THE PHYSICS OF FLUIDS VOLUME 8, NUMBER 5 MAY 1965

Resistive Overstabilities and Anomalous "Diffusion"

FRANCIS F. CHEN
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey
(Received 3 November 1964; final manuscript received 22 January 1965)

A physical picture is given of microinstabilities which can occur in plasmas of finite resistivity with or without an effective gravitational field, and of finite Larmor radius stabilization. The drift waves become overstable because of a phase shift between the density and potential fluctuations. Transport of plasma across a magnetic field also arises from this phase shift; this transport does not necessarily resemble a diffusion process. This mechanism may account for the "anomalous diffusion" commonly observed in fully ionized plasmas. Particular attention is paid to the real part of the frequency and to the phase correlations, which can easily be measured to test this hypothesis. In regard to stellarator "pumpout", it is found that increasing the conductivity will not reduce the part of the loss rate which is due to the "universal" resistive overstability.

I. INTRODUCTION

WHEN both the finite Larmor radius r_L of the ions and the finite resistivity η are taken into account, it has been found[1,2] that drift waves in an inhomogeneous plasma can become overstable and grow in time, driven only by the plasma pressure. The dispersion curve of this mode is shown in Fig. 1. This overstability resembles the "universal" instability of a collisionless plasma in that the wave velocity is nearly the electron pressure-gradient drift velocity v_{de}, but the growth rate is connected with collisions rather than with resonant particles. For this reason, we believe that it is the resistive overstability which has been observed in cesium plasma experiments,[3] rather than the "universal" instability. Our purpose here is twofold: first, to give a physical interpretation of resistive overstabilities based on the calculations, which themselves will be presented in a separate paper; and second, to make use of this physical picture and some plausibility arguments to compute the anomalous transport of plasma across a magnetic field. We find that the process is better described by the term "electrostatic convection" than by "anomalous diffusion."

For simplicity consider a constant and uniform magnetic field $B\hat{z}$ containing a low-density plasma consisting of (a) cold ions distributed so that a constant density gradient n_0'/n_0 exists in the negative x direction and (b) isothermal electrons distributed so that the electric field is zero. Imagine a density perturbation of the form $n_1/n_0 = \nu \exp i(k_\perp y + k_\parallel z - \omega t)$,

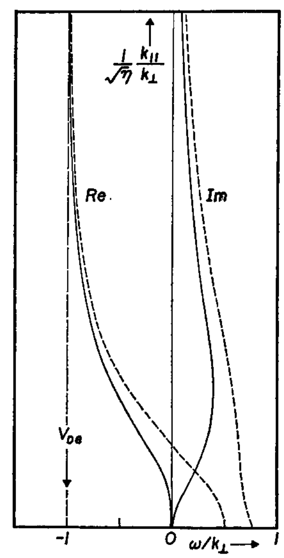

FIG. 1. Behavior of the real and imaginary parts of the electron drift mode phase velocity $\omega/k_\perp$, with (dashed line) and without (solid line) a gravitational field, as the angle of propagation $k_\parallel/k_\perp$ is varied, for $T_i = T_e$. The vertical scale depends on n_0, B, and $k_\perp v_{de}$.

as illustrated in Fig. 2. We first show that v_{de} is a natural velocity for plasma waves, quite independently of finite r_L considerations. Consider a value of $k_\parallel$ sufficiently large or a value of η sufficiently small that electrons flow freely along B; this corresponds to the region at the top of Fig. 1. Electrons are then in equilibrium along each line of force, and we have $n = n_0 \exp(e\phi/KT_e)$, or, to first order, $\nu = \chi \equiv e\phi/KT_e$. There will, in general, be a region in which $k_\parallel$ is large enough for this to be a good approximation but small enough so that ion motions along B can be neglected. Because of the initial gradient, a typical line of constant density will take the form of the curve labeled "isobar." By virtue of $\nu = \chi$, the equipotentials will be identical with the isobars. This distribution of potential gives rise to an electric field E_y and a drift $v_e = E_y/B$ of both electron and ion guiding centers in the x direction. At position 2 on Fig. 2, v_e is maximum and $n_1 = 0$.

[1] S. S. Moiseev and R. Z. Sagdeev, Zh. Eksperim. i Teor. Fiz. **44**, 763 (1963) [English transl.: Soviet Phys.—JETP **17**, 515 (1963)] and Zh. Tekh. Fiz. **34**, 248 (1964)[English transl.: Soviet Phys.—Tech. Phys. **9**, 196 (1964)].
[2] F. F. Chen, Phys. Fluids **7**, 949 (1964).
[3] N. D'Angelo and R. W. Motley, Phys. Fluids **6**, 422 (1963); N. D'Angelo, D. Eckhartt, G. Grieger, E. Guilino, and M. Hashmi, Phys. Rev. Letters **11**, 525 (1963); H. Lashinsky, *ibid.* **12**, 121 (1964).

