u)=-\frac{u}{x}\frac{\mathrm{d}}{\mathrm{d}x}f_i^{(0)}(x)=\frac{2u}{\pi}\left[\frac{1}{x^3}+\pi f_i^{(0)}(x)\right]. \tag{23}$$

For the electrons,

$$f_{e1}\equiv\frac{\mathrm{d}}{\mathrm{d}u}f_e(\eta,\,u)=-\frac{2\varepsilon^3}{\pi^{\frac{1}{2}}}u\exp[-\eta]\exp[-\varepsilon^2u^2]\approx-\frac{2\varepsilon^3}{\pi^{\frac{1}{2}}}u\exp[-\eta]. \tag{24}$$

([8]) O. PENROSE: *Phys. Fluids*, **3**, 258 (1960).

Because of the smallness of ε, the last approximation is always good for the velocities under consideration.

As a consequence of (22), ion waves will be damped by the thermal spread of the ions unless

(25) $$R \equiv \frac{\varepsilon^2 f_{i1}}{-f_{e1}} \leqslant 1 \; .$$

From (23) and (24),

(26) $$R = \varepsilon^{-1}\pi^{-\frac{1}{2}} \exp[\eta]\,[x^{-3} + \pi f_i^{(0)}(x)] \; .$$

Since $f_i^{(0)}(x) \geqslant 0$, $x^2 \leqslant \eta_0$, and $\eta \geqslant 0$ in the plasma region,

(27) $$R \geqslant \varepsilon^{-1}\pi^{-\frac{1}{2}}\eta_0^{-\frac{3}{2}} = 430$$

for Hg. Hence the condition (25) is far from satisfied for any η or u, and the ion stream is stable to this order of approximation. Since it is obvious that an instability will occur near η_0 if it is to occur at all, a more realistic lower limit for R can be obtained by including the factor e^{η_0}, so that

(28) $$R \geqslant 10^3 \quad \text{for Hg} \; .$$

We now consider the effect of the correction terms $f_i^{(1)}$ on R. One can see qualitatively what this effect will be from eq. (16):

(29) $$f_{i1} = 2\frac{\mathrm{d}\eta'}{\mathrm{d}u}\,\frac{\mathrm{d}}{\mathrm{d}\eta'}\left[g(\eta')\,\frac{\mathrm{d}\xi}{\mathrm{d}\eta'}\right] = -4u\left[g\,\frac{\mathrm{d}^2\xi}{\mathrm{d}\eta^2} + \frac{\mathrm{d}g}{\mathrm{d}\eta}\,\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right]' .$$

Since it is evident from Fig. 1 that $\mathrm{d}^2\xi/\mathrm{d}\eta^2$ is negative, we have for constant g:

(30) $$f_{i1} = 4ug\,|\mathrm{d}^2\xi/\mathrm{d}\eta^2|' \; .$$

From the dotted line in Fig. 1 it can be seen that $|\mathrm{d}^2\xi/\mathrm{d}\eta^2|$ is smaller everywhere in the exact solution than in the « plasma solution »; hence f_{i1} is less than $f_{i1}^{(0)}$. The question is whether it can be a factor of 10^3 less. To estimate this, we shall evaluate the correction terms $f_i^{(1)}$, given in eq. (19), using the « plasma solution » of eq. (11). This first-order correction will diverge at η_0, but it will allow us to get a better estimate of f_{i1} near η_0.

It will be convenient for us to adopt a notation in which subscripts on η and ξ indicate the order of differentiation by the other variable; thus $\eta_1 = \mathrm{d}\eta/\mathrm{d}\xi$, $\xi_3 = \mathrm{d}^3\xi/\mathrm{d}\eta^3$, etc. We must now also specialize to the case $g =$ constant. If instead $g \sim e^{-\eta}$, we note that the two terms in (29) are of the same sign and that they must add up to approximately the same amount as for

$g =$ constant, since $f_{i1}^{(0)}$ is independent of g. Now as $\xi(\eta)$ is corrected toward the exact solution, it can be seen from Fig. 1 that $|\mathrm{d}^2\xi/\mathrm{d}\eta^2|$ decreases but $|\mathrm{d}\xi/\mathrm{d}\eta|$ increases at any η. Hence the two terms change in opposite directions, and the correction to f_{i1} will be less for $g \sim e^{-\eta}$ than for $g =$ constant.

