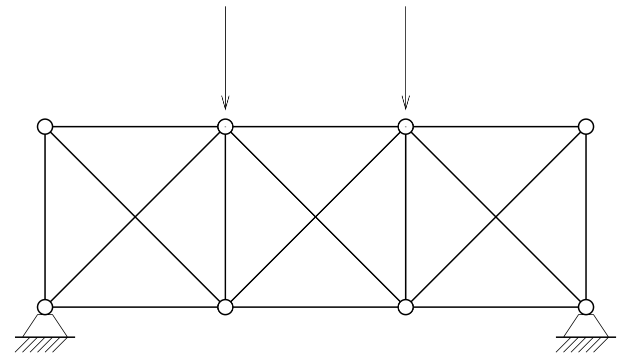
Lecture 9
Structural optimization

- minimum weight truss design
- topology design
- limit analysis

Truss



- m bars (members), N nodes (joints)
- length of bar i is l_i, cross-sectional area x_i
- nodes $n+1, \ldots, N$ are anchored
- external forces $f_i \in \mathbf{R}^2$ at nodes $i = 1, \ldots, n$

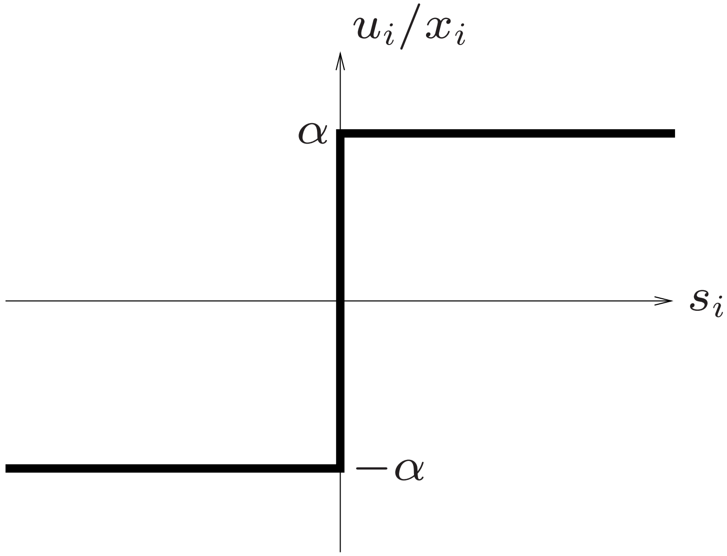
design and analysis problems

- for given topology, find lightest truss that can carry a given load
- find lightest truss that can carry several possible loads
- find best topology
- for a given truss, determine the heaviest load it can carry

Material characteristics

- $u_i \in \mathbf{R}$ is force in bar i ($u_i > 0$: tension, $u_i < 0$: compression)
- $s_i \in \mathbf{R}$ is deformation of bar i ($s_i > 0$: lengthening, $s_i < 0$: shortening)

we assume the material is **rigid/perfectly plastic**:



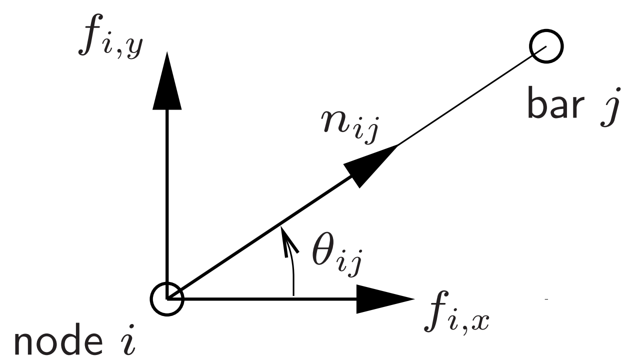
$$
\begin{array}{ll}
s_i = 0 & \text{if } |u_i| < \alpha x_i \\
u_i = \alpha x_i & \text{if } s_i > 0 \\
u_i = -\alpha x_i & \text{if } s_i < 0
\end{array}
$$

α is a material constant

Force equilibrium

force equlibrium equation at free node i ($i = 1, \ldots, n$)

$$\sum_{j=1}^{m} u_j \left[\begin{array}{c} n_{ij,x} \\ n_{ij,y} \end{array} \right] + \left[\begin{array}{c} f_{i,x} \\ f_{i,y} \end{array} \right] = 0$$



2-vectors n_{ij} ($i = 1, \ldots, n$, $j = 1, \ldots, m$) specify geometry of truss

$$n_{ij} = \begin{cases} 0 & \text{if bar } j \text{ is not connected to node } i \\ (\cos\theta_{ij}, \sin\theta_{ij}) & \text{otherwise} \end{cases}$$

