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Cross-field diffusion in low-temperature plasma discharges of finite length

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ABSTRACT

The long-standing problem of plasma diffusion across a magnetic field (B-field) is reviewed, with emphasis on low-temperature linear devices of finite length with the magnetic field aligned along an axis of symmetry. In these partially ionized plasmas, cross-field transport is dominated by ion-neutral collisions and can be treated simply with fluid equations. Nonetheless, electron confinement is complicated by sheath effects at the endplates, and these must be accounted for to get agreement with experiment.

I. Classical diffusion

After almost one century of developments in plasma research, the diffusion of plasma across a magnetic field still remains a fundamental problem of interest in many applications. In the low-temperature plasmas typically used in the industry, like magnetrons or helicons, cross-field diffusion determines the transport phenomena and the properties of the discharge. In the high-temperature magnetically-confined plasmas used for thermonuclear fusion, the cross-field diffusion regulates the fluxes from the hot core to the colder scrape-off layer at the edge, and consequently the fluxes to the plasma facing components, critical for the design of the machine. Interestingly, the same problem is of concern in both low-temperature and high-temperature plasmas, but for different reasons. Being of such large interest, the topic has attracted considerable attention all along the history of plasma physics. The literature available is so extensive that this review will be necessarily limited in scope. It is the aim of the present paper to review the major advancements on both theory and experiment, focusing attention on cross-field diffusion in partially ionized plasmas of finite length.

In fully ionized plasmas, "classical" diffusion arises from electron-ion Coulomb collisions with an arbitrary cut-off. In this case, diffusion is so slow that other effects, such as "Bohm diffusion", arise from instabilities. Theories rarely agree with measured loss rates until nonlinear saturation of instabilities and the final turbulent state are accounted for. In fusion research, diffusion in toroidal devices is further complicated by the magnetic geometry, which spawns banana orbits and magnetic islands. Progress in understanding cross-field diffusion was made possible by the construction of linear machines with uniform B-fields. Such devices include, for instance, weakly ionized plasmas such as helicon discharges, and fully ionized Q-machines, with low electron temperatures KT_e of about 3 eV, and 0.21 eV, respectively. The price one pays for such simple B-fields is that there are sheaths on the endplates, and these have to be treated properly, as is done in the paper.

II. Partially ionized plasmas, fluid perspective

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Cross-field diffusion in cold-plasma theory follows the equation of motion of ions in steady state[1,2,3]:

$$Mn(\mathbf{v}\cdot\nabla)\,\mathbf{v} = Zen(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - \nabla p_i - Mn\mathbf{v}\nu\,, \tag{1}$$

where M is the ion mass, $\mathbf{v}$ the ion fluid velocity, n the plasma density, p_i the ion pressure nKT_i, ν the ions' collision frequency, which is dominated by charge-exchange collisions with neutrals, and the other symbols are obvious. For simplicity, we have omitted the off-diagonal elements of the stress tensor, which contain the viscosity. A similar equation describes the electron fluid, but we shall find that electrons rarely follow that equation; rather, they tend to fall into a Maxwellian distribution and obey the Boltzmann relation. The anomalous mechanism of electron scattering and the surprising Maxwellization of the electron distribution in bounded domains have been observed even at very low collisionalities[4] and interpreted as plasma-boundary oscillations[5], non-local effect of electron kinetics[6], or convective ion-acoustic instabilities[7] near the discharge boundary. The nonlinear convection term $\mathbf{v}\cdot\nabla\mathbf{v}$ will play an important rôle later but can be neglected for now. In the direction of $\mathbf{B}$, or if B = 0, we can define the usual diffusion and mobility coefficients by solving for $\mathbf{v}$ with Z = 1:

$$\mathbf{v} = \frac{1}{Mn\nu}\left(en\mathbf{E} - KT\nabla n\right) \;=\; \frac{e}{M\nu}\mathbf{E} - \frac{KT}{M\nu}\frac{\nabla n}{n}\,. \tag{2}$$

The coefficient of $\mathbf{E}$ is the ion mobility μ_i, and the coefficient of $\nabla n/n$ is the ion diffusion coefficient D_i. Similar definitions obtain for electrons, with μ_e positive.

$$\begin{aligned} D_i &\equiv KT_i / M\nu_i\,, \quad \mu_i \equiv e / M\nu_i \\ D_e &\equiv KT_e / m\nu_e\,, \quad \mu_e \equiv e / m\nu_e \end{aligned} \tag{3}$$

Ambipolar diffusion occurs when both species have the same flux $nv_{i,e}$:

$$\mu_i n\mathbf{E} - D_i\nabla n = -\mu_e n\mathbf{E} - D_e\nabla n\,. \tag{4}$$

