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LINEAR DRIFT WAVES

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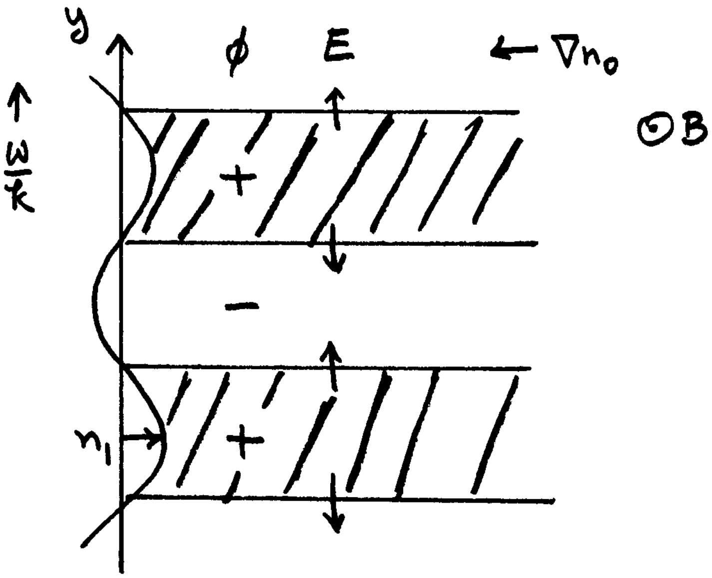
I. INTRODUCTION

In the early days of plasma physics, theoretical studies of plasma waves were invariably made with the assumption of uniform plasma density. On the other hand, experiments almost always required a density gradient, such that the density is higher on the axis of a plasma column than near the edge. It is not surprising, therefore, that the theory of uniform plasmas could not explain the observations of low-frequency oscillations below 50 kHz found almost universally in magnetized laboratory plasmas. These oscillations, typically peaked between 10 and 20 kHz regardless of the type of plasma, are too low in frequency to be explained by ion acoustic waves, the lowest frequency waves predicted by uniform-plasma theory.

The explanation of these low-frequency fluctuations was found when the density gradient was taken into account. Originally, Kadomtsev and Nedospasov proposed a theory, relevant to weakly ionized gases, which explained the waves in a positive column. Since then, it has been found that the instability is a universal one, occurring in fully ionized and even collisionless plasmas. These so-called "drift waves" have been best documented in thermally ionized plasmas (Q-machines), in which this universal instability can be stabilized by ion-ion collisions so that single modes near threshold can be studied.

Drift waves are so called because they have a phase velocity proportional to the electron diamagnetic drift V_{De}. To see this, consider a wave propagating in the y direction in a plasma in which the density gradient is in the -x direction and the uniform magnetic field B_o is in the z direction.



We assume an electrostatic wave, since the frequency is so low, and take a perturbation of the form

$$\phi_1 = \phi\, e^{i(ky + k_z z - \omega t)} , \tag{1}$$

where ϕ_1 is the potential fluctuation of the wave. We also assume $k \gg k_z$, so the wave propagates primarily in the y (azimuthal) direction. However, we must not let $k_z = 0$; k_z must be large enough that the electrons can stay in thermal equilibrium by moving along B_o. In that case, an electric field E_z will arise which is just sufficient to buck out the pressure gradient force ($\propto \partial n/\partial z$) on the electrons, and the potential will obey the Boltzmann relation

$$n = n_o + n_1 = n_o e^{e\phi/KT_e} \approx n_o\left(1 + \frac{e\phi}{KT_e}\right) \tag{2}$$

Thus if there is density variation in the y direction, the potential will follow it. This gives rise to an electric field E_y, which causes ions and electrons to drift in the x direction:

$$V_x = \frac{E_y}{B_o} \tag{3}$$

As the wave passes by, E_y will oscillate, and the plasma will drift back and forth in the x direction, causing n to change, because of ∇n_o. If we focus our attention at a particular point, the rate of increase of density there will be

$$\frac{\partial n}{\partial t} = - V_x \frac{\partial n_o}{\partial x} = - \frac{E_y}{B_o} \frac{\partial n_o}{\partial x} , \tag{4}$$

as given by the equation of continuity and Equation (3). Fourier analyzing according to Equation (1) and substituting for n_1 from Equation (2), we have

$$-i\omega n_1 = - i\omega n_o \frac{e\phi}{KT_e} = i k \frac{\phi}{B_o} n'_o , \tag{5}$$

or

$$\frac{\omega}{k} = - \frac{KT_e}{eB_o} \frac{n'_o}{n_o} \equiv V_{De} . \tag{6}$$

Thus V_{De} is a natural velocity for waves in an inhomogeneous plasma.

If $n'_o/n_o \approx 1/R$, we can get an estimate of V_{De} by using the formula

$$-V_{De} = 10^8 \frac{T(eV)}{B(G)} \frac{1}{R(cm)} \frac{cm}{sec} . \tag{7}$$

For example, if T = 100 eV, B = 10 kG, and R = 3 cm, we have $|V_{De}| = 3 \times 10^5$ cm/sec. For the lowest mode (m = 1), we have $\omega = kV_{De} = \frac{m}{r} V_{De} = V_{De}/r$, and f ≈ 17 kHz, which is of the order of the frequencies observed. Note that Equation (7) is independent of mass and that T/RB tends to remain constant from one experiment to another. This is the reason for the universality of fluctuations in the kilohertz range.

Although it was easy to derive the real part of ω, it is not so easy to see why the wave is unstable. The imaginary part of ω comes from charge-separation effects we did not take into account, First, since E_y is time-varying, ion inertia will cause the ions to drift in the y direction as well as the x direction. This effect will be covered in Section II. Second, since the ions have a much larger Larmor radius than the electrons, and since

E_y varies in space, the ions will see an average E_y smaller than what the electrons see. The E×B drift will be smaller for ions, and there will be a charge separation because this drift is along ∇n_o. This finite-Larmor-radius (FLR) effect will be covered in Section III.

These two small effects both tend to cause a charge buildup and hence an additional electric field E_y. However, if k_z is large enough, the flow of electrons along B_o will cancel this charge buildup so as to satisfy the Boltzmann relation (2). As long as Equation (2) obtains, ω will be real and equal to kV_{De}. In order to have an instability, k_z must be so small that electrons cannot flow unimpeded from wave crest to trough along B_o. Then Equation (2) will be violated, and it will be possible to have E_y and E_x different from what is rigorously prescribed by Equation (2). There are several mechanisms which can affect electron parallel mobility:

1. electon-neutral collisions
2. electon-ion collisions
3. Landau damping
4. inductance
5. electron inertia.

These mechanisms can be considered to "trigger" the instability by allowing Equation (2) to be violated so that charge buildup can take place.

Effect (1) is dominant in weakly ionized gases, such as a positive column. Effect (2) is the finite resistivity of a fully ionized plasma and is the mechanism we shall consider in detail. Effect (3) has to do with a distortion of the electron distribution function due to the interaction with the wave of electrons with $V_z \approx \omega/k_z$; this is the mechanism responsible for drift instabilities in collisionless plasmas. Effect (4) is the impedance to electron flow provided by the magnetic field the flow creates. This effect is important only at high β ($\beta \equiv 8\pi nKT/B_o^2$), when $\omega/k_z \approx V_A$, the Alfvén speed. Effect (5) might be thought to be the dominant mechanism in the absence of collisions, but it turns out that the effect has the wrong phase and does not cause an instability except in higher order. Since (2) is the dominant mechanism in Q-machine experiments, we shall derive the dispersion relation for that case.

II. RESISTIVE DRIFT WAVES WITH $T_i=0$

The simplest drift instability can be found from the fluid equations when the resitivity η is finite and T_i is zero. In this case, there is no FLR effect since $r_L=0$, and ion inertia alone causes instability.

1) Assumptions

a) $T_i = 0$, $T_e > 0$

b) $\eta \neq 0$

c) $\nabla x E = 0$, $B = B_o$ (low-β, electrostatic assumption)

d) $m/M = 0$

e) $n_i = n_e \equiv n$ $\quad$ $(k\lambda_D << 1)$

f) $V_{thi} << \omega/k_z << V_{the}$

g) $\nabla p_e = KT \nabla n$ (isothermal electrons)

2) Equations (e. s. u. - c. g. s.)

$$Mn\left(\frac{\partial V_i}{\partial t} + V_i \cdot \nabla V_i\right) = en\left(-\nabla\phi + V_i x B\right) - n^2 e^2 \eta (V_i - V_e) \tag{8}$$

$$0 = -en(-\nabla\phi + V_e x B) - KT\nabla n - n^2 e^2 \eta (V_e - V_i) \tag{9}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (nV_i) = \frac{\partial n}{\partial t} + \nabla \cdot (nV_e) = 0 \tag{10}$$

The resistivity η is related to the elctron-ion collision frequency ν_{ei}

$$\eta = \frac{m\nu_{ei}}{ne^2} \quad . \tag{11}$$

Since Boltzmann's constant occurs only in the ∇p_e term, an adiabatic equation of state will give the same result as these equations if K is replaced by $\frac{5}{3}$ K in the final answer.