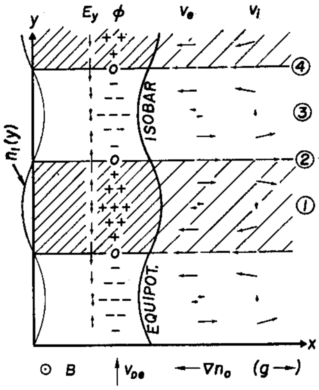

FIG. 2. Schematic of a drift wave $n_1 \approx \exp i\,(k_\perp y + k_\parallel z - \omega t)$, with $k_\parallel \ll k_\perp$, propagating across a density gradient ∇n_0 and possibly also a gravitational field g. For clarity the regions of positive density perturbation have been shaded. The first-order potential ϕ, electric field E_y, and ion and electron velocities v_i, v_e are indicated for the limiting case of "large" $k_\parallel$. The x dependence of these quantities is assumed to be weak.

A little later in time, v_e will have brought in particles from a denser part of the initial distribution, and n_1 (and therefore χ) will be positive at 2. Similarly, at position 4 ν and χ will decrease. Thus the perturbation apparently moves upwards on the diagram, in the same direction as v_{de}. A quarter cycle later, χ becomes maximum at 2, and hence E_y vanishes and v_e changes direction. To find the magnitude of the phase velocity, we note that the rate at which v_e increases the density at any given x is $\partial n/\partial t = -i\omega n_1 = -v_e n_0'$. Since $E_y = -ik_\perp\phi$ and $\phi = (KT_e/e)(n_1/n_0)$, we have

$$v_e = E_y/B = -ik_\perp(KT_e/eB)(n_1/n_0).$$

Hence, $-\omega n_1 = k_\perp(KT_e/eB)(n_1/n_0)n_0'$ and $\omega/k_\perp = -(KT_e/eB)(n_0'/n_0)$, which is just v_{de}. There is no instability because v_e is always 90° out of phase with n_1.

II. "UNIVERSAL" OVERSTABILITY

To see the reason for the overstability, we must consider finite Larmor radius, or, more accurately, finite ion inertia. The behavior of the ion velocity $\mathbf{v}_i$ is shown in Fig. 2. If E_y were steady, each ion would drift at the $\mathbf{E}\times\mathbf{B}/B^2$ velocity, and $\mathbf{v}_i$ would be equal to $\mathbf{v}_e$. However, since E_y is fluctuating, there will be a y component of $\mathbf{v}_i$, while the x component remains almost the same as v_e. This is apparent from the motion of each individual ion undergoing a forced vibration at a frequency $\omega < \omega_c$. As $v_{ix} = E_y/B$ changes in time, ions will be accelerated; their inertia opposes this acceleration and is equivalent to a force $\mathbf{g} = -\dot{\mathbf{v}}_i$. This in turn causes a drift $\mathbf{v} = (M/eB^2)(\mathbf{g}\times\mathbf{B})$ in the y direction. Thus

$$v_{iy} = \frac{M}{eB}\dot{v}_{ix} = \frac{M}{eB^2}\frac{\partial E_y}{\partial t} = -i\frac{\omega}{\omega_c}\frac{E_y}{B}. \qquad (1)$$

At position 2 on Fig. 2, E_y is at a maximum and hence $\dot{E}_y = v_{iy} = 0$. At position 3, v_{ix} vanishes but v_{iy} is maximum. The existence of v_{iy} causes a separation of charge: at position 2 both electrons and ions are brought in from the left by v_e and $v_{ix} \approx v_e$, but ions are depleted by v_{iy}. Hence there a negative charge develops at 2, and, similarly, a positive charge appears at position 4. If $k_\parallel$ is sufficiently large, this charge is easily canceled by electron flow along B. If, however, we now move downwards on Fig. 1 to a region of smaller $k_\parallel$ where electrons have difficulty traversing a half wavelength in the z direction, a negative charge will appear at 2 and a positive charge at 4. These charges build up until the excess E_z (over what normally exists to balance the electron pressure gradient along B) is sufficient to drive the electrons against the frictional drag due to collisions with ions. The net result is a downward shift in the potential distribution, as shown in Fig. 3. In the y–z plane, the potential curves are shifted in the $-z$ direction. This shift causes a component of $v_x = E_y/B$ to appear in phase with n_1. Thus the average v_x is positive over the region where n_1 is positive, and vice versa. Since positive v_x causes an increase in n by virtue of ∇n_0, the perturbation will grow.

As we progress downwards on Fig. 1 to even smaller $k_\parallel$, the electron current along B decreases be-

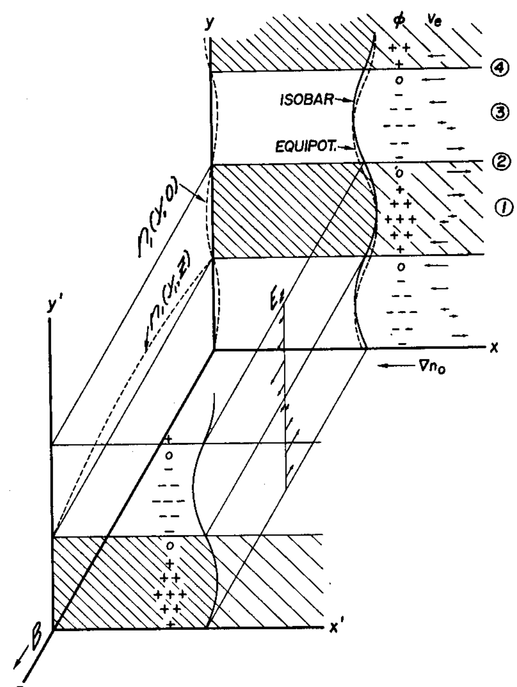

FIG. 3. Schematic of the wave of Fig. 2 for a case of smaller $k_\parallel$, when the potential distribution is shifted downwards to provide the E_z necessary to drive electrons along B. For clarity the indication of E_y and v_i has been omitted. The plane x'–y' is displaced one half-cycle along $\hat{z}$. The value of $k_\parallel$ has been exaggerated.