From (19), we have

$$(31) \qquad f_{i1}^{(1)} = \frac{2}{\pi}\frac{\mathrm{d}\eta'}{\mathrm{d}u}\,\frac{\mathrm{d}}{\mathrm{d}\eta'}\,[\eta'^{-\frac{1}{2}}(\eta_2)_0 + I_2(\eta')] = -\frac{4u}{\pi}\left[-\frac{1}{2}\eta'^{-\frac{3}{2}}(\eta_2)_0 + \frac{\mathrm{d}}{\mathrm{d}\eta'}\,I_2(\eta')\right].$$

For $g =$ constant, the solution of eq. (11) is

$$(32) \qquad \xi = 2\,a\,F(\eta^{\frac{1}{2}})\,, \qquad a \equiv \pi^{-1}g^{-1}.$$

The derivatives of ξ are easily found, and the derivatives of η are given by $\mathrm{d}/\mathrm{d}\xi = \xi_1^{-1}\,\mathrm{d}/\mathrm{d}\eta$. As shown in Appendix III, the quantities appearing in $f_{i1}^{(1)}$ are as follows:

$$(33) \qquad -\tfrac{1}{2}\eta'^{-\frac{3}{2}}(\eta_2)_0 = -\tfrac{1}{4}\pi^2 g^2 \eta'^{-\frac{3}{2}}\,,$$

$$(34) \qquad \frac{\mathrm{d}}{\mathrm{d}\eta'}\,I_2(\eta') = 4\pi^2 g\,\eta'^{-\frac{1}{2}} + I_3(\eta')\,,$$

where

$$(35) \qquad I_3(\eta) = \int_0^{\eta} (\eta - \eta')^{-\frac{1}{2}}\xi_1'^{-5}\cdot (9\xi_1\xi_2\xi_3 - 12\xi_2^3 - \xi_1^2\xi_4)'\,\mathrm{d}\eta'\,.$$

The quantity $K'(\eta')$, defined in eq. (A.22), which appears in the integrand of I_3 can easily be computed as a function of η' by use of the recursion formulas (A.14); this is shown in Fig. 3. The integrand is seen to be positive and to go rapidly to infinity as η' approaches η_0. An upper limit to $I_3(\eta)$ can be obtained by replacing $K'(\eta')$ by its maximum value $K'(\eta)$ and taking it outside the integral. Moreover, $I_3(\eta') \leqslant I_3(\eta)$, so that

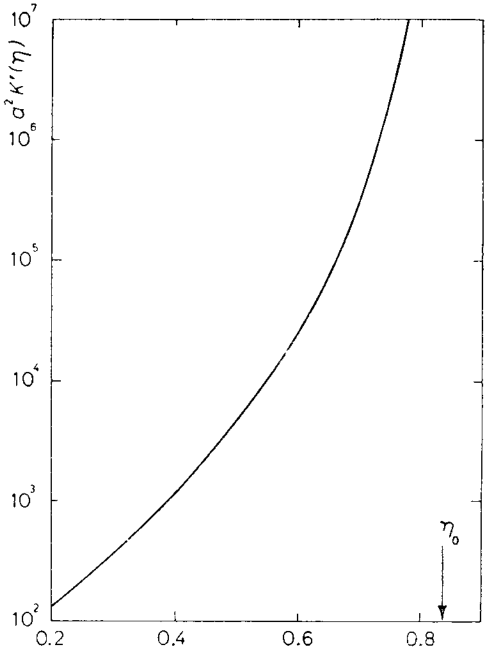

Fig. 3. – The integrand $a^2K'(\eta)$ (see text).

$$(36) \qquad I_3(\eta') \leqslant I_3(\eta) < K'(\eta)\int_0^{\eta}(\eta - \eta')^{-\frac{1}{2}}\,\mathrm{d}\eta' = 2\eta^{\frac{1}{2}}\,K'(\eta)\,.$$

Since $a^2K'(\eta) > 10^3$ for η's of interest, I_3 is greater than $10^3\pi^2g^2\eta^{\frac{1}{2}}$. Thus the other correction terms in (33) and (34) are entirely negligible with respect to I_3.