3) Equilibrium

We assume $V_{oz} = E_o = 0$ and $n_o = n_o(x)$ in the unperturbed state. Setting $\partial/\partial t = 0$, we have

$$Mn_o(V_{io} \cdot \nabla V_{io}) = en_o V_{io} \times B - n_o^2 e^2 \eta(V_{io} - V_{eo}) \tag{12}$$

$$0 = -en_o V_{eo} \times B - KT\nabla n_o - n_o^2 e^2 \eta(V_{eo} - V_{io}). \tag{13}$$

Neglecting $V_{io} \cdot \nabla V_{io}$ for the time being, adding (13) to (12), and taking the cross product with B, we obtain

$$j_o = KT \frac{B \times \nabla n_o}{B^2}, \tag{14}$$

where $j_o \equiv en_o(V_{io} - V_{eo})$ is just diamagnetic current. The y components of (12) and (13) yield

$$(V_{io} \times B)_y = (V_{eo} \times B)_y = \eta j_o .$$

Using (14), we find

$$V_{iox} = V_{eox} = -\frac{\eta KT}{B^2} \eta_o' \tag{15}$$

This is just the "radial" diffusion velocity due to Coulomb collisions. Since this is a slow process compared to the wave frequencies we are considering, we shall neglect V_{ox}. The term $V_{io} \cdot \nabla V_{io}$ automatically vanishes when we set $V_{iox}=0$. The x components of (12) and (13) give

$$V_{ioy} = 0, \quad V_{eoy} = -\frac{KT}{eB}\frac{n_o'}{n_o} \equiv V_{De} . \tag{16}$$

There is only a diamagnetic drift of the electrons in the "azimuthal" direction in equilibrium.

4) Perturbation

We neglect classical diffusion by keeping η only in the z component of the electron equation of motion and Fourier analyse as follows:

$$\phi_1(x,y,z,t) = \phi(x) e^{i(k_y y + k_z z - \omega t)}$$
$$V_1(x,y,z,t) = V(x)\, e^{i(k_y y + k_z z - \omega t)} \qquad (17)$$
$$n_1(x,y,z,t) = \nu(x) n_o(x)\, e^{i(k_y y + k_z z - \omega t)} ,$$

so that $\nu = n_1/n_o$. Linearizing Equation (8) and setting $V_{io}=0$, we have for the ions

$$-i\omega V_i = \frac{e}{M}(-\nabla\phi + V_i \times B) . \qquad (18)$$

for frequencies $\omega \ll \Omega_c \equiv eB/M$, the solution of (18) is

$$iV_{ix} = \frac{k_y \phi}{B} - \frac{\omega}{\Omega_c}\frac{\phi'}{B} \qquad ' = \frac{\partial}{\partial x}$$
$$V_{iy} = \frac{\phi'}{B} - \frac{\omega}{\Omega_c}\frac{k_y \phi}{B} \qquad (19)$$
$$V_{iz} = \frac{ek_z}{M\omega}\phi \approx 0$$

In the last equation we have indicated that we shall consider k_z's so small that V_{iz} can be neglected; this amounts to neglecting the transition from drift waves to ion acoustic waves at "large" k_z/k_y. The first term in V_{ix} and V_{iy} is simply the ExB drift, and the second term is the ion inertia effect mentioned earlier.

Linearizing the x component of Equation (9) for electrons, we have

$$0 = en_o(-\phi' + V_{ey}B) + KT(n_o\nu' + \nu n_o') + en_1 V_{oe} B. \qquad (20)$$

The last two terms cancel by virtue of Equation (16). Defining

$$\chi \equiv \frac{e\phi}{KT} ,$$

we then have

$$V_{ey} = \frac{KT}{eB} (\chi - \nu)' . \tag{21}$$

Similarly, the y and z components of Equation (9) give

$$iV_{ex} = k_y \frac{KT}{eB} (\chi - \nu) \tag{22}$$

$$V_{ez} = ik_z \frac{KT}{eB} \omega_c \tau_{ei} (\chi - \nu), \tag{23}$$

where $\omega_c \tau_{ei} = \dfrac{eB}{m\nu_{ei}} = \dfrac{B}{n_o e\eta}$ is the important dimensionless quantity characterizing the resistivity.

Linearizing the electron continuity equation (10), we have

$$(\omega - k_y V_{oe}) n_1 + in_o (V'_{ex} + ik_y V_{ey} + ik_z V_{ez}) + iV_{ex} n'_o = 0. \tag{24}$$

Substituting for V_{ex}, V_{ey}, and V_{ez} from Equations (21-23), we find that the radial derivatives cancel out (this is a fortunate feature of the fluid equations for massless particles), and we have an equation which does not depend on the "shape" of the perturbation in the x direction:

$$(\omega - k_y V_{De})\nu - (k_y V_{De} + ik_z^2 \frac{KT}{eB} \omega_c \tau_{ei})(\chi - \nu) = 0. \tag{25}$$

At this point we shall adopt a nomenclature which is the best compromise among the multifarious notations used by Russian and American authors. We define

$$\omega^* = k_y V_{De} = - k_y \frac{KT}{eB} \frac{n'_o}{n_o}$$

$$\sigma_\parallel = \frac{k_z^2}{k_y^2} (\omega_c \tau_{ei}) \Omega_c \tag{26}$$

$$a_i^2 = KT_e / M\Omega_c^2$$

$$b = k_y^2 a_i^2 , \quad b\sigma_\parallel = k_z^2 \frac{KT}{eB} \omega_c \tau_{ei} .$$

The quantity $\sigma_\|$ is a frequency proportional to the conductivity and is sometimes called ω_s. The quantity $a_i\sqrt{2}$ is the ion Larmor radius computed with the electron temperature, and b is a common parameter in small Larmor radius expansions. With this notation, Equation (25) becomes

$$\nu = \chi \frac{\omega^* + ib\sigma_\|}{\omega + ib\sigma_\|} \quad . \qquad (27)$$

This relation between the density and potential perturbations is a particularly useful one because it must be obeyed locally (in x) regardless of the complications to be introduced in the following sections. Note that in the limit $b\sigma_\| \to \infty$ Equation (27) becomes $\nu = \chi$, which is simply the Boltzmann relation (2). Finite $b\sigma_\|$ introduces the phase shift between ν and χ which is necessary for instability.

Our last equation is the linearized form of the ion continuity equation (10). This is

$$- i\omega n_1 + n_o(V'_{ix} + ik_y V_{iy}) + V_{ix} n'_o = 0 \ . \qquad (28)$$

Substituting for V_i from Equation (19) and for n_1 from Equation (27), we obtain the dispersion equation

$$\phi'' + \frac{n'_o}{n_o}\phi' - \left(k_y^2 + \frac{\Omega_c}{\omega}\frac{n'_o}{n_o}k_y\right)\phi - B\Omega_c \frac{\omega^* + ib\sigma_\|}{\omega + ib\sigma_\|}\frac{e\phi}{KT} = 0 \ . \qquad (29)$$

5) The Dispersion Relation

Equation (29) is a complex second-order differential equation with variable coefficients and must be solved together with some "radial" boundary conditions — typically, $\phi = 0$ at $x=0$ and $x=R$. Other terms would have appeared had we used cylindrical rather than rectilinear coordinates. The solution of such an equation is the principal problem in drift-wave theory. The eigenfunctions of ϕ give the variation of wave amplitude with x, and the complex eigenvalues of ω give the frequency and growth rate.

If n_o is some arbitrary function of x, as measured in an experiment, it is clear that the wave equation, of which Equation (29) is a simple example, must be solved numerically. However, there are four ways to get an analytic dispersion relation. First, it is sometimes possible by a suitable choice of the function $n_o(x)$ to reduce the wave equation to a standard form, so that the solution $\phi(x)$ can be written in terms of Whittaker's functions or Hermite polynomials, etc. Second, the use of a "quadratic form" (Reference 7) allows one to estimate $\omega(k)$ by using a trial function $\phi(x)$. Third, by a suitable transformation the wave equation can be put into the form

$$W'' + Q(x)W = 0 \ , \tag{30}$$

which can be solved by the WKB approximation. The latter is, in principle, valid only if $\partial\phi/\partial x >> \partial\phi/\partial y$. Since, in practice, the opposite is usually true, it is a better approximation to neglect $\partial/\partial x$ altogether. This amounts to simply using the equation for the turning points of Equation (30). Finally, in the same spirit one can neglect the radial derivatives of ϕ in Equation (29). What remains gives an algebraic equation for $\omega(k)$, which is called the "local" dispersion relation. This is the simplest approximation and is almost universally employed.