cause of the longer resistive path, and it takes longer to transport excess charges away from the x–y plane. Hence the frequency decreases and Im ω decreases with it. Finally, at $k_\parallel = 0$, the wave cannot exist at all, and it grinds to a halt. The point at the origin in Fig. 1 corresponds to a static perturbation: one can build up an arbitrary density perturbation by placing a number of ion guiding centers here and there and adding an equal number of electrons in such a way as to make the plasma neutral. Since we have neglected classical diffusion, these particles do not drift because no electric fields can arise which maintain continuity of current.

Note that the spiraling motion of ions does not appear because we have assumed $T_i = 0$. If T_i is finite, the growth rate is increased because the total plasma pressure, which drives the overstability, is increased. However, there is finite-r_L stabilization, to be discussed below, due to the fact that an ion samples regions of different **E** during the course of a cyclotron gyration; but there is still a net increase in Im ω for $T_i > 0$. Note also that if electron inertia rather than collisions were limiting the electron flow along B, the oscillations would have neutral stability.[2] This is because v_{ez} is then proportional to $(\nu - \chi)$ rather than to its gradient, and the phasing relationships necessary for the overstability are destroyed. It seems plausible from this physical picture that neglected effects such as the x dependence of the perturbation, the variation of KT_e during an oscillation, and an initial gradient in KT_e or n_0'/n_0 are only incidental to the overstability; it will always occur when a frictional drag impedes electron motion along B.

III. GRAVITATIONAL OVERSTABILITY

Although we have discussed only the "universal" mechanism for exciting this overstability, any other mechanism which causes a charge separation can give rise to the phase shift between n_1 and v_x which is responsible for the growth rate. For instance, consider the case of a small curvature in B. This is equivalent to a gravitational force **g** in the positive x direction, say. Referring again to Fig. 2, we see that **g** causes an upward drift of electrons and a downward drift of ions. This causes a build up of negative charge at position 2 and of positive charge at position 4, just as ion inertia did before. Again if $k_\parallel$ is sufficiently large, this charge accumulation is easily dissipated by electron motion along B, and there is only a pure oscillation traveling at v_{de}. For smaller $k_\parallel$, the same overstability occurs, but with a larger growth rate than before. This is shown in the upper portion of Fig. 1 for a typical small value of **g**. As we move toward $k_\parallel = 0$, however, the situation is qualitatively different from before because the growth rate is finite at $k_\parallel = 0$. This is just the Rayleigh–Taylor gravitational instability. The physical explanation of this instability is rather subtle, and at least one previous attempt at a physical picture is erroneous.[4]

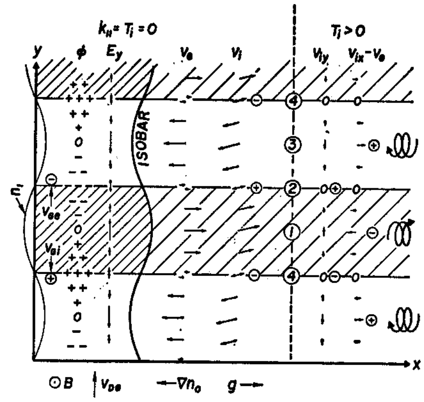

FIG. 4. Schematic of a gravitational instability with $k_\parallel = 0$, with $T_i = 0$ (left), and $T_i > 0$ (right). The ion motions are illustrated for a case of weak finite-r_L stabilization.

Continuing with the case $T_i = 0$, we find the $k_\parallel = 0$ situation to be that shown at the left of Fig. 4. The gravitational drifts v_{gi} and v_{ge} cause an excess of electrons to appear at position 2, and an excess of ions at position 4. The resulting E_y causes both species to drift with the velocity v_e along the zero-order density gradient. Since v_e is in phase with n_1, the density at position 1 increases and that at position 3 decreases, and the perturbation grows. The ordinary explanation stops at this point: no mechanism is given for arresting the separation of charge. In practice, $\nabla \cdot \mathbf{j} = 0$ must be satisfied so that the plasma remains neutral and becomes unstable in an orderly fashion no faster than the acoustic velocity. To see how this is achieved, we must consider the effects of ion inertia discussed above. By Eq. (1) there is a component of $\mathbf{v}_i$ in the y direction given by $v_{iy} = (\omega_c B)^{-1} \partial E_y/\partial t$. If Im $(\omega) \gg$ Re (ω), as is true for small g, v_{iy} is in phase with E_y, as shown in the left part of Fig. 4. It is clear that the divergence of v_{iy} causes positive charges to accumulate at position 2, and negative ones at position 4; and the value of Im (ω) adjusts itself so that this rate of charge accumulation just cancels that due to v_g.