In order to evaluate I_3, we must find the magnitude of g. This can be done by considering the ion flux to the wall. Since g is constant, this is just

(37) $$j_i = \int_0^{L/h} g\,\mathrm{d}\xi = gLh^{-1}\,.$$

In terms of $f_i^{(0)}$, it is also

(38) $$j_i = \int_0^{\eta_0^{\frac{1}{2}}} u f_i^{(0)}(\eta_0, u)\,\mathrm{d}u = \frac{2}{\pi}\int_0^{\eta_0^{\frac{1}{2}}}\left[\frac{1}{x} - 2F(x)\right] x\,\mathrm{d}x = \frac{2}{\pi}F(\eta_0^{\frac{1}{2}}) \approx \frac{1}{\pi}\,,$$

by eq. (A.10). Therefore

(39) $$a^{-1} = \pi g = L^{-1}h \approx 10^{-2}$$

for $L \approx 1$ cm and $h \approx 10^{-2}$ cm.

Using (31), (34), and (36), we find that $f_{i1}^{(1)}$ is negative and that

(40) $$-f_{i1}^{(1)} < \frac{8u}{\pi}\,\eta^{\frac{1}{2}}a^{-2}[a^2K'(\eta)]\,.$$

Comparing this with $f_{i1}^{(0)}$ for $a = 10^2$, we find that $f_{i1}^{(1)}$ and $f_{i1}^{(0)}$ are of the same order of magnitude for $\eta \sim 0.6$. Therefore $f_{i1} \ll 10^3$ and $R \leqslant 1$ only for $\eta \geqslant 0.6$, and no ion waves are excited in the plasma up to $\eta = 0.6$. This result is not surprising, since the average ion velocity there is less than the acoustic velocity. Beyond this point the theory is not sufficiently accurate to tell. However, eq. (30) indicates that the $\eta - \xi$ curve must be very nearly linear in order to decrease f_{i1} by a factor of 10^3, and it is extremely unlikely to be so for any appreciable distance (in terms of a Debye length) unless $q(z)$ were especially tailored to give this.

Our neglect of the erf term in the electron density, eq. (2), can have little effect in the plasma, where $V - V_w$ is large. This term can have no effect on f_{e1}, since the latter relates to electrons traveling toward the wall, whereas the erf term describes the deficiency of electrons traveling away from the wall.

5. – Stability in the sheath.

In the sheath region, $\eta > \eta_0$, Poisson's eq. (6) must be solved more exactly. Let us first consider the case in which no ions are produced in the sheath.

At $\eta = \eta_0$, the ion distribution is approximately

$$f_i^{(0)}(\eta_0, u) = \frac{2}{\pi}\left[\frac{1}{x} - 2F(x)\right], \tag{41}$$

where $x^2 = \eta_0 - u^2$. If there is no ion production in the sheath, the ion distribution there will be given by the same expression (41), except that now x is given by $x^2 = \eta - u^2$, and $f_i^{(0)}$ vanishes for $u^2 < \eta - \eta_0$ and $u^2 > \eta$. This represents a simple acceleration of each ion in the sheath.