If we neglect ϕ'' and ϕ' in Equation (29) and multiply through by $a_i^2\omega(\omega + ib\sigma_\parallel)/\phi$, we obtain the desired dispersion relation for $T_i = 0$:

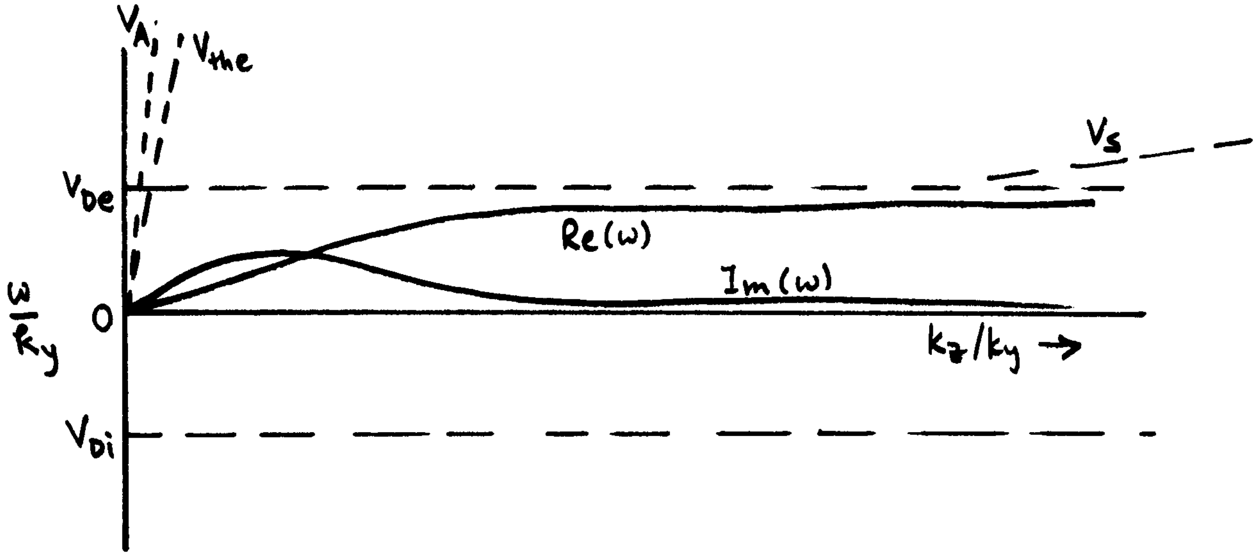
$$\omega^2 + i\sigma_\parallel(\omega - \omega^*) = 0 \ . \tag{31}$$

In the limit of large $\sigma_\parallel/\omega^*$, it is clear that $\omega \approx \omega^*$. Solving for $\omega - \omega^*$, we obtain an expression for the growth rate in this limit:

$$\omega - \omega^* = i\,\frac{\omega^{*2}}{\sigma_\parallel} \ , \tag{32}$$

where we have substituted ω^* for ω on the right-hand side. Note that the growth rate $\gamma \equiv \text{Im}(\omega)$ is proportional to the resistivity. The other root of Equation (31) is heavily damped in this limit and is unimportant.

The unstable root of Equation (31) behaves qualitatively as shown in the following diagram.



Inclusion of the V_{zi} terms would have made Re(ω) approach the line V_s (acoustic velocity) at large k_z/k_y. The lines representing the Alfvén velocity and the electron thermal velocity are drawn to indicate that other equations are valid only for $\omega/k_z \ll V_A, V_{the}$. A physical picture of this instability is given in Reference 4b.

III. RESISTIVE DRIFT WAVES WITH $T_i > 0$

1. Equations

For $T_i > 0$, we shall replace Equation (8) with

$$Mn\left(\frac{\partial v}{\partial t} + V \cdot \nabla V\right) = en(-\nabla\phi + VxB) - \nabla p - \nabla \cdot \pi + Mng. \qquad (33)$$

The subscript i has been suppressed. We have neglected classical diffusion and have split the divergence of the ion stress tensor into an isotropic part $\nabla p = \gamma_i KT_i \nabla n$ and an anisotropic part $\nabla \cdot \pi$. The tensor π is called the magnetic viscosity tensor and consists of a part connected with ion-ion collisions and a part which remains in the limit $\nu_{ii} \rightarrow 0$. This collisionless viscosity is simply the FLR effect mentioned earlier: the ions have a modified E×B drift in a nonuniform E field. We have also included in Equation (33) the term Mng due to a "gravitational" field $\underset{\sim}{g}$, because it does not complicate the analysis to do so. The "gravitational" force can represent the centrifugal force of particles moving along a curved magnetic field or the centrifugal force of a rotating plasma column.

Deferring collisional viscosity to the next section, we may write the relevant components of π in the collisionless case as follows, correct to first order in $k^2 r_L^2$.

$$\pi_{yy} = -\pi_{xx} = \frac{1}{2} \frac{nKT_i}{\Omega_c} \left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y} \right)$$
$$\pi_{xy} = \pi_{yx} = \frac{1}{2} \frac{nKT_i}{\Omega_c} \left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} \right) . \tag{34}$$

We shall neglect V_{zi}.

2. Equilibrium

In addition to V_{eo}, there is now an ion drift V_{io} given by

$$\underset{\sim}{V}_{io} = (V_{Di} - V_G)\hat{y} , \tag{35}$$

where

$$V_{Di} \equiv \frac{KT_i}{eB} \frac{n_o'}{n_o} \quad \text{and} \quad V_G \equiv \frac{g}{\Omega_c} .$$

We have had to assume n_o'/n_o = constant and $\underset{\sim}{g} = \hat{g}x$ = constant, so that V_{io} = constant and $V_o \cdot \nabla V_o = \pi_o = 0$. Equation (35) is the only place where g will appear.

3. Perturbation

The linear part of Equation (33) is

$$Mn_o \left(\frac{\partial V}{\partial t} + V_o \cdot \nabla V \right) = en_o (-\nabla\phi + V \times B) - KT_i \nabla n - \nabla \cdot \pi_1 \tag{36}$$

We shall make the local approximation at the outset and neglect x derivatives of V, ν, and ϕ. In evaluating π_1, there will be terms in $n_o \nabla V$ and $n_1 \nabla V_o$;

the latter vanish because we have taken V_o constant. With these assumptions, Equation (34) yields

$$\pi_{yx} = -\frac{1}{2}\frac{n_o KT_i}{\Omega_c} ikV_y, \qquad \pi'_{yx} = -\frac{1}{2} ik \frac{KT_i}{\Omega_c} n'_o V_y$$

$$\pi_{xx} = -\pi_{yy} = -\frac{1}{2}\frac{n_o KT_i}{\Omega_c} ikV_x, \qquad \pi'_{xx} = -\frac{1}{2} ik \frac{KT_i}{\Omega_c} n'_o V_x, \tag{37}$$

where the subscript y on k has been suppressed. We define

$$\omega_i \equiv kV_{Di} = k\frac{KT_i}{M\Omega_c}\frac{n'_o}{n_o}$$

$$r_L^2 \equiv \frac{2KT_i}{M\Omega_c^2}, \qquad \theta \equiv \frac{T_i}{T_e}, \tag{38}$$

so that $\omega_i = -\theta\omega^*$ and $b = k^2 r_L^2/2\theta$. The x and y components of Equation (36) then give

$$(\omega - kV_o + \tfrac{1}{2}\omega_i)V_x = i\Omega_c(1 - \tfrac{1}{2}\theta b)V_y$$

$$(\omega - kV_o + \tfrac{1}{2}\omega_i)V_y = k\frac{KT_e}{M}(\chi + \theta\nu) - i\Omega_c(1 - \tfrac{1}{2}\theta b)V_x. \tag{39}$$

These equations for V_x and V_y may be solved by writing $\alpha = \omega - kV_o + \frac{1}{2}\omega_i$, $\beta = 1 - \frac{1}{2}\theta b$, and $\gamma = k(KT_e/M)(\chi + \theta\nu)$ and eliminating V_y and V_x, respectively. We then have

$$(\alpha^2 - \beta^2\Omega_c^2)V_x = i\beta\Omega_c\gamma$$

$$(\alpha^2 - \beta^2\Omega_c^2)V_y = \alpha\gamma. \tag{40}$$

The term α^2 can now be dropped because it is of order b^2. This can be seen by writing

$$\frac{\alpha}{\beta\Omega_c} \approx \frac{kV_o}{\Omega_c} = k\frac{KT_i}{M\Omega_c^2}\frac{1}{R} = \frac{k^2 r_L^2}{2kR} \lesssim b \quad \text{for} \quad kR > 1, \quad \theta < 1.$$

With this simplification, Equation (40) becomes

$$V_x = -i\,\frac{k\,\frac{KT_e}{M}\,(\chi+\theta\nu)}{\Omega_c\left(1-\frac{1}{2}\theta b\right)}$$

$$V_y = -\,\frac{(\omega-kV_o+\frac{1}{2}\omega_i)k\,\frac{KT_e}{M}\,(\chi+\theta\nu)}{\Omega_c^2\left(1-\frac{1}{2}\theta b\right)^2}\ . \qquad (41)$$

The linearized ion equation of continuity is now written

$$(\omega-kV_o)\nu - kV_y + iV_x\,\frac{n_o'}{n_o} = 0\ . \qquad (42)$$

Inserting Equation (41) into (42), we find a fortunate cancellation of terms (due to fluid drifts which are not real particle drifts) and end up with

$$(\omega-kV_o-\theta\omega^*)\nu + [b(\omega-kV_o)-\omega^*]\,\chi = 0\ . \qquad (43)$$

The electron equations are unchanged, so we use Equation (27) to eliminate ν. We also use Equation (35) for V_o.