In practice, v_e is not exactly in phase with n_1, and there is a small Re (ω) of the order of kv_g. If **g** is a real gravitational force, so that $v_{gi} \gg v_{ge}$, the wave

[4] F. C. Hoh, Phys. Fluids **6**, 1359 (1963).

velocity is in the direction of $\mathbf{v}_{gi}$. If $\mathbf{g}$ is actually a force due to a curvature in B, we have $v_{gi} = 0$ for $T_i = 0$, and the wave travels in the direction of $\mathbf{v}_{ge}$. These statements hold in the frame in which the equilibrium electric field vanishes.

IV. FINITE LARMOR RADIUS EFFECTS

When T_i is finite, the ion drift motions are modified by the finite size of the cyclotron orbits. Consider an ion moving in an inhomogeneous electric field $\mathbf{E} = E_y\hat{y}\cos ky$; typical spiral orbits are depicted at the right of Fig. 4. An ion with its guiding center at a maximum of $\mathbf{E}$ spends some of its time in the regions of smaller E; hence its drift speed v_{ix} is less than $v_e = E_y/B$. An ion with its guiding center at $\mathbf{E} = 0$ spends as much time in a region of $E > 0$ as of $E < 0$; hence its (vanishing) drift speed is unaffected. From such considerations one can anticipate the rigorous result, which can be obtained quite readily, that $v_{ix} - v_e$ is always proportional to and opposite in direction to $\mathbf{v}_e$ and depends on the second derivative of $\mathbf{E}$, that is, on k^2. When a large number of ions rather than a single one is considered, an additional effect arises because the distribution of guiding centers differs from the distribution of ions. Some of the ions at a maximum of $\mathbf{E}$ have their guiding centers at points of smaller $\mathbf{E}$, and this further reduces the average drift velocity v_{ix}. Schmidt[5] has shown that these two effects are of equal magnitude; the resulting ion drift is $v_{ix} = (1 - \frac{1}{2}k^2r_L^2)E_y/B$.

If $k^2r_L^2$ or T_i/T_e is small, so that finite-r_L effects are weak, the situation is that depicted at the right of Fig. 4. Since ω is essentially imaginary, v_{iy} is about the same as in the $T_i = 0$ case. The difference in ion and electron drifts along ∇n_0 now brings in an excess of ions at position 3 and of electrons at position 1. This has the effect of shifting the point of maximum ion accumulation from 2 to a position between 2 and 3. The whole pattern of ϕ, E_y, v_i, and v_e must now shift downwards so that the maximum of ion accumulation again occurs at 2, where the maximum electron accumulation due to v_{ge} occurs. This downward shift puts v_e and n_1 slightly out of phase and therefore decreases the growth rate of the instability. More important, it gives ω a real part, because the maximum of v_e now occurs below the maximum of n_1; since v_e increases the density by bringing in plasma of higher n_0, the maximum of n_1 will move downward in time; that is, the wave travels in the direction of v_{di}. Unless T_i is so small that $T_i/T_e \ll$

[5] G. Schmidt, Stevens Institute of Technology Report SIT–131 (1964).

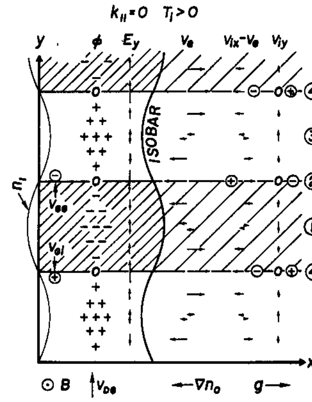

FIG. 5. Schematic of a gravitational wave completely stabilized by finite Larmor-radius effects.

$(Rn_0'/n_0)^{-1}$, R being the radius of curvature of B, the wave velocity is always $\frac{1}{2}v_{di}$. We are unable to give a simple reason for the factor $\frac{1}{2}$. Since the charge accumulation rate due to $v_{ix} - v_e$ is proportional to n_0'/n_0, finite-r_L stabilization is larger the steeper the initial gradient, as is well known.[6]

We now consider a case of large T_i or large $k^2r_L^2$, in which the gravitational instability is completely stabilized by finite Larmor radius; this is shown in Fig. 5. The difference $v_{ix} - v_e$ is now so large that it dominates over v_{iy} in providing ions at position 2 to cancel the electron accumulation due to v_{ge}. The maximum of $v_{ix} - v_e$ then occurs at 2, so that v_e and n_1 are 90° out of phase. Two pure oscillations can now occur, both traveling in the direction of v_{di}. Since the wave travels downwards, $\partial E_y/\partial t$ is negative at position 1, say, and by Eq. (1) v_{iy} is negative there. From Fig. 5 it is seen that v_{iy} actually aids the gravitational drift in providing negative charges at 2. As T_i or $k^2r_L^2$ is changed, ω, and hence v_{iy}, changes in such a way as to keep the net charge buildup rate zero. The explanation of Hoh[4] is in error on two counts: first, it depended on the x dependence of both E_y and ϕ, a dependence which was ignored in the theoretical results[6] under examination; second, the effect of ion inertia on v_{iy} was not found.