At $\eta = \eta_0$, $f_i^{(0)}$ is zero for $u = 0$, as is shown in Fig. 2. At $\eta > \eta_0$, $f_i^{(0)}$ is still zero for $u = u_{\min} = (\eta - \eta_0)^{\frac{1}{2}}$, and the curve looks very much the same except that it is slightly compressed. This is not true, however, for the derivative of $f_i^{(0)}$. From (23) we see that

$$f_{i1}^{(0)} = \frac{2u}{\pi}\left[x^{-3} + \pi f_i^{(0)}(x)\right]. \tag{42}$$

At $\eta = \eta_0$, $f_{i1}^{(0)} = 0$ for $u = 0$; however, for $\eta > \eta_0$, u cannot vanish. At $u = u_{\min}$,

$$f_{i1}^{(0)} = \frac{2}{\pi}(\eta - \eta_0)^{\frac{1}{2}}\eta_0^{-\frac{3}{2}} > 0. \tag{43}$$

Thus the slope of the shifted distribution is finite at its lower end. The expression (27) for R still obtains, so that at $u = u_{\min}$, $R > 10^3$.

This situation is shown in Fig. 4, in which the derivatives $f_{i1}^{(0)}(\eta_0, u)$, $f_{i1}^{(0)}(\eta, u)$, and $f_{e1}(\eta, u)$ are schematically plotted (solid lines). The dashed line shows what f_{i1} might look like in the sheath if the correction terms (19) were taken into account. If there were strictly no ions produced in the sheath, the curve would drop sharply to zero below $u_{\min}$ and would cross the line $\varepsilon^{-2} f_{e1}$, giving a solution to eq. (22). However, the discontinuity is not physically real; in practice, a small account of ionization in the sheath will extend f_{i1} smoothly to zero as shown either by the dashed line, which does not cross $\varepsilon^{-2} f_{e1}$ or by the dotted line, which does. Since the magnitude of $\varepsilon^{-2} f_{e1}$ has been exaggerated a factor of 10^3 in Fig. 4, we would expect that the amount of ionization in the sheath must be indeed small before an intersection can exist.

We next wish to show that an intersection cannot occur if the rate of ionization in the sheath is the same as in the plasma, for the case g = constant. From eq. (30), we see that

$$f_{i1} = 4ug\xi_2' = 4ug\eta_1'^{-3}\eta_2'. \tag{44}$$

It is evident from Fig. 4 that if an intersection is to occur, it will occur for a velocity u close to zero. Therefore $\eta_1'^{-3}\eta_2'$ in (44) must be evaluated at a point

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very close to η. The most unstable case is found by replacing η' by η. The point, then, is to see whether or not there is a place in the sheath where $\eta_1^{-3}\eta_2$ is small enough to make $R=1$.

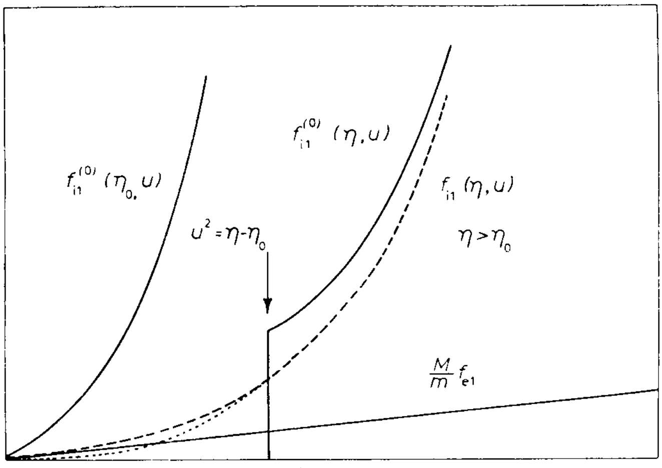

Fig. 4. – The derivatives of the velocity distributions at the edge of the sheath and in the sheath, drawn schematically. The slope of the electron curve has been greatly exaggerated. The dotted and dashed portions of $f_{i1}(\eta, u)$ show possible behaviors near $u = 0$ for varying amounts of ionization in the sheath.