4. Dispersion Relation

Equation (43) then yields the following dispersion relation:

$$\omega(\omega-\omega_i)+kV_G(\omega^* b^{-1}+\omega)+i\sigma_\parallel[\omega-\omega^*+kV_G+b(\omega-\omega_i+kV_G)] = 0\ . \qquad (44)$$

The last term in b can safely be neglected here; this will not be true in the presence of ion-ion collisions.

a) Pure drift wave. For $V_G = 0$, we have the following dispersion relation for resistive drift waves:

$$\omega(\omega-\omega_i)+i\sigma_\parallel(\omega-\omega^*) = 0\ . \qquad (45)$$

In the limit $\sigma_\parallel/\omega^* >> 1$, we have

$$\omega-\omega^* \approx i\,\frac{\omega^*(\omega^*-\omega_i)}{\sigma_\parallel} = i\,\frac{\omega^{*2}(1+\theta)}{\sigma_\parallel}\ . \qquad (46)$$

Compared to Equation (32), the growth rate is increased by the factor $1+\theta$, representing the increase in total plasma pressure. This is reasonable from an energetic viewpoint. Kinematically, the growth rate is increased through the FLR effect in the ion drifts.

b) Resistive-g mode. If we retain the kV_G terms in Equation (44) and take the same limit $\sigma_\parallel/\omega^* >> 1$, noting that $kb^{-1}\omega^* = -\Omega_c n_o'/n_o$, we find

$$\omega \approx \omega^* - kV_G + i\,\frac{\omega^*(\omega^*-\omega_i) - g\,n_o'/n_o}{\sigma_\parallel} \quad . \tag{47}$$

If g is due to a curvature in B, we may set it equal to $2KT/MR_c$, where R_c is the radius of curvature. Depending on your point of view, Equation (47) then shows either the increase in drift wave growth rate when $gn_o' < 0$ or the decrease in gravitational instability growth rate (see below) due to finite k_z.

c) FLR Stabilization of Flute Modes. When $k_z = 0$, Equation (44) becomes

$$\omega^2 + (kV_G - \omega_i)\omega - gn_o'/n_o = 0 \tag{48}$$

In the absence of the middle term, one obtains the usual growth rate $\gamma = (-gn_o'/n_o)^{1/2}$ for what is known as the interchange, Rayleigh-Taylor, or gravitational flute instability. The kV_G term arises from finite ion inertia, and the ω_i term from finite Larmor radius. Together, these terms lower the growth rate and reduce it to zero for sufficiently large k. This is known as FLR stabilization. Note that the real part of ω is $\frac{1}{2}kV_{oi}$; flute modes (k_z=0) travel in the direction of the ion diamagnetic drift.

IV. DRIFT WAVES WITH ION VISCOSITY

Equation (45) predicts that drift waves are always unstable regardless of k_y and k_z, as long as they are both finite. Experimentally, it was found that drift waves could be stabilized by lowering B, and hence increasing $k^2r_L^2$. To explain this, we must include the collision terms in the magnetic viscosity tensor π. If one takes the full tensor as given by, say, Bernstein and Trehan [Nuclear Fusion 1, 3(1960)] and takes the limit $\Omega_c^2 \tau_{ii}^2 >> 1$, one finds that for motions perpendicular to B one can write $\nabla \cdot \pi$ as

$$\nabla \cdot \pi = -\mu_\perp \nabla_\perp^2 V, \qquad \mu_\perp \equiv \frac{nKT_i}{4\Omega_c^2 \tau_{ii}},$$

$$\tau_{ii} = C \frac{5}{8\sqrt{\pi}} \left(\frac{M}{KT_i}\right)^{1/2} \frac{(KT_i)^2}{ne^4 \ln \Lambda}, \tag{49}$$

where $\ln \Lambda$ is the usual Coulomb logarithm and C is a constant between 1 and 2 which depends on numerical calculation. To account for ion-ion collisions, we merely add a term $\mu_{\perp o} \nabla_\perp^2 V$ to the right-hand side of Equation (36).

This term, after Equation (36) is divided by Mn_o, becomes

$$-k^2 \mu_\perp / Mn_o = -\frac{1}{4} k^2 \theta a_i^2 / \tau = -\frac{1}{4} \theta b \nu_{ii} \equiv -\sigma_\perp . \tag{50}$$

The solution of Equation (36) which replaces Equation (41) is now

$$V_x \frac{n_o'}{n_o} = i\omega * \frac{(\chi + \theta\nu)}{1 - \frac{1}{2}\theta b} \qquad \text{(unchanged)}$$

and (51)

$$V_y = -ka_i^2 (\omega - kV_o + \frac{1}{2}\omega_i + i\sigma_\perp) \frac{(\chi + \theta\nu)}{\left(1 - \frac{1}{2}\theta b\right)^2} .$$

We have assumed $|\omega + i\sigma_\perp|^2 << \Omega_c^2$. The $\sigma_\perp$ term represents an ion flux in the y direction which can short-circuit the fluctuating electric field in that direction if b is large enough; when this happens, the wave is damped out.

Equation (51) is now inserted into the ion equation of continuity (42), and ν is eliminated by virtue of the electron equation (27). Straightforward algebra yields the local dispersion relation

$$(\omega + i\sigma_\perp)(\omega - \omega_i) + i\sigma_\parallel [\omega - \omega^* + b(\omega - \omega_i) + ib\sigma_\perp (1+\theta)] = 0 . \tag{52}$$

In Reference 5 it is pointed out that a simplification of the algebra is achieved by noting that the terms $Mn_o V_o \cdot \nabla V$ and $\nabla \cdot \pi_1$ (collisionless part) in Equation (36) happen to cancel to the order desired. While this is not exactly true, since the term $b(\omega - \omega_i)$ in Equation (52) would be replaced by $b(\omega + \theta\omega)$, there is no practical difference for $\omega \approx \omega^*$. Equation (52) can be simplified in the long-$\lambda_\parallel$ and short-$\lambda_\parallel$ limits.

<u>Long-$\lambda_\parallel$ limit.</u> We assume $b\sigma_\parallel \ll \omega^*$. The two terms in b in Equation (52) can then be neglected compared to the first term in the equation, since we consider $\omega \leqq 0(\omega^*)$ and $\sigma_\perp \leqq 0(\omega^*)$. The dispersion relation becomes

$$(\omega + i\sigma_\perp)(\omega - \omega_i) + i\sigma_\parallel(\omega - \omega^*) = 0 \quad , \tag{53}$$

which clearly reduces to Equation (45) for $\nu_{ii} \to 0$. For that case, we obtain an explicit expression for the growth rate in the $\sigma_\parallel \gg \omega^*$, or short-$\lambda_\parallel$, limit. We can still do this, provided b can satisfy the inequality $\sigma_\parallel \ll \omega^* \ll b\sigma_\parallel$. Solving (53) for $\omega - \omega^*$ and letting $\omega \approx \omega^*$ on the right-hand side, we obtain

$$\omega - \omega^* = -\frac{\sigma_\perp}{\sigma_\parallel}(\omega^* - \omega_i) + \frac{i\omega^*(\omega^* - \omega_i)}{\sigma_\parallel} . \tag{54}$$

In this limit, the growth rate is not affected by $\sigma_\perp$, but the real part of $\omega - \omega^*$ shows that the assumption $\omega \approx \omega^*$ that we made is valid only for $\sigma_\perp \ll \sigma_\parallel$. To obtain the condition for viscous stabilization, we rewrite Equation (53) as follows:

$$\omega^2 - (\omega_i - i\sigma)\omega - i\omega^*(\sigma_\parallel - \theta\sigma_\perp) = 0 \; , \tag{55}$$

where $\sigma \equiv \sigma_\parallel + \sigma_\perp$. If the quantity $\Delta\sigma \equiv \sigma_\parallel - \theta\sigma_\perp$ vanishes, we have a real solution $\omega = 0$ and a damped solution $\omega = \omega_i - i\sigma$. To examine the stability threshold, we should therefore consider $\omega \approx 0$. Assuming $\omega \ll \omega^*$ and neglecting the ω^2 term, we obtain

$$\omega = \frac{i\omega^* \Delta\sigma}{i\sigma - \omega} \quad . \tag{56}$$

for $\sigma << |\omega_i|$, this simplifies to

$$\omega = \frac{\omega^*}{\omega_i^2}\,\sigma\,\Delta\sigma + \frac{i\Delta\sigma}{\theta} \quad . \tag{57}$$

For $\sigma >> |\omega_i|$, we obtain instead

$$\omega = \omega^* \;\frac{\Delta\sigma}{\sigma} + \frac{i\theta\,\omega^{*2}}{\sigma}\;\frac{\Delta\sigma}{\sigma} \quad . \tag{58}$$

In either case, $\sigma_\| = \theta\sigma_\perp$ is a necessary and sufficient condition for marginal stability. As $\sigma_\|$ increases or $\sigma_\perp$ decreases from this value, both Im(ω) and Re(ω) increase from 0. The general solution of Equation (53) for $\sigma_\| >> \omega^* >> b\sigma_\|$, found by straightforward expansion, is

$$\omega = \omega^* \;\frac{\Delta\sigma}{\sigma} + \frac{i(1+\theta)\omega^{*2}}{\sigma}\;\frac{\sigma_\|\Delta\sigma}{\theta\sigma^2} \quad , \tag{59}$$

which reduces to Equation (54), (57), and (58) in the proper limits.