In order for this to happen, there must be an electric field $\mathbf{E}$ given by

$$\mathbf{E} = \frac{D_i - D_e}{\mu_i + \mu_e}\frac{\nabla n}{n}\,. \tag{5}$$

When this E-field is inserted into Eq. (4), the common ambipolar diffusion coefficient D_a is found to be

$$D_a \equiv \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}\,. \tag{6}$$

This result applies only to diffusion *along* $\mathbf{B}$, or when $\mathbf{B}$ = 0. For diffusion *across* $\mathbf{B}$, we must use the perpendicular components of Eq. (1). Again with the nonlinear term neglected, these are, for either species,

$$v_x = \pm\frac{eE_x}{m\nu} - \frac{KT}{m\nu}\frac{1}{n}\frac{\partial n}{\partial x} \pm \frac{eB}{m\nu}v_y = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} \pm \frac{\omega_c}{\nu}v_y$$
$$v_y = \pm\frac{eE_y}{m\nu} - \frac{KT}{m\nu}\frac{1}{n}\frac{\partial n}{\partial y} \mp \frac{eB}{m\nu}v_x = \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} \mp \frac{\omega_c}{\nu}v_x \tag{7}$$

where ± stands for the sign of the charge, and where we have introduced the cyclotron frequency $\omega_c \equiv eB/m$, defined as positive for either species. It is customary now to replace ν with its reciprocal $1/\tau$ to obtain the familiar factor $\omega_c\tau$, whose size tells whether collisions destroy the cyclotron orbits or not. Simultaneous solution of Eqs. (7) for v_x and v_y yields

$$v_y\left(1+\omega_c^2\tau^2\right) = \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \omega_c^2\tau^2\frac{E_x}{B} \pm \omega_c^2\tau^2\frac{KT}{eB}\frac{1}{n}\frac{\partial n}{\partial x}$$
$$v_x\left(1+\omega_c^2\tau^2\right) = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \omega_c^2\tau^2\frac{E_y}{B} \mp \omega_c^2\tau^2\frac{KT}{eB}\frac{1}{n}\frac{\partial n}{\partial y} \tag{8}$$

The first two terms of each equation show that the mobility and diffusion coefficients for transport perpendicular to **B** are reduced by the magnetic field:

$$\mu_\perp = \frac{\mu}{1+\omega_c^2\tau^2}, \quad D_\perp = \frac{D}{1+\omega_c^2\tau^2} \quad . \tag{9}$$

Electrons normally have large ω_c's and diffuse slowly across **B**. The last two terms of each equation (8) are the **E** × **B** and diamagnetic drifts, $\mathbf{v}_E$ and $\mathbf{v}_D$, reduced by a collisional factor.

$$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}, \quad \mathbf{v}_D = \pm\frac{\mathbf{B}\times\nabla p}{enB^2} \tag{10}$$

Thus, the cross-field velocity for either species can be written succinctly as

$$\mathbf{v}_\perp = \pm\mu_\perp\mathbf{E} - D_\perp\frac{\nabla n}{n} + \frac{\mathbf{v}_E + \mathbf{v}_D}{1+\left(\nu^2/\omega_c^2\right)} \tag{11}$$

Many classic papers have been written on variations and applications of these results. For instance, there are books by Delcroix[8], Rozhansky and Tsendin[9], Shkarofsky et al.[10], and others. Robson *et al.*[11] have discussed collision cross sections. Fruchtman[12] has extended such fluid theories to ambipolar and non-ambipolar diffusion in both linear and nonlinear regimes. Most theories assume infinite, one-dimensional half-spaces or infinite cylinders, neither of which exists in reality. In a long cylinder uniformly ionized at the edge by, say, radiofrequency (RF) antennas, one would expect that, in a strong B-field, the plasma density would be peaked at the edge because the electrons diffuse so slowly. However, this is rarely observed because finite plasmas have boundaries, and collisionless sheaths are needed there to equalize ion and electron losses and maintain quasineutrality of the plasma.