To do this we shall solve an approximate sheath equation, in which the ion distribution is replaced by a monoenergetic stream with the same average velocity as the actual one. It can be shown that if the ion distribution has a width δ, the ion density differs only by a term proportional to δ^2, so that the monoenergetic approximation is a good one for computing n_i. We shall also neglect the contribution to n_i of the ions produced in the sheath, although of course these ions are all-important as far as the factor g in (44) is concerned. With these approximations, Poisson's equation becomes:

$$\eta_2 - \eta_{2s} = \tfrac{1}{2} e^{-\eta_s}[1 + \mathrm{erf}\,(\eta_w - \eta_s)][1 + u_s^{-2}(\eta - \eta_s)]^{-\frac{1}{2}} - \tfrac{1}{2} e^{-\eta}[1 + \mathrm{erf}\,(\eta_w - \eta)]\,, \tag{45}$$

where we have used the full expression (2) for n_e, η_s is the potential at which we shall start the solution, and u_s is the average ion velocity there. The first term on the right will be recognized as the usual expression for the density of a monoenergetic ion stream with velocity u_s; the factor in front is the electron (or ion) density at η_s.

Since we have found that the plasma solution breaks down around $\eta = 0.6$, we shall choose $\eta_s = 0.6$ and match both η_{1s} and η_{2s} to the plasma solution there. For Hg, $\eta_w \approx 6.2$, so that we may safely replace the erf $(\eta_w - \eta_s)$ by unity. The average velocity u_s is found as follows:

$$(46)\qquad \bar{u}(\eta) = \int_0^{\eta^{\frac{1}{2}}} u f_i^{(0)}(\eta, u)\,du \Big/ \int_0^{\eta^{\frac{1}{2}}} f_i^{(0)}(\eta, u)\,du =$$

$$= \exp[\eta] \int_0^{\eta^{\frac{1}{2}}} \frac{2}{\pi}[1 - 2xF(x)]\,dx = \frac{2}{\pi}\exp[\eta]\,F(\eta^{\frac{1}{2}})\,,$$

by eq. (A.10). Therefore

$$(47)\qquad u_s = \bar{u}(\eta_s) = \frac{2}{\pi}\exp[\eta_s]\,F(\eta_s^{\frac{1}{2}}) = 0.6123 \qquad \text{for} \qquad \eta_s = 0.6\,.$$

If we let $\varphi = \eta - \eta_s$ and $b = \eta_w - \eta_s$, eq. (45) becomes

$$(48)\qquad e^{\eta_s}\varphi_2 = (1 + u_s^{-2}\varphi)^{-\frac{1}{2}} - \tfrac{1}{2}e^{-\varphi}[1 + \operatorname{erf}(b - \varphi)] + e^{\eta_s}\varphi_{2s}\,.$$

Multiplying by φ_1 and integrating from ξ_s to ξ, we have, after some algebra,

$$(49)\qquad \tfrac{1}{2}\exp[\eta_s(\varphi_1^2 - \varphi_{1s}^2)] = 2u_s^2[(1 + \varphi u_s^{-2})^{\frac{1}{2}} - 1] + \tfrac{1}{2}(e^{-\varphi} - 1) +$$
$$+ \tfrac{1}{2}[e^{-\varphi}\operatorname{erf}(b - \varphi) - \operatorname{erf} b] + \tfrac{1}{2}e^{-b+\frac{1}{4}}[\operatorname{erf}(b - \tfrac{1}{2}) - \operatorname{erf}(b - \tfrac{1}{2} - \varphi)] + e^{\eta_s}\varphi_{2s}\varphi\,.$$

Because of the size of b, the term in $e^{-b+\frac{1}{4}}$ can be neglected, and erf b can be replaced by 1. Thus (49) becomes

$$(50)\qquad e^{\eta_s}[\varphi_1^2 - \varphi_{1s}^2 - 2\varphi_{2s}\varphi) =$$
$$= 4u_s^2[(1 + \varphi u_s^{-2})^{\frac{1}{2}} - 1] +$$
$$+ e^{-\varphi}[1 + \operatorname{erf}(b - \varphi)] - 2\,.$$

