Lecture 5
Alternatives

- theorem of alternatives for linear inequalities
- Farkas' lemma and other variants

Theorem of alternatives for linear inequalities

for given A, b, **exactly one** of the following two statements is true

1. there exists an x that satisfies $Ax \leq b$
2. there exists a z that satisfies $z \geq 0$, $A^T z = 0$, $b^T z < 0$

- it is clear that 1 and 2 cannot be both true:

$$\begin{array}{rcl} Ax \leq b, \quad z \geq 0 & \Longrightarrow & z^T(Ax - b) \leq 0 \\ A^T z = 0, \quad b^T z < 0 & \Longrightarrow & z^T(Ax - b) > 0 \end{array}$$

- proof that 1 and 2 cannot be both false is less obvious (see page 5–7)
- z in statement 2 is a **certificate** of infeasibility of $Ax \leq b$

Farkas' lemma

for given A, b, exactly one of the following statements is true:

1. there exists an x with with $Ax = b$, $x \geq 0$
2. there exists a y with $A^T y \geq 0$, $b^T y < 0$

proof: apply previous theorem to

$$\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

- this system is infeasible if and only if there exist u, v, w such that

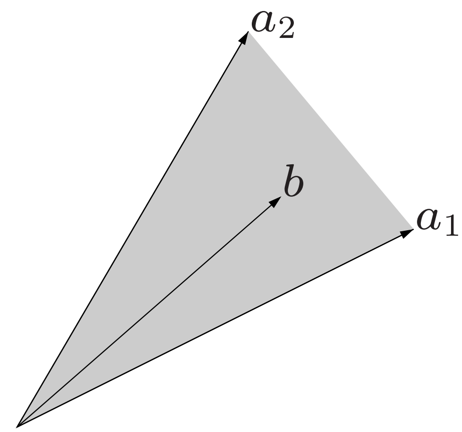
$$u \geq 0,\ v \geq 0,\ w \geq 0, \qquad A^T(u-v) = w, \qquad b^T(u-v) < 0$$

- in simpler notation (defining $y = u - v$): $A^T y \geq 0$, $b^T y < 0$

Geometric interpretation of Farkas' lemma

assume A is $m \times n$ with columns a_i

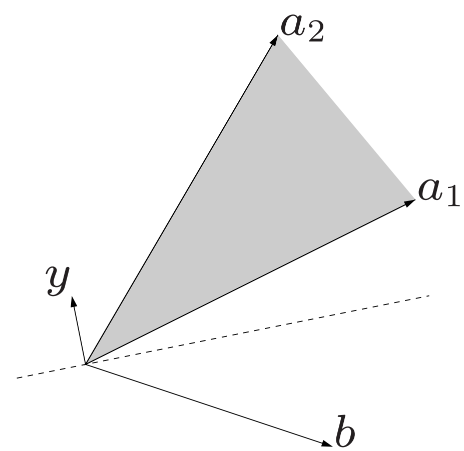
first alternative



$$b = \sum_{i=1}^{n} x_i a_i, \qquad x_i \geq 0, \quad i = 1, \ldots, n$$

b is in the cone generated by the columns of A

second alternative



$$y^T a_i \geq 0, \quad i = 1, \ldots, m, \qquad y^T b < 0$$

the hyperplane $y^T z = 0$ separates b from a_1, . . . , a_m

Mixed inequalities and equalities

given A, b, C, d, exactly one of the following statements is true

1. there exists an x that satisfies

$$Ax \leq b, \qquad Cx = d$$

2. there exist y, z that satisfy

$$z \geq 0, \qquad A^T z + C^T y = 0, \qquad b^T z + d^T y < 0$$

proof: apply theorem of page 5–2 to

$$\begin{bmatrix} A \\ C \\ -C \end{bmatrix} x \leq \begin{bmatrix} b \\ d \\ -d \end{bmatrix}$$