Particularly important are the sheaths on the surfaces ("endplates") which intersect the field lines of the B-field. Electrons can escape along **B** in nanoseconds unless they are repelled by the Coulomb barrier of a sheath. Ions enter the sheath at the Bohm speed[13], which is equal to the acoustic velocity $c_s = (KT_e / M)^{1/2}$, where M is the ion mass. Electrons enter with their one-dimensional thermal velocity $v_{th} = (2KT_e / \pi m)^{1/2}$ and are repelled by a factor e^{eV/KT_e}, where V is the (negative) sheath drop. Equating electron and ion fluxes to the endplates yields a sheath drop of $-T_{eV}[\frac{1}{2}\ln(M/2\pi m)] \approx -4.7T_{eV}$ in argon (with $T_i << T_e$), where T_{eV} is KT_e in electron-volts. Thus, if KT_e is about 3eV in a gas discharge, there is a sheath drop of about 15eV at the endplates. A similar calculation could be made for losses to the side boundary of a cylinder, but it would be inaccurate because electrons can oscillate at immeasurably high frequency, and the cylinder may not be exactly aligned along **B**. At the endplates, however, an interesting mechanism called the Simon[14] "short-circuit" effect occurs.

To illustrate this effect, consider a cylinder of radius a and length $2L$ such that the ion flux at KT_i to the endplates is slow and can be neglected. Let the ionization source be localized at the radial edge. We wish to show that the density peak is nonetheless located on axis. Figure 1 shows one end of the discharge and the sheath conditions there. Initially, density n is higher in magnetic tube 1 near the edge. The sheath there must be thicker to repel more electrons, and that

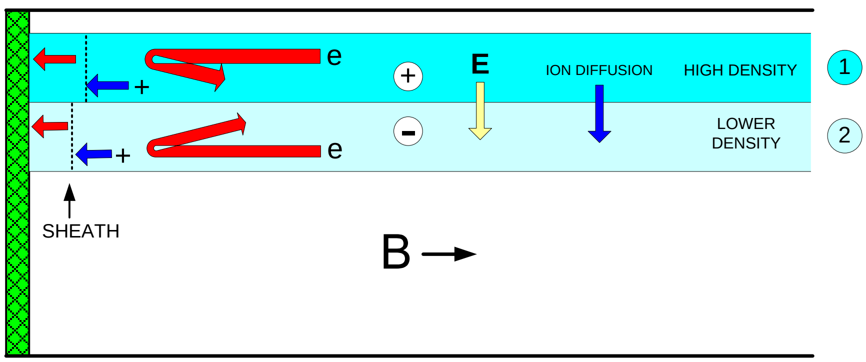

Fig. 1. Initial sheath conditions at an endplate.

generates an inward-pointing E-field which drives ions inwards. By adjustments of the sheath drop, the electron density can follow the ion density without actually moving across **B**. This allows electrons to be Maxwellian and follow the Boltzmann relation everywhere, but at the local temperature. The short-circuit effect cannot equalize KT_e across B-lines. Ultimately, the plasma is quasineutral and peaks on axis, even though ionization is at the edge[15]. By the Boltzmann relation, the plasma potential must also peak on axis, corresponding to an outward-pointing E-field, which drives the loss of ions to the edge. Thus, "classical diffusion" in infinite cylinders is not easy to achieve because of the sheath effects at endplates.

If one assumes that the electrons are Maxwellian everywhere, a consequence is that the plasma density profile $n(r)$ has a "universal" shape independent of the pressure p and discharge radius a. Eq. (1) in its full form is[16]:

$$M\mathbf{v}\nabla\cdot(n\mathbf{v}) + Mn\mathbf{v}\cdot\nabla\mathbf{v} - en\mathbf{E} + Mn\nu_{io}\mathbf{v} = en(\mathbf{v}\times\mathbf{B}) - KT_i\nabla n \approx 0 \tag{12}$$

The first term represents drag due to injection of slow ions by ionization, ν_{io} is the frequency of ion-neutral charge-exchange collisions, and the terms on the right can be neglected. The ion temperature is small; and the $(\mathbf{v}\times\mathbf{B})$ term, which describes ion Larmor orbiting, is small if the ion Larmor radius at the electron temperature is much larger than a. T_e is used here because ions are accelerated by electric fields scaled to T_e. Defining the ionization and collision probabilities as

$$P_i(r) \equiv <\sigma v>_{ion}(r), \quad P_c(r) \equiv <\sigma v>_{cx}(r) = \nu_{io}/n_n, \tag{13}$$

where n_n is the density of neutrals, we can combine Eq. (12) with the equation of continuity

$$\nabla \cdot (n\mathbf{v}) = nn_n P_i(r)\,. \tag{14}$$

to obtain

$$M\mathbf{v}\cdot\nabla\mathbf{v} - e\mathbf{E} + Mn_n(P_i + P_c)\mathbf{v} = 0\,. \tag{15}$$