The boundary values of φ_{1s} and φ_{2s} are found from (A.14) and (A.15) to be $\varphi_{1s} = 4.25 \cdot 10^{-2}$ and $\varphi_{2s} = 10^{-2}$. For any η, it is then easy to com-

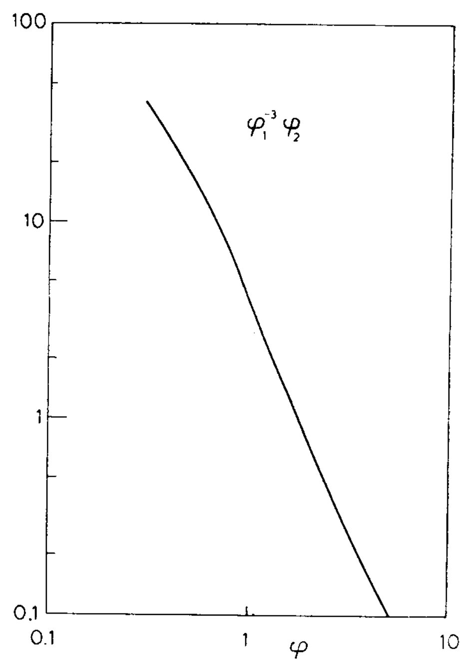

Fig. 5. – The quantity $\varphi_1^{-3}\varphi_2$, which is proportional to the derivative of the ion distribution, as computed from an approximate sheath equation for the case of constant g.

pute φ_2 from (48) and φ_1 from (50). The resulting values for $\varphi_1^{-3}\varphi_2$ (or $\eta_1^{-3}\eta_2$) are shown in Fig. 5. It should be noted that ordinarily no monotonic solution of (48) is possible for $u_s < \sqrt{2}$ because of the sheath criterion [7]. In this case, however, the boundary values of φ_1 and φ_2 were not assumed to be zero and, in fact, were large enough to allow a solution.

The ratio R, from (44) and (24), is

$$R = 2\varepsilon^{-1}\pi^{\frac{1}{2}} g e^{\eta} \eta_1^{-3} \eta_2 . \tag{51}$$

Using the value 10^{-2} for g, from (39), and replacing e^{η} by e^{η_s}, we see that

$$\eta_1^{-3}\eta_2 < 0.1$$

for $R < 1$, the condition for instability. Figure 5 shows that this condition is not fulfilled up to $\eta = 5$, which is almost at the wall.

6. – Conclusion.

We have shown that the classical « plasma solution » for a collisionless plane discharge leads to an ion velocity distribution which is independent of the mechanism of ion production, and that this distribution nowhere satisfies the necessary condition for an ion wave instability. Consideration of the first-order correction to the plasma solution showed that this result is accurate only up to $\eta = 0.6$. However, the solution of an approximate sheath equation showed that the ion stream is stable essentially everywhere, in the case g = constant. If, on the other hand, ion production in the sheath is severely minimized, it is conceivably possible for the necessary condition for instability (that the total distribution function have a minimum) to be satisfied. It seems reasonable that the sufficient part of Penrose's criterion [8] can be eventually satisfied as the ions are accelerated in the sheath. Therefore, if ion production in the sheath is eliminated, it is not impossible for ion waves to grow in the sheath. Such waves will travel at a small velocity in the laboratory system and will therefore appear as low frequency oscillations.

In the numerical computations performed in this paper we have assumed a 2 cm mercury discharge with a Debye length of 10^{-2} cm, corresponding to $kT = 2$ eV and $n_0 = 10^{10}\ \text{cm}^{-3}$. We believe the conclusions are essentially correct also for other elements and other reasonable values of the parameters.

In this paper we have not considered the high-frequency oscillations reported by GABOR *et al.* [6]. Such oscillations would not be predicted by this theory. At the same time, if such oscillations do exist, they would not affect our calculations, since the motion of the ions would be unaffected by oscillations in the 10 GHz range. The electron distribution could be thermalized by such oscillations, but we have already assumed a Maxwellian distribution except very near the wall. One possible effect of large amplitude high-frequency oscillations might be to increase ionization in the sheath to the extent that low-frequency oscillations cannot arise.