Since $\omega = 0$ at threshold, the phase shift formula (27) reduces to

$$\nu = \chi\frac{\omega^* + ib\sigma_\|}{ib\sigma_\|} \approx \frac{\omega^*}{ib\sigma_\|}\chi \;. \tag{60}$$

This shows that in the long-$\lambda_\|$ limit, n_1 and ϕ_1 are 90^o out of phase at threshold. In the zero viscousity case, n_1 and ϕ_1 were in phase when Im(ω) $\rightarrow$ 0.

Short-$\lambda_\|$ limit. If we keep the b terms in Equation (52), we obtain another stabilization threshold for short $\lambda_\|$. For large $\sigma_\|/\omega^*$, we would expect $\omega \approx \omega^*$; so we let $\omega = (1+\epsilon)\omega^*$ and expand in ϵ to obtain

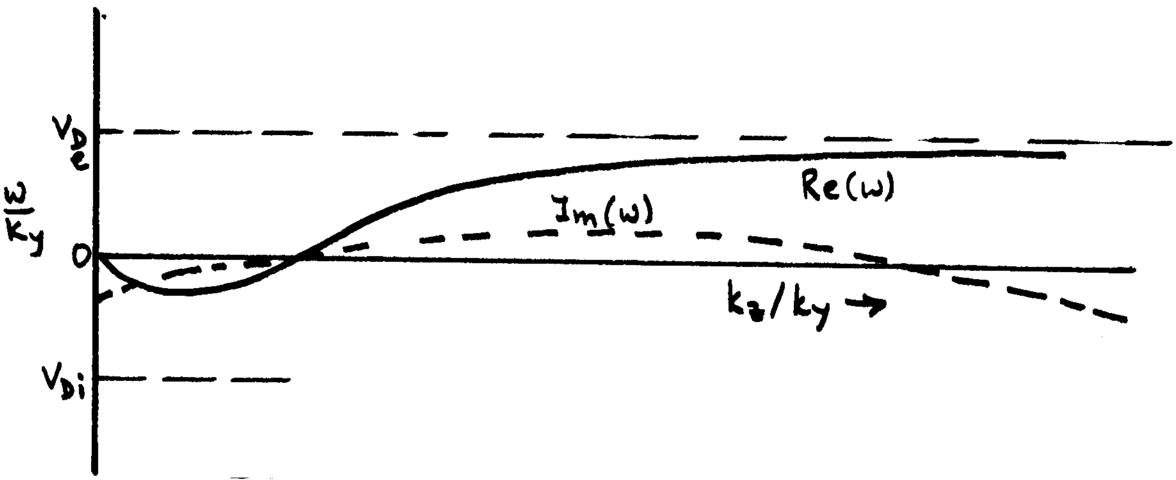
$$-\epsilon\omega^* = (1+\theta)\,\frac{\omega^{*2} - b\sigma_\|\sigma_\perp + i(\sigma_\perp + b\sigma_\|)\omega^*}{i\sigma_\|\,(1+b) + (2+\theta)\omega^* + i\sigma_\perp} \quad . \tag{61}$$

for $\sigma_\| >> \omega^*$, $\sigma_\| >> \sigma_\perp$, this reduces, after some algebra, to

$$\omega - \omega^* = \frac{1+\theta}{\sigma_\|}[-(\sigma_\perp + b\sigma_\|)\omega^* + i(\omega^{*2} - b\sigma_\|\sigma_\perp)]. \tag{62}$$

There is therefore a stability threshold at $b\sigma_\|\sigma_\perp = \omega^{*2}$.

The dispersion diagram with viscosity looks as follows.



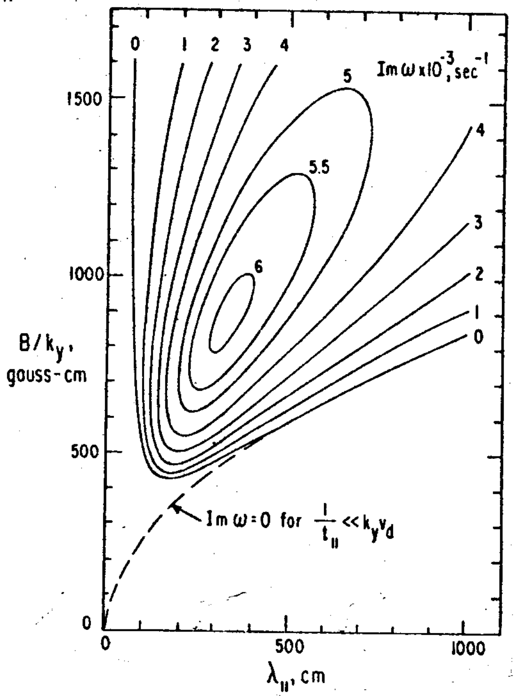
The long-$\lambda_\parallel$ and short-$\lambda_\parallel$ stabilization points can be seen. The long-$\lambda_\parallel$ stabilization criterion, $\theta\sigma > \sigma_\parallel$, with the definitions (26) and (50), works out to be

$$\frac{1}{4}\,\frac{T_i}{T_e}\,\frac{k_y^4}{k_z^2}\,\frac{KT_i}{M\Omega_c^2}\,\frac{m}{M\Omega_c^2}\,\nu_{ii}\nu_{ei} > 1, \tag{63}$$

or

$$B/k_y = \lambda_\parallel \text{ x const.} \tag{64}$$

Contours of constant growth rate for Equation (52) can be displayed on a plot of B/k_y vs. $\lambda_\parallel$.



For given $\lambda_\parallel$, the wave is stabilized at sufficiently small B/k_y. The long-$\lambda_\parallel$ approximation Equation (53) is shown as the dashed line.

The short-$\lambda_\parallel$ stabilization criterion, $b\sigma_\parallel \sigma_\perp > \omega^{*2}$, works out to be

$$\frac{1}{4} b^2 \frac{T_i^2}{T_e^2} \frac{k_z^2}{k_y^2} \omega_c \Omega_c \frac{\nu_{ii}}{\nu_{ei}} > \omega^{*2} . \tag{65}$$

The ratio ν_{ii}/ν_{ei} depends only on the mass ratio. Taking $\nu_{ii}/\nu_{ei} \approx 4(m/M)^{1/2}$ and $T_i = T_e$, we obtain

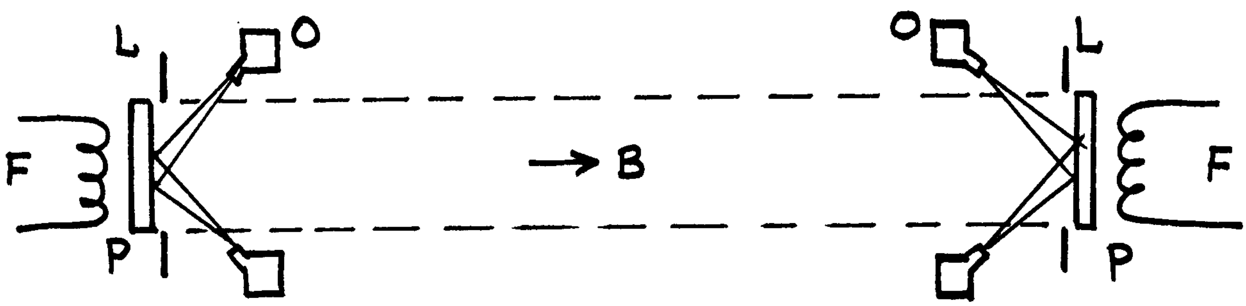
$$k_\parallel^2 > \left(\frac{m}{M}\right)^2 \left(\frac{n'_o}{n_o}\right)^2 . \tag{66}$$

The short-$\lambda_\parallel$ stabilization criterion is independent of B and k_y. This peculiar result has never been observed.

V. DRIFT WAVES IN EXPERIMENT

1. Q-machine Geometry

In a typical alkali-metal plasma, a cylindrical plasma column is confined radially by a uniform magnetic field between 1 and 10 kG. Axially, the plasma is terminated by thermionically emitting metal plates, and one charged species can be confined by the sheath drop at the plates. The other species is not confined and must be continuously replenished. Understanding the operation of the hot plates and of the sheaths there is crucial to experiments in such "quiescent" plasmas.