We can now easily find the effect of finite T_i on the "universal" overstability of Sec. II. Referring to Fig. 2, we see that since $v_{ix} - v_e$ is opposite in direction to $\mathbf{v}_e$, the finite-r_L effect will bring in negative charges at 2 and positive charges at 4. This is such as to add to the charge accumulation due to v_{iy}. Hence finite r_L will increase the growth rate of the overstability. We find that the rate is increased by the factor $(1 + T_i/T_e)$.

Experimentally, the resistive overstability of Fig. 2 with $k_\parallel \neq 0$, with or without $\mathbf{g}$, can easily be dis-

[6] M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Nucl. Fusion Suppl. Pt. 1, 143 (1962).

tinguished from a nearly stabilized gravitational instability with $k_\parallel = 0$ (Fig. 5), because the waves travel in opposite directions. Even if a zero-order electric field makes the direction of propagation difficult to determine, it is clear that the correlation between ϕ and n_1 has opposite sign in the two cases.

V. THE DIFFUSION COEFFICIENT

From the physical picture given above, it is clear that rapid convection of plasma across a magnetic field is produced automatically by these overstabilities: the oscillations grow because the first-order electric field causes regions of high initial density to drift into regions where the density perturbation is already positive, and vice versa. This continues until the initial density gradient is destroyed. During the time of growth, plasma is convected "outwards" on the average; that is, toward the region of low initial density. The usual way[1] to compute the diffusion coefficient $D_\perp$ is to use a dimensional argument: $D_\perp$ is a length squared divided by a time. If the fluctuations are taken to be random, the "correlation time" taken to be $(\text{Im } \omega)^{-1}$, and the "correlation length" taken to be $k_\perp^{-1}$, which in turn is of order r (the plasma radius), then one obtains the Bohm diffusion coefficient $D_B = \alpha KT_e/eB$, where α is a constant smaller than unity. This method of computing D_B, similar to that originally suggested by Spitzer,[7] is unsatisfactory because the electric-field fluctuations are assumed to be random. Drift wave instabilities, on the other hand, grow precisely because the electric-field fluctuations are not random but are correlated with density fluctuations; the process is not a random walk at all. We therefore propose to calculate the anomalous loss rate in a more plausible manner by taking these correlations into account.

In the overstabilities considered here the growth rate is caused by the component of $v_x = E_y/B$ which is in phase with n_1. This can be seen from the linearized equation of continuity of the ion fluid, which, with v_z and $\partial v_x/\partial x$ neglected, reads

$$dn_1/dt + v_x n_0' + ik_\perp n_0 v_y = 0. \tag{2}$$

The real part is, for real n_1,

$$\text{Im } (\omega) n_1 + \text{Re } (v_x) n_0' - k_\perp n_0 \text{ Im } (v_y) = 0. \tag{3}$$

The third term contains the ion inertia effect which causes the overstability, but it can be neglected in Eq. (3) in comparison with the second term because if ω is primarily real, Eq. (1) says that $\text{Im } (v_y) \approx (\omega/\omega_c) \text{ Re } (v_x)$. We have implicitly assumed $\omega/\omega_c \ll (n_0'/n_0)/k_\perp$, which is necessary in any case to eliminate a high-frequency root of the complete dispersion relation. Thus we have

$$\text{Re } (v_x) \approx -\text{Im } (\omega)(n_1/n_0'). \tag{4}$$

Consider now the average flux j_x across a given plane $x = x_0$. This is given by

$$j_x = \langle (n_0 + n_1) v_x \rangle = \tfrac{1}{2} n_1 \text{ Re } (v_x)$$
$$= \tfrac{1}{2} n_0 (-n_0/n_0')(\text{Im } \omega)(n_1/n_0)^2. \tag{5}$$

This is a second-order quantity, but it is given to first-order accuracy by Eq. (5) if the value of Im (ω) from the linear theory is used, since the perturbation amplitude n_1 is assumed known. Second-order terms in Re (v_x) would appear in third-order in j_x.

As long as Im (ω) is positive, there is a net average transport of plasma in the $-\nabla n_0$ direction even if the oscillations are coherent. For small amplitudes the flux can be evaluated from the known expressions for Im (ω). For the "universal" overstability, the growth rate[8] in the upper region of Fig. 1 is (in esu)

$$\text{Im } (\omega/\omega_c) = \tfrac{1}{2}(en_0\eta/B)(k_\perp^2/k_\parallel^2)(k_\perp^2 r_L^2)(r_L n_0'/n_0)^2. \tag{6}$$

Inserted in Eq. (5) this gives a j_x proportional to n_0', and hence this overstability has an effective diffusion coefficient $D_\perp \equiv -j_x/n_0'$ dependent on $k_\perp$, $k_\parallel$, and n_1. For the gravitational overstability the "large" $k_\parallel$ growth rate is given[8] by

$$\text{Im } (\omega/\omega_c) = -(en_0\eta/B)(k_\perp^2/k_\parallel^2)(r_L/R)(r_L n_0'/n_0). \tag{7}$$

Now j_x is independent of n_0', and no diffusion coefficient can be defined. The maximum growth rate in the absence of curvature[8] is approximately

$$\text{Im } (\omega/\omega_c) = 0.4(k_\perp v_D/\omega_c) = 0.2(k_\perp r_L)\ |r_L n_0'/n_0|, \tag{8}$$

independent of resistivity. If the corresponding value of $k_\parallel$ were dominant, the escape flux would be independent of n_0'. The maximum growth rate in a strongly curved field is

$$\text{Im } (\omega/\omega_c) = [-(r_L n_0'/n_0)(r_L/R)]^{\frac{1}{2}}, \tag{9}$$

occurring at $k_\parallel = 0$; hence the escape flux can even be proportional to $(n_0')^{-\frac{1}{2}}$.