* * *

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APPENDIX I

If we let $\eta = y^2$ and $\eta' = y^2 \sin^2\theta$, eq. (6) becomes

$$\exp[-y^2] + \frac{d^2(y^2)}{d\xi^2} = \int_0^{\pi/2} 2y \sin\theta\, g(y^2\sin^2\theta)\, \frac{d\xi}{d(y^2\sin^2\theta)}\, d\theta\,. \tag{A.1}$$

This equation can be inverted by using Schlömilch's transformation (⁹), which states that if

$$\psi(y) = \frac{2}{\pi}\int_0^{\pi/2} \varphi(y\sin\theta)\, d\theta\,, \tag{A.2}$$

then

$$\varphi(y) = \psi(0) + y\int_0^{\pi/2} \psi'(y\sin\theta)\, d\theta\,. \tag{A.3}$$

Letting

$$\psi(y) = \exp[-y^2]\,\frac{d^2(y^2)}{d\xi^2}\,, \tag{A.4}$$

and

$$\varphi(y) = \pi y g(y^2)\,\frac{d\xi}{dy^2}\,, \tag{A.5}$$

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(9) E. T. WHITTAKER and G. N. WATSON: *A Course of Modern Analysis*, 4th ed. (Cambridge, 1952), p. 229.

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we see that eq. (A.1) is in the form of (A.2). Thus (A.3) becomes

$$\text{(A.6)}\qquad \pi y g(y^2)\frac{\mathrm{d}\xi}{\mathrm{d}y^2} = 1 + \left(\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2}\right)_{\eta=0} + y\int_0^{\pi/2}(-2y\sin\theta)\exp[-y^2\sin^2\theta]\,\mathrm{d}\theta + {} + y\int_0^{\pi/2}\frac{\mathrm{d}}{\mathrm{d}(y\sin\theta)}\left(\frac{\mathrm{d}^2(y^2\sin^2\theta)}{\mathrm{d}\xi^2}\right)\mathrm{d}\theta\,.$$

The first integral, I_1, can be evaluated by the substitution $t^2=y^2\cos^2\theta$, whereupon it becomes

$$\text{(A.7)}\qquad I_1 = -2\exp[-\eta]\int_0^{\eta^{\frac{1}{2}}}\exp[t^2]\,\mathrm{d}t = -2F(\eta^{\frac{1}{2}})\,.$$

The second integral, I_2, when expressed in terms of η', is just

$$\text{(A.8)}\qquad I_2 = \int_0^{\eta^{\frac{1}{2}}}\frac{\mathrm{d}^3\eta'}{\mathrm{d}\xi^3}\,\frac{\mathrm{d}\xi}{\mathrm{d}\eta'}\,\frac{\mathrm{d}\eta'}{(\eta-\eta')^{\frac{1}{2}}}\,,$$

where we have used $\mathrm{d}/\mathrm{d}\eta' = (\mathrm{d}\xi/\mathrm{d}\eta')(\mathrm{d}/\mathrm{d}\xi)$.

Appendix II

Properties of the function $F(x)$ (10):

$$\text{(A.9)}\qquad F(x)_{\max} = \frac{1}{2x_m} = 0.541 \quad\text{at}\quad x_m = 0.924\,,$$

$$\text{(A.10)}\qquad \frac{\mathrm{d}F(x)}{\mathrm{d}x} = 1 - 2xF(x)\,,$$

$$\text{(A.11)}\qquad \int F(\eta^{\frac{1}{2}})\,\mathrm{d}\eta = \eta^{\frac{1}{2}} - F(\eta^{\frac{1}{2}})\,,$$

$$\text{(A.12)}\qquad \int_0^{\eta} F(\eta'^{\frac{1}{2}})\,\frac{\mathrm{d}\eta'}{(\eta-\eta')^{\frac{1}{2}}} = \frac{\pi}{2}\left(1-\exp[-\eta]\right),$$

$$\text{(A.13)}\qquad F(x) = x\left(1 - \frac{2}{3}x^2 + \frac{4}{15}x^4 - \ldots\right).$$

(10) W. L. Miller and A. R. Gordon: *Journ. Phys. Chem.*, **35**, 2875 (1931).