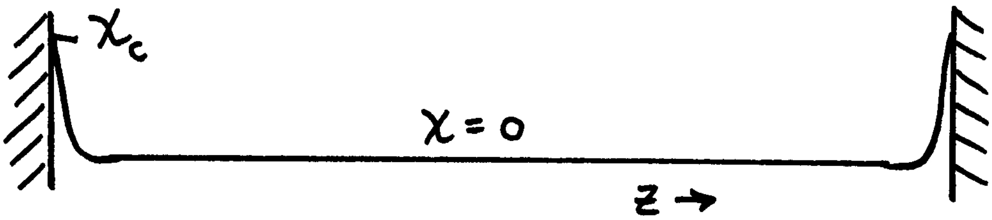
The figure shows an idealized arrangement. The endplates P, which may be tungsten, tantalum, or rhenium, are heated to emission temperatures

(2000-3000 ^{o}K) by electron bombardment from the filaments F. Atomic beam ovens O, operating at 400-800^{o}C, direct a collimated neutral beam of Cs, K, Li, or Na atoms onto the plates. Since alkali metals have a low ionization potential and the endplate has a high work function, there is a high probability (> 95% for the Cs-W combination) that the plate will snatch an electron from the atom and ionize it. A stream of positive ions is then emitted from the plates. To neutralize them, the plates must also supply an equal flux of thermionic electrons. If the plates are so hot that the electron supply is more than sufficient, the plasma potential will be negative, and the sheath drop will be just large enough to reflect the excess of electrons back to the hot plate. This sheath drop will also confine the ions in the plasma. If the electron supply is insufficient, the plasma potential will be positive, and emitted ions will be reflected back to the hot plate. The electrons in the plasma will then be confined axially by a Coulomb barrier. These are called electron-rich and ion-rich operating conditions, respectively.

By controlling the collimation of neutral atoms, the density gradient can be made smooth, as required in careful studies of drift waves. Since stray neutrals are condensed on the cool vacuum chamber walls, the degree of ionization is nearly 100%. The plasma is thermally ionized and is at the same temperature as the endplates ($kT \approx 0.2$ eV). Since no currents are driven through the plasma to ionize it, no instabilities are excited thereby; this allows the weak drift instabilities to be observed. The aperture limiters L are normally used to define the plasma edge and to prevent high energy electrons from the filament F from entering the plasma. Because the limiters are cool, a thermoelectric potential exists between L and P. This causes a local instability called the "edge oscillation". To avoid the complicating effects of this instability, it is best to make the density fall radially to a very small value at the edge. In "single-ended" operation, one of the hot plates is replaced by a cold, negative collector. Experiments in this configuration are more difficult to analyze, because the sheath conditions are not symmetrical, because there is a flow of plasma axially, and because the lifetime of the ions is not long compared to the growth rates of drift waves.

2. Sheath Conditions

In electron-rich operation, the axial potential distribution looks as follows.



Let $\chi = e\phi/KT_e$, and suppose that between the sheaths there is a uniform density n. The random flux of electrons entering the sheath from the plasma is

$$j_r = nV_{the}, \qquad V_{the}^2 \equiv KT_e/2\pi m. \tag{67}$$

In steady state this flux must be replaced exactly by those emitted electrons which penetrate the Coulomb barrier. Let the hot plates emit a half-Maxwellian distribution at temperature T_e, and let the Richardson current at this temperature be j_T. The flux getting into the plasma is then $j_T \exp(-\chi_c)$, and this must equal j_r. In general (nonsteady-state), the flux into the plasma is

$$j_e = j_T \exp(-\chi_c) - nV_{the} \quad . \tag{68}$$

The condition $j_e = 0$ for steady-state prescribes the sheath drop χ_c for given n and T_e. It can be shown (Reference 8) that $f(V_e)$ is Maxwellian everywhere, even in the sheaths, and n_e simply follows the Boltzmann relation. This is because the acceleration of plasma electrons falling toward the hot plate produces a high-velocity stream which exactly compensates for those emitted electrons which get over the Coulomb barrier and cause a deficit of high-velocity electrons reflected by the barrier.

The situation for the ions is slightly more complicated. Ions in the plasma are confined by the sheath, and the flux reaching the hot plate is therefore $nV_{thi} \exp(-\chi_c)$. However, these ions are not all lost, for they have a probability p of being reionized, and only the fraction (1-p) is lost. This is replenished by the inward flux pj_o, where j_o is the flux of neutral atoms directed at the plate. Thus the net ion flux into the plasma is

$$j_i = p j_o - (1-p)nV_{thi} \exp(-\chi_c). \tag{69}$$

If there are no radial or volume losses, the ions also will have a Maxwellian distribution everywhere independent of the collision rate. This is obvious from thermodynamic arguments. However, if there are losses, one must rely on collisions to keep the distributions Maxwellian. At low densities, when the plasma is essentially collisionless, the incoming ions will be accelerated by the sheaths, and the ion distribution will consist of two high-velocity streams. These can cause instabilities. Clearly, for drift wave studies, one would want a large ratio of collision rate to loss rate; this usually requires $n > 10^{10} cm^{-3}$.

3. Axial Boundary Conditions

To apply our theory of drift waves to experiment, we consider the y direction to be the azimuthal direction, so that the waves propagate primarily around the column. In the radial or x direction, the waves tend to be standing waves, with nodes at the axis and at the edge. The x variation is usually neglected, but this approximation is valid only for large m, where m is the azimuthal wave number ($\phi \sim \exp im\theta$). The x dependence has been computed numerically in Reference 6.

In the z direction the boundary condition is complicated by the fact that it is the sheath, not the conducting hot plate, which bounds the column. Since the sheath has a finite conductivity, we cannot simply set $\phi_1=0$ at the sheath edge. Instead, what must be preserved is the continuity of current through the sheath. We need only consider electrons, since V_{zi} is negligible for the frequencies in question. As the wave propagates toward the sheath, the potential of the sheath edge will oscillate. The frequency is low enough that the sheath adjusts adiabatically to this potential change. A current j_e of electrons will be injected into the plasma, according to Equation (68). However, to be consistent with the wave, j_e must equal nV_{ez}, where V_{ez} is the electron fluid velocity in the wave, as given by Equations (23) and (27). Reference 8 shows that this procedure yields the axial boundary condition for ion-rich operation:

$$k_z L \tan k_z L = \frac{L}{a_i}\left(\frac{n_o e\eta}{B}\right)\left(\frac{M}{2\pi m}\right)^{1/2}\left(\frac{j_T}{n_o V_{the}}\right), \qquad (70)$$

where 2L is the distance between the sheaths. The last factor is to be replaced by 1 for electron-rich operation.

Equation (70) shows that when the right-hand side is large, as in electron-rich operation, $k_z L \simeq \pi/2$, and the wave is a standing wave in the z direction with a node at each end. The sheaths act as perfect conductors in this case. When the right-hand side is small, as in ion-rich operation, $k_z L < \pi/2$. The wave is still a standing wave with a maximum at the mid plane, but the oscillation amplitude at the sheath edge is now finite, corresponding to an axial wavelength greater than 4L. The effect of the hot plates is to fix k_z in this manner. Equation (70) has been verified experimentally (Reference 10).

4. Doppler Shifts and Phase Shifts

The dispersion formulas we have worked out are valid in the frame in which $E_o = 0$. In Q-machines there is usually a radial electric field in the equilibrium. If this field E_r is proportional to r, the plasma rotates as a solid body, and the wave frequencies will be Doppler shifted by this rotation. In addition to this effect, there will be a centrifugal force caused by the rotation, and a "gravitational" instability may arise; in Q-machines the centrifugal effect is usually negligible.

That Q-machines have an E_r can be seen from Equation (68) when j_e=0. Since n_o is a function of r, clearly $\chi_c = e(\phi_c - \phi)/KT_e$ must vary with r. The potential is high where the density is high, so E_r is positive. This is true if T_c, the hot plate temperature, is constant. In practice, T_c also varies with r; and since j_T varies exponentially with T_c, a small temperature gradient in the hot plate can cause an appreciable variation in j_T and hence in $\phi(r)$. Usually, T_c falls with radius; in that case, E_r is decreased from the constant-T_c value, and may even be reversed. Most experiments are performed in this last condition.

If care is taken to make T_c constant, then it can be found from Equation (68) that the E_r/B Doppler shift is equal and opposite to V_{De}. The measured drift wave frequency will then be small compared to ω^*. Being the difference of two large numbers, ω^* and kE_r/B, the measured frequency can be compared with theory only if there are accurate measurements of both $n_o(r)$ and $\phi_o(r)$.

When E_o is not 0, Equation (27) relating density to potential is changed to

$$\nu = \chi \frac{\omega^* + ib\sigma_\parallel}{\omega - \omega_E + ib\sigma_\parallel} , \qquad (71)$$

where

$$\omega_E \equiv - (m/r)(E_r/B).$$

In general, ω^*, ω_E, and $\sigma_\parallel$ can all vary with r; but Equation (71) will still be valid locally, regardless of FLR and viscosity effects (which do not affect the electron equations) and of the x dependence of ν and χ (which cancels out in this equation).

When ω_E is not constant, the phase and amplitude difference between ν and χ will vary with r. Furthermore, there will be Kelvin-Helmholtz effects due to the zero-order shear in velocity. These will be discussed in the next section.