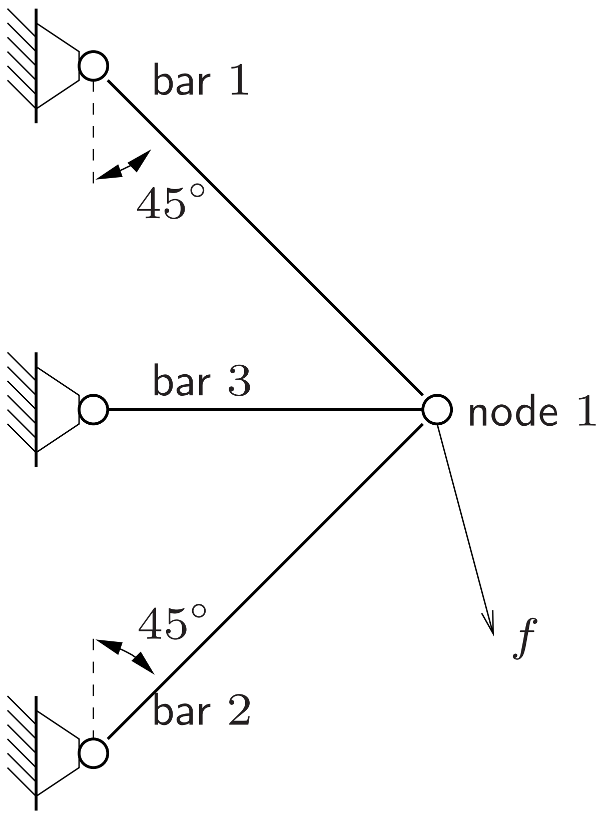
Minimum weight truss

$$
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{m} l_j x_j \\
\text{subject to} & \sum_{j=1}^{m} u_j n_{ij} + f_i = 0, \quad i = 1, \ldots, n \\
& -\alpha x_j \leq u_j \leq \alpha x_j, \quad j = 1, \ldots, m
\end{array}
$$

- an LP with variables x_j, u_j, $j = 1, \ldots, m$
- eliminating x_j gives ℓ_1-norm optimization problem

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{m} l_j |u_j| \\
\text{subject to} & \sum_{j=1}^{m} u_j n_{ij} + f_i = 0, \quad i = 1, \ldots, n
\end{array}
$$

Example



$$
\begin{array}{ll}
\text{mimimize} & l_1x_1 + l_2x_2 + l_3x_3 \\
\text{subject to} & -u_1/\sqrt{2} - u_2/\sqrt{2} - u_3 + f_x = 0 \\
& u_1/\sqrt{2} - u_2/\sqrt{2} + f_y = 0 \\
& -\alpha x_1 \leq u_1 \leq \alpha x_1 \\
& -\alpha x_2 \leq u_2 \leq \alpha x_2 \\
& -\alpha x_3 \leq u_3 \leq \alpha x_3
\end{array}
$$

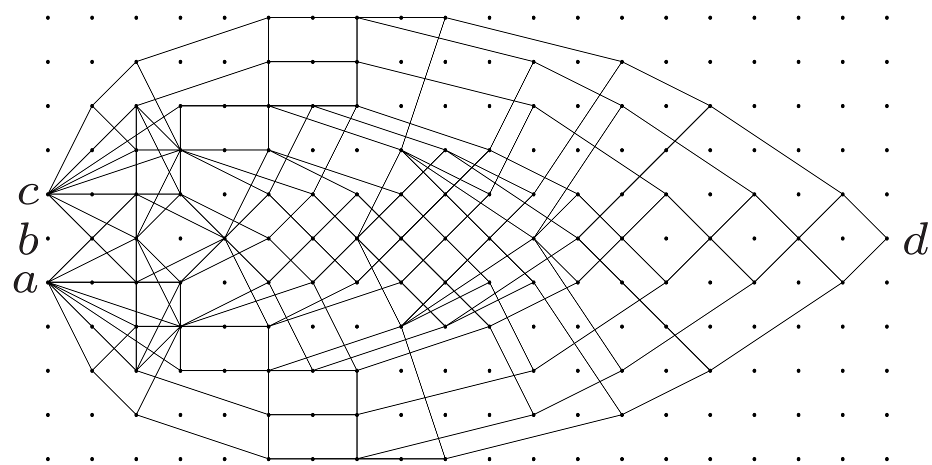
Topology design

- start with grid of nodes; all pairs of nodes are connected
- design the minimum weight truss: at optimum, $u_i = 0$ for most bars
- exclude bars with $u_i = 0$ from topology

example

- 20×11 grid: 220 (potential) nodes, 24,090 (potential) bars
- nodes a, b, c are fixed; unit vertical force at node d

designed topology uses 289 bars



Multiple loading scenarios

minimum weight truss that can carry M possible loads $f_i^1, \ldots, f_i^M$

$$\begin{array}{ll}\text{minimize} & \sum_{j=1}^{m} l_j x_j \\ \text{subject to} & \sum_{j=1}^{m} u_j^k n_{ij} + f_i^k = 0, \quad i = 1, \ldots, n, \quad k = 1, \ldots, M \\ & -\alpha x_j \leq u_j^k \leq \alpha x_j, \quad j = 1, \ldots, m, \quad k = 1, \ldots, M\end{array}$$

- an LP with variables x_j, $u_j^1, \ldots, u_j^M$
- adds robustness: truss can carry any convex combination of $f^1, \ldots, f^M$

Limit analysis

- truss with given geometry and given cross-sectional areas x_i
- forces f_i are given up to a multiple:

$$f_i = \gamma g_i, \quad i = 1, \ldots, n$$

with given $g_i \in \mathbf{R}^2$ and $\gamma > 0$

analysis problem: find heaviest load (largest γ) that the truss can carry

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & \sum_{j=1}^{m} u_j n_{ij} + \gamma g_i = 0, \quad i = 1, \ldots, n \\ & -\alpha x_j \leq u_j \leq \alpha x_j, \quad j = 1, \ldots, m \end{array}$$

- an LP in the variables γ, u_j
- maximum γ is the *safety factor*