So far we have considered only small values of n_1/n_0, for which the results are as rigorous as the linear theory. We now abandon all attempts at rigor in order to make the transition to the nonlinear regime. Note that if n_1 and $k_\perp$ had maximum values of n_0 and r_L^{-1}, respectively, which could be achieved simultaneously, Eqs. (5) and (8) would give $j_x =$

[7] L. Spitzer, Jr., Phys. Fluids 3, 659 (1960).

[8] F. F. Chen (to be published); see also J. D. Jukes, Phys. Fluids 7, 1468 (1964).

$0.1n_0r_L\omega_c$, and there would be no magnetic confinement. To get a more realistic estimate of the escape flux, we must consider the distribution of amplitudes n_1 in the nonlinear limit. Physically, the growth of the oscillations corresponds to a drift of alternate layers of plasma to the left and right, respectively, as depicted, for instance, in Fig. 4. Clearly, when n_1 becomes so large that $\partial n_1/\partial y$ is comparable with $\partial n_0/\partial x$, the direction of the local gradient will be changed, and the wave cannot continue growing as before. This nonlinear behavior is reached when

$$n_1 = n_0'/k_\perp \equiv n_{1\max}. \qquad (10)$$

What happens beyond this point is of no concern to us, because either the local gradient is so small that the oscillation does not grow, or the propagation vector is turned so that the correlated drifts are in the y (or azimuthal) direction. The escape flux occurs during the growth of the waves, and as long as the boundary conditions impose an average prevailing density gradient in the $-x$ direction, there will be a tendency for waves to start in the y direction, giving a net flux in the x (or radial) direction. We therefore approximate the escape flux by inserting the linear growth rate Im (ω) in Eq. (5) and taking its average over time. Letting

$$n_1 = n_{10} \exp [\text{Im}\ (\omega)t],$$

we find the average n_1^2 over a growth period (from $n_1 = n_{10}$ to $n_1 = n_{1\max}$) to be

$$\langle n_1^2 \rangle = \tfrac{1}{2}n_{1\max}^2[\ln (n_{1\max}/n_{10})]^{-1} \equiv \tfrac{1}{2}\epsilon n_{1\max}^2.$$

The size of the initial perturbation n_{10} is unknown, but it enters only weakly. With the use of Eq. (10), we obtain

$$j_x = \tfrac{1}{4}n_0\epsilon k_\perp^{-2}(-n_0'/n_0)\ \text{Im}\ (\omega), \qquad (11)$$

where n_0'/n_0 is the prevailing density gradient averaged over the eddies in the turbulent state.

If $k_\parallel$ is allowed to adjust itself to give the maximum growth rate, use of Eq. (8) yields

$$j_x = 0.05n_0\epsilon r_L^2\omega_c k_\perp^{-1}(n_0'/n_0)^2, \qquad (12)$$

where $r_L^2 = K(T_i + T_e)/M\omega_c^2$. Thus for each mode characterized by $k_\perp$, $D_\perp$ varies as KT/B and is itself proportional to n_0'. In a curved field Eq. (9) should be used instead for the lowest modes; however, finite-r_L stabilization strongly damps the gravitational overstability for larger values of $k_\perp r_L$, and Eq. (12) is valid for the higher modes even in a curved field. If there is a constraint on $k_\parallel$, Eq. (6) or (7) must be inserted in Eq. (11). For a field with curvature R we obtain

$$j_x = \tfrac{1}{4}\epsilon e^2\eta r_L^2 n_0'^2/(MRk_\parallel^2), \qquad (13)$$

which shows $D_\perp \propto KTn_0'/B^2$. Any convective instability with Im $(\omega) \propto n_0'$ would give $j_x \propto n_0'^2$. If $k_\parallel$ is fixed by the topology of the experiment, Eq. (13) gives the flux for any $k_\perp$. For a straight field we obtain

$$j_x = -\tfrac{1}{8}n_0\epsilon(n_0e\eta/B)(k_\perp/k_\parallel)^2 r_L^4\omega_c(n_0'/n_0)^3, \qquad (14)$$

whence $D_\perp \propto (KT)^2n_0'^2/B^4$.

From the foregoing it can be seen that resistive overstabilities can give transport rates considerably faster than classical diffusion by a process not resembling diffusion at all. Our description of thir process does not depend sensitively on the randoms ness of the oscillations or on their radial wavelengths. The main assumption about the nonlinear behavio⁻ is the distribution of amplitudes given by Eq. (10). This is a reasonable assumption which yields a power spectrum in qualitative agreement with experiment. Our philosophy has been to ignore the escape flux caused by a wave after it has grown to the limit given by Eq. (10) on the grounds that this flux will be small (and we do not know how to compute it anyway) and to confine our attention to the flux occurring during the growth of a wave, when the linearized value of Im (ω) is apt to be a good approximation.