The phase angle δ by which ν leads χ has the following significance. Drift instabilities can give rise to "anomalous transport" of plasma across magnetic fields. The radial flux is the time average $\langle n_1 v_r \rangle$, where

$$n_1 v_r = n_1 \frac{E_\theta}{B} = - \frac{im}{r} \frac{n_1 \phi_1}{B} . \tag{72}$$

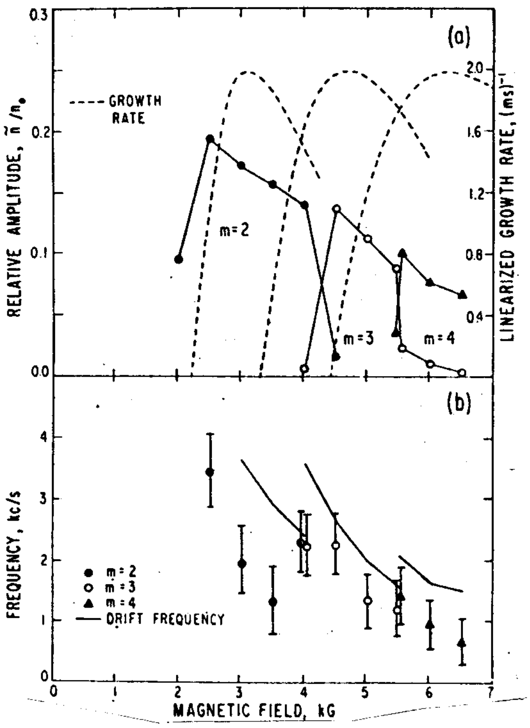
When $b\sigma_\parallel$ is either very large or very small, ω is real, and ν and χ are nearly in phase. Then n_1 and V_r are 90° out of phase, and $\langle n_1 v_r \rangle \approx 0$. The amount of anomalous loss thus depends on the phase shift between ν and χ. The phase shift is also proportional to the growth rate (Reference 4B). Separating Equation (71) into real and imaginary parts, we obtain

$$\text{sgn}\omega \tan \delta = \frac{b\sigma_\parallel [\omega^* - (\omega - \omega_E)]}{(b\sigma_\parallel)^2 + \omega^*(\omega - \omega_E)} . \tag{73}$$

This shows that δ, and hence $\langle n_1 v_r \rangle$, depends only on ω in the E=0 frame, as it should. The sgnω factor takes into account the fact that δ may be reversed in the lab frame by a sufficiently large ω_E . When ω_E varies with r, Equation (73) shows that δ, and hence $\langle n_1 v_r \rangle$, must also vary with r.

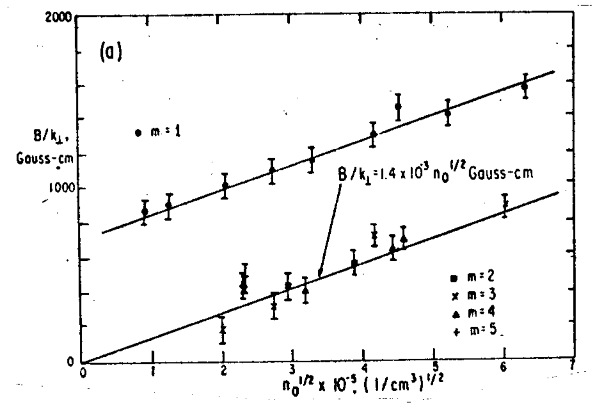
5. Observations of Resistive Drift Waves

The most complete study of drift waves was made by Hendel, Chu, and Politzer (Reference 5) whose Q-machine data we shall show. By working at high densities ($10^{11} < n < 10^{12}$ cm^{-3}), they were able to suppress all large-m modes ($m=rk_y$) by ion viscosity. Thus one steady, sinusoidal wave at a time could be observed in detail. After making a local correction for ω_E, it was found that $\omega \approx \frac{1}{2} \omega^*$ in the E=0 frame, which agrees roughly with the frequency expected at maximum growth rate. No exact check could be made because ω_E varied with radius. The oscillation amplitude was peaked in the region of large n_o'/n_o, as one would expect. The next figure shows the wave amplitude n_1/n_o as a function of B. It is seen that the m=2, 3, and 4 modes occur one at a time as B is increased. From the B/k_y vs. $\lambda_\parallel$ contours shown previously, it is seen that for each B there is a k_y for maximum growth rate. Apparently, it is this mode which grows to saturation at the expense of all the others. The threshold for excitation of each mode is

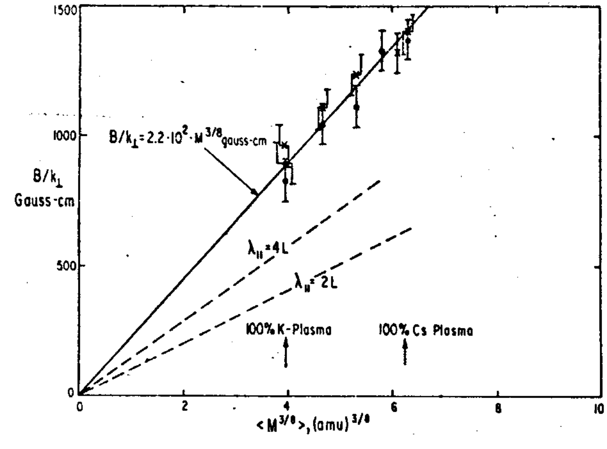

(a)
(b)
GROWTH RATE
RELATIVE AMPLITUDE, ñ/n₀
LINEARIZED GROWTH RATE, (ms)⁻¹
m=2
m=3
m=4
FREQUENCY, kc/s
m=2
m=3
m=4
DRIFT FREQUENCY
MAGNETIC FIELD, kG

given by the condition $\sigma_\perp = \sigma_\parallel$ $(\theta=1)$, which is given in Equation (63). The measured thresholds agree with Equation (63), but only if B is scaled down by a factor 1.5. From Equation (63) it is seen that ν_{ii} scales as B^4; hence the measured thresholds correspond to a value of ν_{ii} which is $(1.5)^4 = 5$ times what it should be. Recent theoretical work taking fluctuations in T_e into account has removed part of this discrepancy.

From Equations (11) and (49) we see that ν_{ii} and ν_{ei} are proportional to n. Equation (63) then shows that B/k_y scales as $n_o^{1/2}$. This dependence is verified in the next figure.



From Equations (49) and (63), one sees also that $B/k_\perp$ scales as $M^{3/8}$. By injecting Cs atoms at one end and K atoms at the other, it was possible to vary M from 39 to 132. The results shown below are consistent with the $M^{3/8}$ dependence, although the constant of proportionality is off by a factor of 2.



VI. OTHER RELATED INSTABILITIES

1. Collisionless Drift Wave

In the absence of collisions a truly universal instability is predicted to occur. Here the impedance to electron flow along B is caused by the slowing down, by the wave, of electrons moving slightly faster than the wave. Since a distortion of the distribution function is involved, a kinetic theory derivation is required. We give the local dispersion relation in a form in which the effects of FLR can be separated out from the effects of ion inertia by setting $\omega_i=0$:

$$\omega - \omega^* + b(\omega - \omega_i) + i\sqrt{\pi}\,\frac{\omega}{k_z V_{the}}\,(\omega - \omega^*) = 0\,. \qquad (74)$$

For $\omega \approx \omega^*$ and $\omega/k_z \ll V_{the}$, we have

$$\mathrm{Im}(\omega) \approx \sqrt{\pi}\,(1+\theta)b\,\frac{\omega^{*2}}{k_z V_{the}}\,, \quad (\theta = T_i/T_e) \qquad (75)$$

This is a very weak instability requiring very large $\lambda_{\parallel}$. Whether or not it has been identified in the laboratory is debatable, but Politzer (Reference 12) claims to have seen it.

Kinetic theory also gives results for arbitrary $b \equiv k_y^2 a_i^2$. Defining $\beta \equiv e^{-b} I_o(b)$, where I_o is a Bessel function, we then have (for $T_i = T_e$)

$$\mathrm{Re}(\omega) = \omega^* \frac{\beta}{2-\beta}\,, \quad \mathrm{Im}(\omega) = 2\sqrt{\pi}\,\frac{\omega^{*2}}{k_z V_{the}}\,\frac{\beta(1-\beta)}{(2-\beta)^2}\,. \qquad (76)$$

2. Kinetic g-mode

The resonant electron effect mentioned above can also destabilize a gravitational instability even if k_z is required to be finite. This instability has been calculated by Yoshikawa.

3. Centrifugal Instability

Reference 6 given the results for resistive drift modes when plasma rotation (due to E_r) adds a centrifugal "gravitational" force. It is found that

$E_r > 0$ is more destabilizing than $E_r < 0$ due to an asymmetry caused by the Coriolis force.

4. Temperature Gradients (References 1 and 11)

Since it is the radial pressure gradient, rather than density gradient, which supplies the free energy to drive a drift mode, one would expect that a gradient ∇T in the equilibrium would be destabilizing. The results for ∇T_i are complicated, and we quote here only the results for ∇T_e. If we define

$$\xi = \frac{d \ln T_{eo}}{d \ln n_o} , \tag{77}$$

Equation (74) for the collisionless drift wave becomes

$$\omega - \omega^* + b(\omega - \omega_i) + \frac{i\sqrt{\pi}}{k_z V_{the}} \omega\left(\omega - \omega^* + \frac{1}{2}\omega^* \xi\right) = 0 \tag{78}$$

From this it can be shown that ξ is destabilizing only if it lies outside the range $0 < \xi < 2$.