Ions will move slowly along $B\hat{\mathbf{z}}$ because electron conductivity prevents any large E_z from arising. Thus, we need consider only the r component of Eq. (15). With the usual definitions

$$\mathbf{E} = -\nabla\phi, \quad \eta \equiv -e\phi / KT_e, \quad c_s \equiv (KT_e / M)^{1/2}\,, \tag{16}$$

the radial components of Eqs. (15) and (14) become

$$v\frac{dv}{dr} = c_s^2\frac{d\eta}{dr} - n_n(P_c + P_i)v \tag{17}$$

$$\frac{dv}{dr} + v\frac{d(\ln n)}{dr} + \frac{v}{r} = n_n P_i(r)\,, \tag{18}$$

where we have dropped the subscript from v_r. The local electron Boltzmann relation, given by

$$n = n_0 e^{e\phi/KT_e} = n_0 e^{-\eta}\,, \qquad \frac{d(\ln n)}{dr} = -\frac{d\eta}{dr}\,, \tag{19}$$

can be inserted into Eq. (18) to obtain

$$\frac{dv}{dr} - v\frac{d\eta}{dr} + \frac{v}{r} = n_n P_i(r)\,. \tag{20}$$

Finally, $d\eta/dr$ can be inserted from Eq. (17) to obtain, after a few steps,

$$\frac{dv}{dr} = \frac{c_s^2}{c_s^2 - v^2}\left[-\frac{v}{r} + n_n P_i(r) + \frac{v^2}{c_s^2}n_n(P_i + P_c)\right]. \tag{21}$$

Note that dv/dr becomes infinite when v approaches c_s, leading to a natural transition to the thin, collisionless Debye sheath. The nature of Eq. (21) becomes clear if we introduce the dimensionless quantities

$$u \equiv v / c_s, \quad \rho \equiv (n_n P_i / c_s)r, \text{ and } k(r) \equiv 1 + P_c(r) / P_i(r)\,, \tag{22}$$

obtaining

$$\frac{du}{d\rho} = \frac{1}{1 - u^2}\left[1 + ku^2 - \frac{u}{\rho}\right]. \tag{23}$$

Solution of this equation will yield the radial profiles of n, v, and ϕ. The properties of the plasma are contained in $k(r)$, and these properties are only the ratio P_c/P_i. Eq. (23) has been solved in Ref. 16 together with equations for neutral depletion and input-output balance in each radial shell. Here we show results for uniform T_e and n_n. Figure 2 shows $u(\rho)$ for three values of k. The slope becomes infinite at different values ρ_a, which should be identified with the sheath edge at $r = a$. Rescaling ρ to the same r/a, one obtains the plot of u vs. r/a in Fig. 3, together with corresponding profiles of n/n_0 and η. Since P_c and P_i, and hence k, are independent of n_n, these profiles are independent of pressure and of the numerical value of a. They are "universal" in that sense, but they will vary if KT_e changes.

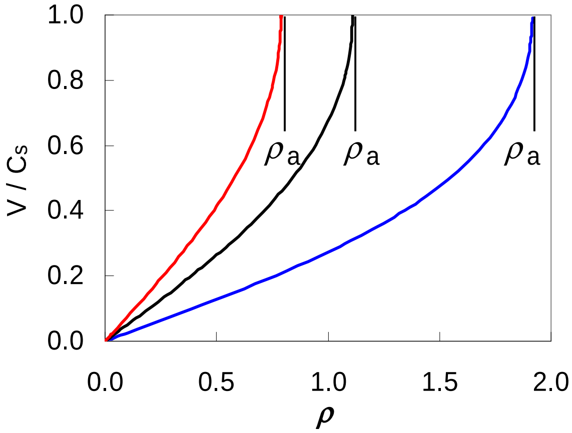

Fig. 2. Solution of Eq. (23) for three values of k.

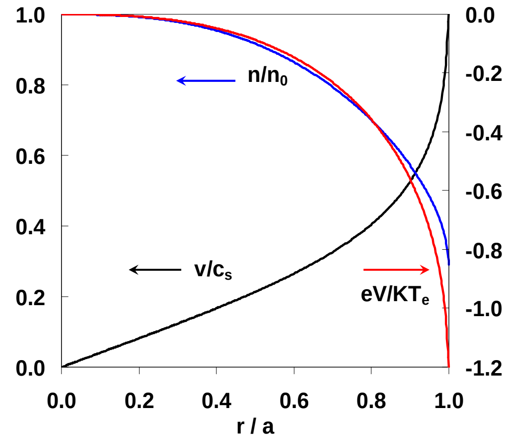

Fig. 3. Renormalized profiles of n, v, and ϕ.