VI. RELATION TO EXPERIMENT

Although the anomalous "diffusion" coefficient has been derived theoretically numerous times in the literature, we know of no good measurement of it in a fully ionized gas. To see the reason for this, consider a plasma in a conducting cylinder. Since $E_y = 0$ near the wall, fluctuating electric fields, random or otherwise, cannot directly bring particles all the way to the wall: near the wall all drifts are in the azimuthal direction. What drift instabilities can do is make it impossible for a large density gradient to exist in the body of the plasma, so that the loss rate is controlled by classical diffusion in a steep gradient near the wall. The small gradient necessary to maintain the turbulence in the region of anomalous diffusion is difficult to measure. We are acquainted with two devices producing a hydromagnetically stable, fully ionized plasma of sufficient duration for study of electrostatic "diffusion." In a thermally ionized cesium plasma,[3] the losses by recombination, mostly at the endplates, are so large that diffusion losses are masked. In the stellarator, an aperture limiter forms the wall of the plasma, and it is observed[9] that the density profile is almost flat inside the aperture and falls rapidly outside. This indicates that the loss rate is not limited solely by the anomalous transport in the interior region. Rather, the flux is controlled by ambipolar flow along B to the limiter in the exterior region, together with a small amount of anomalous radial transport. The latter is caused by small amplitude oscillations which occur in spite of the existence of conducting endplates outside the aperture. The density gradient in the interior is only that necessary to produce this flux by electrostatic convection. In most cases this small gradient cannot be measured accurately, and the B and KT dependence of the loss rate is characteristic of the exterior region rather than the interior region. At present one can test the theory only by asking whether Im (ω) is large enough to account for the growth time of the oscillations, and by measuring the phase between n_1 and ϕ, and hence between n_1 and v_x, to see if $j_x = \langle n_1 v_x \rangle$ is large enough to account for the observed loss rate. Preliminary observations[10] of the latter sort in the stellarator indicate that this is so.

Our method of computing the escape flux can be used with any of the "drift" instabilities. To see whether the resistive overstabilities described here are relevant to stellarator "pumpout", we consider numerically the "low-current" case[9] in which the Ohmic heating current appears to play no role. Thus we take $KT = 5$ eV, $n_0 = 10^{13}$ cm^{-3}, $B = 3 \times 10^4$ G, $r = 5$ cm, $L = 1200$ cm, $R = 100$ cm, with hydrogen as the gas. We assume $k_\perp \approx m/r$ and $n_0'/n_0 \approx 1/r$. The gravitational overstability depends on the sign of n_0'/n_0; hence the lower m numbers are presumably eliminated by the rotational transform. For $\lambda_\parallel \lesssim \frac{1}{2}L$, the transform does not help, and for large m the growth rate is given by Eq. (7). This gives Im $(\omega) \approx m^2$ sec^{-1}, or $\tau \approx 100$ μsec for $m = 100$, corresponding to $k_\perp r_L = 0.2$. Im (ω) decreases linearly with η and with L, if R is the same order as L. The "universal" overstability, on the other hand, does not depend on the sign of n_0'/n_0, and is not affected by the rotational transform. The latter, however, imposes a periodicity condition because $\lambda_\parallel$'s longer than L will generate a periodicity in the azimuthal direction, which fixes $k_\parallel k_\perp$. We therefore write, approximately, $\frac{1}{2}\lambda_\parallel \leq mL$. If this condition allows the $k_\parallel$ corresponding to the max mum growth rate to occur, Eq. (8) gives Im $(\omega) = 240m$ sec^{-1}, independent of η. If the upper limit to m is fixed by $k_\perp r_L = 1$, we obtain $\tau \approx 10$ μsec. As η is decreased, Im (ω) remains constant but $\lambda_\parallel$ becomes longer and longer. Finally, the periodicity condition above limits $\lambda_\parallel$, and a further decrease in η decreases Im (ω) linearly; the maximum Im (ω) is then given by Eq. (6), with $k_\parallel k_\perp$ fixed and $k_\perp r_L = 1$. This dependence on η is not achieved until $KT = 1.3$ keV. T_e and T_i have separate effects: increasing T_e decreases η, while increasing T_i increases r_L, reducing the maximum value of m. At 5 eV, the maximum growth rate is reached for all $m > 13$; for $m < 13$, Eq. (6) and the periodicity condition give Im $(\omega) = 6 \times 10^{-4} m^6$ sec^{-1}, and $\tau = 1.7$ msec at $m = 10$. Since the observed confinement times are several milliseconds, we conclude that all but the lowest modes grow sufficiently fast; and we proceed to compute the escape flux.