Resistive drift waves with finite ∇T_e and finite heat conductivity κ have been calculated by Moiseev and Sagdeev (Reference 3). If we define

$$\omega_T \equiv k \frac{KT_e}{eB} \frac{T'_e}{T_e} , \tag{79}$$

Equation (45) is replaced by

$$\omega(\omega - \omega_i) + i\sigma_{\parallel} \left(\omega - \omega^* - \frac{(1+\alpha)\omega_T}{1 - \frac{2}{3}\frac{ik_z^2 \kappa}{\omega n_o}} \right) . \tag{80}$$

Here $\alpha = 0.71$ is a term arising from a thermal force proportional to $\partial T_e / \partial z$ For small κ, it can easily be seen from Equation (80) that T'_{eo} is destabilizing if $d \ln T_{eo} / d \ln n_o < 0$, as in the collisionless case.

5. Parallel Drifts

If the electron and ion fluids have a relative drift along B in equilibrium, the energy in this drift can be coupled to a drift wave if V_{zi} is taken into account. There is then an enhanced growth rate. In the collisionless case, if $\omega/k_z V_{thi}$ is large enough for ion Landau damping to be neglected, the dispersion relation becomes

$$\omega-\omega^* - \frac{k_z^2 v_s^2}{\omega^2}(\omega-\omega_i) + \frac{i\sqrt{\pi}}{k_z V_{the}}\,\omega(\omega-\omega^*+k_z V_{oz}) = 0\ , \qquad (81)$$

where V_{oz} is the ion drift in the frame in which the electrons are at rest in equilibrium.

For the resistive drift case, the dispersion relation becomes

$$(\bar{\omega}-\omega)\omega^* + b\omega\bar{\omega} + i b\sigma_{\|}(\bar{\omega}-\omega^*) = 0\ , \qquad (82)$$

where $\bar{\omega} \equiv \omega - k_z V_{oz}$. For $\bar{\omega} \approx \omega^*$, we then have the growth rate

$$\bar{\omega}-\omega^* \approx \frac{i}{\sigma_{\|}}\left(\omega^{*2} - \frac{k_z V_{oz}\omega^*}{b}\right)\ , \qquad (83)$$

which indicates that the traveling wave with $k_z V_{oz} < 0$ is unstable.

6. Drift-cyclotron Instability

If we had not made the low-frequency approximation $\omega^2 \ll \Omega_c^2$, the retained cyclotron terms would have given two additional roots $\omega \approx \pm\Omega_c$. These are electrostatic ion cyclotron waves modified by the density gradient. The real part of ω is given approximately by

$$(\omega-k_y V_{Di})^2 = \Omega_c^2 + \left(1+\frac{T_e}{T_i}\right) k_y^2\,(V_{Di}^2 + V_{thi}^2). \qquad (84)$$

The wave which travels in the electron diamagnetic drift direction is unstable for finite k_z if $k_y < n_o'/n_o$. This drift-cyclotron instability arises when the cyclotron wave has nearly the same velocity as the drift wave; i.e., when $\Omega_c/k_y \approx \omega^*/k_y$. Thus the interaction is strong for $\Omega_c \approx \omega^* = -k_y a_i^2 \Omega_c n_o'/n_o$, or $k_y a_i^2 n_o'/n_o \approx 1$. The small Larmor radius expansion breaks down in this case, and one has to use kinetic theory for the exact growth rate [c.f. Mikhailovsky and Timofeev, ZETΦ 44, 919 (1963)].

These authors found that if electromagnetic effects are included ($B_1 \neq 0$), there is an instability even if $k_z = 0$. The unstable condition is

$$a_i \frac{n_o'}{n_o} > 2 \left(\frac{m}{M} + \frac{V_A^2}{c^2} \right)^{1/2} , \qquad (85)$$

where $V_A = cB_o/(4\pi nM)^{1/2}$ is the Alfvén velocity.

Experimentally, drift-cyclotron instabilities have been detected and are important only in magnetic mirrors and multipoles. In these devices the magnetic field is highly nonuniform, and the wave is greatly affected by the ∇B drifts.

7. Drift-Alfvén Waves

As mentioned at the beginning, finite-β effects can hinder electron motion along B via the inductance effect, and this can lead to an instability of Alfvén waves in an inhomogeneous plasma. The coupling between drift waves and Alfvén waves occurs for $\omega^*/k_z \approx V_A$.

8. Trapped Particle Modes

Recent work on multipoles has revealed still another mechanism for impeding electron motion along B. This occurs when B_z varies along z, so that some electrons are trapped in the regions of low B by the magnetic mirror effect. Drift-type instabilities can then arise even in the absence of collisions. However, many other modes, some with $k_z = 0$, also arise; and the situation is too complicated for us to summarize here.

9. Kelvin-Helmholtz Instabilities

For ordinary hydrodynamics, a shear flow can excite a Kelvin-Helmholtz instability. The same phenomenon occurs in a plasma, but the physical mechanisms are quite different for shear in V_{oz} and $V_{o\perp}$. If V_{oz}

varies in the x direction, it is a simple matter to include V'_{oz} in our equations to obtain, for $\eta=0$,

$$\omega = k_z V_{oz} + \frac{1}{2}(\omega^* + \omega_i) \pm \left[\frac{1}{4}(\omega^* - \omega_i)^2 + (1+\theta)\frac{KT_e}{M} k_z\left(k_z - k_y \frac{V'_o}{\Omega_c}\right)\right]^{1/2} . \quad (86)$$

In this instability, the density gradient is a stabilizing effect; by neglecting η, we neglected the destabilizing effect of n'_o.

If it is V_{oy} which has shear, so that $V_o = V_{oy}(x)\hat{y}$, we have the case of nonuniform E_o. In this problem, the radial or x variation of the wave is essential, and one cannot make the local approximation. For $k_z=0$, the radial wave equation is

$$(T\psi')' - k^2(T - g\rho')\psi = 0 , \quad (87)$$

where

$$T = \rho(\omega - kV_E)(\omega - kV_E - \omega_i) ,$$

and $\rho=nM$, $V_E=E_{ox}(x)/B_o$, g is a gravitational potential, and $\psi=V_x/(\omega-kV_E)$ is the Lagrangian displacement of the ion fluid. If FLR effects are neglected by setting $\omega_i=0$, the zeroes in T would coalesce, and we would have the same equations as in classical hydrodynamics. Even in this case, Equation (87) can be solved analytically only for very simple functions $\rho(x)$.

For $k_z \neq 0$ and $\eta \neq 0$, we can combine this result [due to Rosenbluth and Simon, Phys. Fluids 8, 1300 (1965)] with the electron continuity Equation (27) to obtain

$$(T\psi')' + k^2[-T + g\rho' + i\sigma_{\parallel}\rho(\omega^* - \omega + kV_E)]\psi = 0. \quad (88)$$

An equation similar to this but including also cylindrical geometry and ion viscosity effects has been analyzed recently by Perkins and Jassby (Reference 15).

The first identification of the transverse Kelvin-Helmholtz instability in a plasma was made by Kent et al. (Reference 14) in explaining the "edge oscillation" in a Q-machine. A more careful experiment by Jassby (Reference 15) has confirmed this identification.

VII. FEEDBACK STABILIZATION

Drift instabilities can be cured by supplying shear or minimum-B geometry to the magnetic field. Recent work has been concerned with a simpler method: feedback stabilization. The wave is detected by a probe or by an optical method, and the signal is amplified and applied to the plasma locally by one of several methods. If the applied signal bears the right phase relationship to the detected signal and is sufficiently large, the drift wave can be suppressed.

If a probe is used to apply the feedback signal, the probe current can be used to aid electron flow along B and cancel the resistive drag. An oscillating source of electrons can be added to the electron equation of continuity, and the phase and amplitude for stabilization calculated (References 16 and 18). This has been verified experimentally (Reference 18).

If a beam of microwaves is used to apply the feedback signal, the interior of the plasma can be reached without an electrode because the microwaves interact with the plasma only at the place where their frequency equals the upper hybrid frequency. This has been done experimentally by Hendel et al., (Reference 19) and Wong et al., (Reference 20). Stabilization of the drift waves occurs through a nonlinear effect.

A beam of neutral atoms can also be used to reach the interior of a high temperature plasma (Reference 17). The plasma must be dense enough to ionize the neutrals by electron impact. There are then two different mechanisms for applying the feedback signal. First, if the beam is modulated so that it is ionized at the density minima, then n_1/n_o can be decreased. By Equation (27), if $\nu \rightarrow 0$, so does χ. Second, as the beam is ionized, it also imparts an impulse to the plasma because of its initial momentum. This impulse can be phased in such a way that the plasma is pushed back whenever it oscillates outwards. This theory cannot easily be checked experimentally without a thermonuclear-type plasma.

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