Exercise: strict inequalities

show that exactly one of the following statements is true

1. there exists an x that satisfies

$$Ax < b, \qquad Bx \leq c$$

2. there exist y, z that satisfy

$$y \geq 0, \qquad z \geq 0, \qquad A^T y + B^T z = 0,$$

and

$$b^T y + c^T z < 0 \qquad \text{or} \qquad b^T y + c^T z = 0, \quad y \neq 0$$

hint. statement 1 is equivalent to: there exist u, t such that

$$Au \leq tb - \mathbf{1}, \qquad Bu \leq tc, \qquad t \geq 1$$

Proof of the theorem of alternatives

- we show that if statement 1 on page 5–2 is false, then 2 is true
- the proof is by induction on the column dimension of A

basic case: if A has zero columns, the alternatives are

1. $b \geq 0$
2. there exists a $z \geq 0$ with $b^T z < 0$

clearly, if 1 is false ($b_i < 0$ for some i), then 2 is true (take $z = e_i$)

induction step

- assume the theorem holds for sets of inequalities with $n - 1$ variables
- consider an inequality $Ax \leq b$ with an $m \times n$ matrix A

- we divide the inequalities $Ax \leq b$ in three groups:

$$I_+ = \{i \mid A_{in} > 0\}, \qquad I_0 = \{i \mid A_{in} = 0\}, \qquad I_- = \{i \mid A_{in} < 0\}$$

- scale the inequalities with $A_{in} \neq 0$ to get an equivalent system

$$\begin{aligned} \sum_{k=1}^{n-1} C_{ik} x_k + x_n &\leq d_i && \text{for } i \in I_+ \\ \sum_{k=1}^{n-1} C_{ik} x_k - x_n &\leq d_i && \text{for } i \in I_- \\ \sum_{k=1}^{n-1} A_{ik} x_k &\leq b_i && \text{for } i \in I_0 \end{aligned}$$

where

$$C_{ik} = \begin{cases} A_{ik}/A_{in} & i \in I_+ \\ -A_{ik}/A_{in} & i \in I_- \end{cases} \qquad d_i = \begin{cases} b_i/A_{in} & i \in I_+ \\ -b_i/A_{in} & i \in I_- \end{cases}$$

- the inequalities indexed by I_+ and I_- hold for some x_n if and only if

$$\max_{i \in I_-} \left(\sum_{k=1}^{n-1} C_{ik} x_k - d_i \right) \leq \min_{i \in I_+} \left(d_i - \sum_{k=1}^{n-1} C_{ik} x_k \right)$$

- therefore $Ax \leq b$ is solvable if and only if there exist $(x_1, \ldots, x_{n-1})$ s.t.

$$\sum_{k=1}^{n-1} (C_{ik} + C_{jk}) x_k \leq d_i + d_j \qquad \text{for all } i \in I_-,\ j \in I_+$$

$$\sum_{k=1}^{n-1} A_{ik} x_k \leq b_i \qquad \text{for all } i \in I_0$$

this is a system of inequalities with $n - 1$ variables

- if this system is infeasible, there exist u_{ij} $(i \in I_-,\ j \in I_+)$, v_i $(i \in I_0)$,

$$u_{ij} \geq 0 \quad \text{for } i \in I_-,\ j \in I_+, \qquad v_i \geq 0 \quad \text{for } i \in I_0$$

$$\sum_{i \in I_-, j \in I_+} (C_{ik} + C_{jk}) u_{ij} + \sum_{i \in I_0} v_i A_{ik} = 0, \quad k = 1, \ldots, n-1$$

$$\sum_{i \in I_-, j \in I_+} (d_i + d_j) u_{ij} + \sum_{i \in I_0} b_i v_i < 0$$

- now define

$$z_i = \frac{1}{-A_{in}} \sum_{j \in I_+} u_{ij} \qquad \text{for } i \in I_-$$

$$z_j = \frac{1}{A_{jn}} \sum_{i \in I_-} u_{ij} \qquad \text{for } j \in I_+$$

$$z_i = v_i \qquad \text{for } i \in I_0$$

to get a vector z that satisfies $z \geq 0$, $A^T z = 0$, $b^T z < 0$