The result is that classical diffusion in discharges of reasonable length is quite different from that in an infinite cylinder. The density, and hence potential, has to be peaked on axis in order to drive the ions out radially. They cannot be driven *toward* the axis in steady-state equilibrium because there is no fast way to escape from there.

Experimentally, the unexpected central peaking of density in edge-ionized plasmas was reported by Evans and Chen[17]. This was explained by the time-averaged Lorentz force on the electrons in the RF field, which caused the electron trajectories to cross through the central region, building up their density there and attracting the ions with the resulting E-field. That mechanism was for an infinite cylinder and would be changed by the presence of endplates.

III. Fully ionized plasmas, fluid perspective

Maintaining a fully ionized plasma in equilibrium normally requires a fusion confinement device in the form of a torus or magnetic mirror. In these machines, unavoidable instabilities enhance diffusion well above the classical rate based on ion-electron collisions. The diffusion rates, originally calculated by L. Spitzer[18], have been summarized by D. Book and published by J.D. Huba[19]. In thermal plasmas with temperatures T_i and T_e and no beams, the classical diffusion coefficient for drifts across **B** is given by[19]

$$D_\perp \approx 10^{-4} Zn \ln\Lambda (KT_e + KT_i) / B^2 \ \Omega - \mathrm{m}\,, \tag{24}$$

where $\ln\Lambda$ is the Coulomb logarithm, a weakly varying number of the order of 10. Note that $D_\perp$ decreases as $1/B^2$, in contrast with Bohm diffusion, which goes at the rate

$$D_B = \frac{1}{16}\frac{KT_e}{eB}\,, \tag{25}$$

which decreases more slowly with B and could be several orders of magnitude larger. In early experiments plasma losses occurred at the faster Bohm rate than the classical rate. Spitzer once tracked down David Bohm in Brazil and asked him how the factor 1/16 was derived. Bohm did not remember!

Though classical diffusion in fully ionized plasmas has not been achieved in fusion research, it has been observed in two experimental inventions: Q-machines[20] and machines with levitated internal conductors, such as octopoles[21,22]. Multipole machines, which have internal ring conductors either levitated magnetically or supported by thin wires, can produce minimum-*B* volumes where |B| increases in all directions. Plasmas created there are stable and can diffuse classically. Q-machines produce plasma by thermal injection. Electrons are emitted from tungsten "cathodes" heated to 2400K at the ends of uniform B-field lines. Atoms of sodium or potassium are injected toward the cathodes, where they are ionized upon contact, since their ionization potentials are smaller than the work function of the tungsten. Thus, a thermal plasma at 2400K is formed without any applied electric fields. Nonetheless, spontaneous oscillations were observed[23] which caused enhanced diffusion[24], as explained below. By applying magnetic shear with a current through a 1-cm diam aluminum tube strung through holes in the hot cathodes, Chen and Mosher[25] quenched the oscillations and brought the diffusion down to within a factor 2 of classical.

IV. Fully ionized plasmas, kinetic perspective

Fluid models of cross-field transport, as those described above, are convenient for rapid but approximate descriptions of macroscopic behavior. However, plasmas exhibit features that cannot entirely be taken into account within the fluid framework. For example, there are cases where the electron distribution function can be driven far from a Maxwellian state. In these cases, a kinetic approach is required.

The full kinetic behavior of N particles (both charged and neutrals) immersed in a magnetic field is formally described by Liouville's theorem of the time evolution of the phase-space distribution function. The Liouville theorem can opportunely be rewritten (Bogolyubov[26], Born and Green[27], Kirkwood[28,29], Yvon[30]) as a hierarchy of integro-differential equations relating the k-particle distribution function f_k to the k+1 particle distribution function f_{k+1}:

$$D_k f_k = C_k\left(f_{k+1}\right) \tag{26}$$

for $k = 1, 2, \ldots, N$-2, where the derivative operator D_k of order k in the phase-space (t,$\mathbf{r}$,$\mathbf{v}$) and the collision operator C_k of order k are

$$D_k = \frac{\partial}{\partial t} + \sum_{i=1}^{k} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} + \sum_{i=1}^{k}\sum_{j=1}^{k} \mathbf{a}_{i,j} \cdot \nabla_{\mathbf{v}_i} \tag{27}$$

$$C_k\left(f_{k+1}\right) = -\frac{N-k}{V}\sum_{i=1}^{k}\int d\mathbf{r}_{k+1} d\mathbf{v}_{k+1} \mathbf{a}_{i,k+1} \cdot \nabla_{\mathbf{v}_i} f_{k+1} \tag{28}$$