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APPENDIX III

By using eq. (A.10) one finds for the derivatives of ξ the following:

$$\text{(A.14)} \qquad \begin{cases} \xi = 2aF(\eta^{\frac{1}{2}}) \,, \\ \xi_1 = a[\eta^{-\frac{1}{2}} - 2F(\eta^{\frac{1}{2}})] \,, \\ \xi_2 = -\frac{1}{2}\, a\eta^{-\frac{3}{2}} - \xi_1 \,, \\ \vdots \\ \xi_n = 2a \dfrac{\mathrm{d}^n}{\mathrm{d}\eta^n} (\eta^{\frac{1}{2}}) - \xi_{n-1} \,. \end{cases}$$

Thus

$$\text{(A.15)} \qquad \eta_2 = \xi_1^{-1} \frac{\mathrm{d}}{\mathrm{d}\eta} (\xi_1^{-1}) = -\xi_1^{-3}\xi_2 \,.$$

Since $F(0)=0$,

$$\text{(A.16)} \qquad (\eta_2)_0 = (2a^2)^{-1} \,.$$

From eq. (10) we have

$$\text{(A.17)} \qquad I_2(\eta) = \int_0^{\eta^{\frac{1}{2}}} \frac{\eta_3'\xi_1' \,\mathrm{d}\eta'}{(\eta - \eta')^{\frac{1}{2}}} \,.$$

This can be integrated by parts to give

$$\text{(A.18)} \qquad I_2(\eta) = 2\eta^{\frac{1}{2}}(\eta_3'\xi_1')_0 + 2\int_0^{\eta} (\eta - \eta')^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\eta'} (\eta_3'\xi_1') \,\mathrm{d}\eta'$$

Therefore

$$\text{(A.19)} \qquad \frac{\mathrm{d}}{\mathrm{d}\eta} I_2(\eta) = \eta^{-\frac{1}{2}}(\eta_3'\xi_1')_0 + \int_0^{\eta} (\eta - \eta')^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\eta'} (\eta_3'\xi_1') \,\mathrm{d}\eta' \,.$$

In terms of ξ_n's, it is easily found that

$$\text{(A.20)} \qquad \eta_3\xi_1 = \xi_1^{-4}(3\xi_2^2 - \xi_1\xi_3) \equiv K(\eta) \,.$$

For small η only the leading term, in η^{-2}, has to be retained. Using (A.14) and (A.13), we have finally

$$\text{(A.21)} \qquad \lim_{\eta\to 0} \eta_3\xi_1 = 4a^{-2} \,.$$

The integral in (A.19) can similarly be expressed in terms of the ξ_n's as follows:

$$\int_0^{\eta}(\eta-\eta')^{-\frac{1}{2}}K'(\eta')\,\mathrm{d}\eta'=\int_0^{\eta}(\eta-\eta')^{-\frac{1}{2}}\xi_1'^{-5}(9\xi_1\xi_2\xi_3-12\xi_2^3-\xi_1^2\xi_4)'\,\mathrm{d}\eta' . \tag{A.22}$$

RIASSUNTO (*)

Si trova che in una scarica piana senza collisioni la corrente di ioni che cadono sulle pareti è stabile rispetto alle instabilità dell'onda ionica nel caso in cui la funzione di ionizzazione è costante. Se si sopprime la ionizzazione nella guaina, invece, si possono eccitare nella stessa oscillazioni ioniche. Tali oscillazioni rassomiglieranno a quelle osservate recentemente da altri autori.

(*) *Traduzione a cura della Redazione.*