Although Im (ω) is largest for large $k_\perp$, we have seen that j_x decreases with $k_\perp$ because the amplitudes of the higher modes are smaller. Whether the loss rate is determined primarily by high or low m numbers depends on the distribution of initial values of $k_\perp$, and this requires some assumption about the turbulent state. We imagine that a perturbation with $\mathbf{k} = k_\perp \hat{y}$ grows until the local gradient is in the y direction; then waves with smaller $k_\perp$ in the x direction can grow in this local gradient. These do not contribute to j_x, but as they hit the nonlinear limit, they allow waves of still smaller wavelength to grow with $\mathbf{k}$ in the y direction. Thus large-scale perturbations break up into smaller ones until $\lambda_\perp$ is of order r_L and classical diffusion damps the growth of smaller wavelengths. In this process there is no reason to assume that all values of m between $m_{\max} = 2\pi r/r_L$ are not equally probable. Hence we estimate the total flux by taking only one value of $k_\parallel$ for each $k_\perp$, namely, the one giving the largest growth rate, assuming $k_x \approx 0$, putting $k_\perp = m/r$ in Eq. (12), and summing over m from 1 to $m_{\max}$. We then obtain

$$j_{\text{total}} = 0.05 n_0 \epsilon r_L^2 r \omega_c \ln (2\pi r/r_L)(n_0'/n_0)^2, \qquad (15$$

approximately proportional to KT/B. It is clear that most of the flux comes from the higher modes. Thus although our picture of the turbulent state is similar to Kadomtsev's,[11] we come to the conclusion that the particle loss comes as much from the large m numbers as from the small because of the larger number of possible modes of short wavelength. Note that $m_{\max}$ enters only logarithmically. The observed flux is, for $\tau \approx 4$ msec, $j \approx 5 \times 10^{15}$ ions/cm^2 sec; from Eq. (15) we obtain a similar flux if we set $\epsilon \equiv$

[9] S. Yoshikawa, W. L. Harries, R. M. Sinclair, and J. C. Young, Princeton University Plasma Physics Laboratory Report, MATT–Q–21 (1963).
[10] S. Yoshikawa (private communication).
[11] B. B. Kadomtsev, J. Nucl. Energy, Pt. C 5, 31 (1963).

$1/\ln(n_{max}/n_{10}) = 0.3$ and $n_0'/n_0 = 1/1.2r$. This value of n_0'/n_0 is larger than the measured value, and this value of ϵ seems somewhat high. Hence we conclude that not all the presently observed loss rate is due to the "universal" overstability. However, the fraction of it which is cannot be reduced by increasing the conductivity. In addition, there is a possible contribution from the gravitational overstabilities with $\lambda_\parallel \leq \frac{1}{2}L$. Multiplying Eq. (13) by the number of modes, which is $2\pi r/r_L$, we obtain a flux of the same order of magnitude as above. This flux also varies as B^{-1}, but it can be reduced by reducing η.

Thus far we have ignored the effect of shear in the magnetic field. This will put a lower limit on the value of k_x. However, consideration of the x dependence[1] shows that Im (ω) is not greatly affected by k_x for $k_x \leq k_y$, so that if the large values of k_y are important for the loss rate, it will not be greatly affected by the shear-imposed minimum value of k_x. On the other hand, shear can also limit the value of $k_\parallel$. If these waves can be localized to regions of x several Larmor radii thick, in which the shear is negligible, then the growth rates and escape fluxes computed above are not greatly changed. If the waves are not localized, then $k_\parallel$ cannot be small everywhere, and one can set $k_\parallel/k_y \geq r/L$. This condition does not allow the maximum growth rate for the "universal" mode to be attained in the stellarator, and we must use Eq. (6) for the growth rate. Setting $k_\parallel$ equal to its minimum value in Eq. (14) and summing over $k_\perp$, we obtain $j_x = 3\times10^{13}\epsilon(n_0'/n_0)^3$, which is about 100 times less than the shearless value. This flux is proportional to $n_0^2\eta(KT)^{\frac{3}{2}}/B^3$, so that it is independent of temperature. The gravitational mode is more severely affected by shear because if $k_\parallel$ is set equal to $k_\perp r/L$ in Eq. (13), j_x is proportional to $k_\perp^{-2}$. It is then the small $k_\perp$ modes that contribute most to the sum over $k_\perp$, and these modes are not allowed by the condition on k_x.

At large Ohmic heating currents, an increased loss rate is observed[9] in the stellarator. The overstabilities considered here are not greatly affected by a zero-order current, but instabilities associated with a temperature gradient[12] can then arise. In particular, the "rippling" mode[13] may increase the loss rate by making thinner the classical-diffusion layer just inside the conducting aperture limiter. In addition, an instability due to resonant particles[11] can be driven by the current, but as the electron–ion mean free path is only $3 \times 10^{-2}L$ in the high-current discharge, we do not believe resonant particles can be important. The current-driven anomalous flux can be eliminated by other methods of heating and by increasing KT_e. We have seen, however, that the loss due to the "universal" overstability cannot be reduced by an increase in temperature and therefore may be more serious in the long run.

In addition to their importance in the stellarator, these resistive overstabilities can explain oscillations observed in cesium plasmas.[3] Here the effects of the endplates are all-important; our analysis of this problem will appear elsewhere.[14] Endplate stabilization in cesium plasmas makes possible careful studies of the onset of these overstabilities.

ACKNOWLEDGMENTS

We have benefited from a number of interesting conversations with T. E. Stringer and J. L. Johnson, and we wish to thank S. Yoshikawa, R. Kulsrud, L. Spitzer, Jr., and J. M. Greene for critical comments on the manuscript.

This work was supported by the U. S. Atomic Energy Commission.

[12] See, for instance, F. C. Hoh, Phys. Fluids **7**, 956 (1964).
[13] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).
[14] F. F. Chen, presented at the Symposium on Diffusion of Plasma Across a Magnetic Field, Feldafing, Germany, 1964 (to be published).