Here the volume V is the phase-space volume where the number of particles is evaluated, and the acceleration $\mathbf{a}_{i,j}$ of i relative to j is obtained from the solution of the Maxwell-Lorentz set of equations. Even if the BBGKY hierarchy does not have any more information than Liouville's theorem, it is much more useful. It allows the chain to be cut off at some stage, and to make clear the error which occurs in the cut-off. Typical truncations of the hierarchy are at the order zero, giving the Vlasov equation,

$$\frac{\partial f_0}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f_0 + \mathbf{a} \cdot \nabla_{\mathbf{v}} f_0 = 0 \tag{29}$$

or at the first order, giving the Boltzmann equation:

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}} f_1+\mathbf{a}\cdot\nabla_{\mathbf{v}} f_1=C_1(f_2) \tag{30}$$

The Chapman-Enskog method[31] and Grad's moment method[32] are two powerful analytical tools to find solutions of the Boltzmann equation. The first is based on the expansion of the distribution function in terms of associated Laguerre polynomials[33]. The latter expands the distribution function in series of tensorial Hermite polynomials. In the Boltzmann model, only two-body interactions are included for the evaluation of the collision integral $C_1(f_2)$. The termination of the chain at such low order is however critical. When the distribution function is not too far from a state of statistical equilibrium (Maxwellian state), the error committed in the truncation of the chain is of order g^k, where k is the order of truncation, and $g \propto (n/T_e^3)^{1/2}$ is the inverse of the classical plasma parameter. The parameter g is proportional to the ratio of the average interaction energy between particles to their average kinetic energy. When g is increased, the distribution function departs from a Maxwellian, and kinetic effects become gradually more relevant. At high g's ($g >> 1$) the truncation of the hierarchy at the first order (Boltzmann equation) loses its validity, and higher orders of the hierarchy are required.

V. Partially ionized plasmas, kinetic perspective

The Boltzmann kinetic equation allows evaluating both electron and ion cross-field transport rates in a weakly ionized plasma with considerable high fidelity. Druyvesteyn[34] first derived the electron velocity distribution function in a uniform electric field for the simple case of impacts with velocity-independent cross sections; *c.f.* Smirnov[35]. In weakly ionized conditions, electron-neutral collisions are by far the most frequent collision event: $\nu_{eo} >> \nu_{ei}$. In these conditions, the density of electrons is much smaller than the density of their parent gas, so that the electron-electron collision frequency can be neglected with respect to the electron-neutral collision frequency: $\nu_{ee} << \nu_{eo}$.

Druyvesteyn's result can be generalized to the case of crossed electric *and* magnetic fields by solving the stationary and spatially uniform Boltzmann problem for the case $\mathbf{E} = E_x\,\hat{\mathbf{x}}$ and $\mathbf{B}_0 = B_z\,\hat{\mathbf{z}}$. Expanding the distribution function into three terms (spherical contribution, electric field contribution, and $\mathbf{E}\times\mathbf{B}$ drift),

$$f(v)=f_0(v)+v_x f_1(v)+v_y f_2(v) \tag{31}$$

and expressing the collision integral as

$$C(f)=C(f_0)-\nu_{eo}v_x f_1-\nu_{eo}v_y f_2\,, \tag{32}$$

where the collision integral $C(f_0)$ for elastic collisions of electrons with the background neutral gas at temperature T_o is obtained by solving the Fokker-Planck equation, we have

$$C(f_0)=\frac{m}{M}\frac{KT_o}{v^2}\frac{\partial}{\partial v}\left[v^3\nu_{eo}\left(\frac{f_0}{KT_o}+\frac{1}{mv}\frac{\partial f_0}{\partial v}\right)\right], \tag{33}$$

Substituting Eqs. (31), (32), and (33) into the Vlasov equation (29), the following solution is found for the electron velocity distribution function:

$$f_0(v) = a_o \exp\left[-\int_0^v \frac{mv}{KT_o + e^2 M E_x^2 / \left(3m^2 (\nu_{eo}^2 + \omega_{ce}^2)\right)} dv \right] \tag{34}$$

$$f_1(v) = -\left(\frac{eE_x \nu_{eo}}{\nu_{eo}^2 + \omega_{ce}^2} \right) \frac{1}{mv} \frac{\partial f_0}{\partial v} \tag{35}$$

$$f_2(v) = -\frac{\omega_{ce}}{\nu_{eo}} f_1(v) \tag{36}$$

where the normalization constant a_0 is a function of the density. When the magnetic field is zero ($\omega_{ce} \to 0$), the solution (34)-(36) returns the classical Druyvesteyn distribution in absence of magnetic field, with $f_2(v)=0$. Integrating Eqs. (34)-(36) over the spherically symmetric part $f_0(v)$ of the distribution function, the drift velocity of electrons in direction perpendicular to the B-field can be derived:

$$u_{e,x} = \frac{eE_x}{3m} \int \frac{1}{v^2} \frac{d}{dv} \left(\frac{\nu_{eo} v^3}{\nu_{eo}^2 + \omega_{ce}^2} \right) f_0(v) dv \tag{37}$$

$$u_{e,y} = -\frac{eE_x \omega_{ce}}{3m} \int \frac{1}{v^2} \frac{d}{dv} \left(\frac{v^3}{\nu_{eo}^2 + \omega_{ce}^2} \right) dv \tag{38}$$

Eqs. (37) and (38) are the solution of electrons cross-field diffusion in weakly ionized conditions. The two components x and y are both in the direction perpendicular to the magnetic field, with x being along the electric field and y along the $\mathbf{E} \times \mathbf{B}$ direction. The limiting case of Eqs. (37) and (38) for strong magnetic fields $\omega_{ce} \to \infty$ gives:

$$u_{e,x} << u_{e,y}, \qquad u_{e,y} = -eE_x / m\omega_{ce} . \tag{39}$$

Eq. (39) shows that at strong magnetizations the main drift motion is along the $\mathbf{E} \times \mathbf{B}$ direction and is independent of the collision frequency. A similar procedure can be extended also to weakly ionized ions[36]. Dedicated treatments of ion-neutral collisions and of charge-exchange processes are necessary in this case. The equal mass of ions and neutrals involve significant momentum transfer among the two species. Charge-exchange usually determines a significant deceleration of the ion fluid, causing the so-called charge-exchange drag. The concurrent effect of charge-exchange deceleration and ionization (new ions generated at rest) can be seen at the macroscopic fluid level from the parameter $k(r)$ in Eq. (22). Kinetic calculations including finite-size effects[37,38] show the possibility of controlling the discharge parameters by modifying the short-circuiting conditions at the plasma wall. An extensive review giving experimental evidence of the relevance of the short-circuiting conditions is given by Zhilinskii and Tsendin[39].

VI. Anomalous diffusion

In actual devices, the electron cross-field diffusion is non-classical. Anomalous electron diffusion has been observed in a broad variety of plasmas with a wide range of ionization fractions. Studies on anomalous diffusion are related to the instabilities occurring in a plasma.

In normal conditions, any plasma with a density gradient, which includes any confined plasma, suffers from a universal instability discovered by Sagdeev[40] and Chen[41] called a resistive drift wave instability. A Langmuir probe inserted into a Q-machine would show a turbulent spectrum of electrostatic oscillations which caused anomalously rapid diffusion across **B**. By applying a voltage on an aperture limiter, the instability could be brought near threshold so that it could be seen as a sine wave oscillation[42]. In a cylinder, a drift wave propagates azimuthally in the electron diamagnetic drift direction and has a long wavelength in the direction of **B**. Its azimuthal electric field causes an oscillating radial drift of both ions and electrons. When the wave is growing, the radial drift is out of phase with the density perturbation, such that the drift is outward when the density is high and inward when the density is low. Thus, there is a net transport of plasma toward the boundary. In the nonlinear state, the plasma can be envisioned as escaping in "blobs"[43], and these have been observed in fusion devices[44].

Anomalous diffusion was initially observed in elongated partially-ionized plasma tubes. Hoh and Lehnert[45] did a famous series of measurements using electropositive gases, confirmed also by Allen et al.[46], showing that the agreement between experiment and classical diffusion theory was valid only up to a certain critical magnetic field. At magnetic fields higher than the critical value, a much higher diffusion was present. Above the critical magnetic field, the discharge loses its axial symmetry, and exhibits regular oscillations appearing as rotating helical structures[47]. In their experiment they used tubes long enough (length bigger than 50 times the diameter) so that any short-circuit effect as described by Simon[14] was suppressed. Hoh advanced the idea that above the critical magnetic field, a wall sheath instability might occur. Kadomtsev and Nedospasov[47] explained the phenomenon from a different standpoint. They studied harmonic perturbations of the form $\exp(im\varphi + ikz - i\omega t)$ in a diffusion model including fluid magnetized electrons. Their dispersion relation led to the calculation of the correct analytical value of the critical magnetic field, to the correct description of the onset of anomalous diffusion, and to the calculation of the frequency of oscillation of the unstable mode. Their model was in successful agreement with the measurements done by Hoh and Lehnert. The limitations of using a hydromagnetic fluid model in low-density collisionless plasmas were described later by Schmidt[48], who used a "self-consistent" guiding-center model to describe the plasma motion across the magnetic field. His approach gave more detailed information on plasma behavior (e.g. the discussion of a polarization layer), but no comparison with experiments was given.

Despite the success of the Hoh-Lehnert experiments and of the Kadomtsev-Nedospasov theory, the measurements were still controversial. Other authors[49,50] reported diffusion in agreement with classical rates ($D \sim 1/B^2$) in shorter cylindrical discharges at high levels of ionization even in strong magnetic fields. The $1/B$ "Bohm diffusion" was interpreted by Bryan Taylor[51] as the maximum value which the transverse diffusion can ever attain. At that time Taylor was investigating the correlation function of the fluctuating electric field in a plasma[52] as a means for the calculation of the mean force on a slowly moving test charge, and ultimately obtain the transport coefficients. Taylor applied his method to the problem of diffusion of ions across a magnetic field[53]. He found that ion diffusion was driven by three contributions, two of which arose from the mean force on the ion, and the third one from the fluctuations. For the case of different ion and electron temperatures, $T_i \neq T_e$, Taylor's method suggested results substantially different than that of the classical Chapman-Enskog expansion. Guest and Simon[54] extended the Kadomtsev-Nedospasov model to the cases of B-fields higher than a critical threshold or of background pressures below a critical value. According to their interpretation, in a magnetized plasma column at low pressure the different azimuthal streams of ions and electrons are the main driving force of the instability. The difference in the streams generates an oscillating electric field E_θ in the azimuthal direction that in turn increases radial diffusion via the $E_\theta \times B$ drift. The frequency of the oscillating azimuthal E_θ field was approximately equal to $\omega_{rot} \approx \pi/(4L)\, u_i (1 + T_e/T_i)$, where L is the discharge length and u_i is the average ion velocity to the end walls. Despite the evident role of the plasma angular momentum, no explicit calculation of this quantity was done. Further experiments using cesium plasmas and comparison with calculations supported the transition from classical to anomalous diffusion, also confirming Bohm diffusion as the maximum limiting value of the process.

Chen[55,56] found that even the reflex arc works only because of cross-field diffusion caused by the **E** × **B** instability. In the reflex arc, the arc is in a DC magnetic field and the cathodes (either cold or thermionic) are at each end. The anode is surrounds the plasma; as a consequence, the discharge current can flow only across the B-field. A further important step in understanding the instabilities in crossed electric and magnetic fields was made by A. Simon[57] (Simon-Hoh instability). Simon showed that when a strong electric field is applied across the magnetic field in a non-uniform plasma, the discharge becomes unstable. The instability occurs when the electric field is *in the same direction* as the density gradient. In the reflex arc this is the direction of the applied ionizing potential, and only the instability allows the discharge current to flow. A different, original approach to the problem of cross-field diffusion was proposed by Kurşunoğlu[58], who used stochastic theory to sample the Langevin equation and follow the Brownian motion of charged particles across the field lines. He derived expressions of both classical and "enhanced" diffusion coefficients.

The subject of anomalous diffusion became so important that an Anomalous Absorption Conference series was started and has persisted for over four decades. At the end of the century Bohm's anomalous transport was predominantly interpreted as an upper limiting value of cross-field diffusion[59], with low-frequency drift-wave fluctuations playing the major role in enhancing classical diffusion[60]. At small amplitudes of the fluctuating azimuthal E-field the diffusion remains classical; when the amplitude of the oscillation is increased, the diffusion gradually shifts to the anomalous $1/B$ trend[61]. In fusion devices, electron drift instabilities can be suppressed by such mechanisms as shear, minimum-B, and short connection length. Similar instabilities can be driven by other energy sources, such as ion temperature gradients.

VII. Summary

In discharges in which the B-field intersects endplates, the sheaths on the endplates will self-adjust to equalize ion and electron fluxes on each field line. The appearance that electrons have crossed the B-field to follow the faster cross-field motion of the ions is called the Simon short-circuit effect. In very long discharges or those with closed magnetic surfaces, the azimuthal **E** × **B** drifts of the electrons and ions are not equal because of finite Larmor radius effects. In the presence of the necessary pressure gradient in a confined plasma, a drift-wave instability can then arise with an azimuthal wavelength. The wave causes an oscillating radial drift of the plasma such that the drift is outward when the density is high and inwards when the density is low, leading to enhanced plasma losses to the wall. This loss rate can be verified by measuring the phase shift between the $\tilde{n}$ and $\tilde{\phi}$ oscillations of the drift wave.

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