1. Introduction

- mathematical optimization
- least squares and linear programming
- convex optimization
- example
- course information

Mathematical optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $x = (x_1, \ldots, x_n)$: optimization variables
- f_0: objective function
- $f_1, \ldots, f_m$, $h_1, \ldots, h_p$: inequality and equality constraint functions

Examples

Optimal design and control

- variables represent design parameters, decisions, control actions
- objective function measures performance, cost, deviation from desired outcome
- constraints represent design specifications, restrict allowable choices

Model fitting and approximation

- variables are model parameters
- objective includes approximation or prediction error, regularization terms
- constraints represent prior knowledge, restrictions on possible values

Solving optimization problems

General optimization problem

- very difficult to solve with guarantees of global optimality
- good suboptimal solutions are often sufficient in applications

Exceptions: important classes of problems can be solved globally and efficiently

- least squares
- linear programming
- convex optimization

Least squares

$$\text{minimize} \quad \|Ax - b\|_2^2 = \sum_i \left(\sum_j A_{ij} x_j - b_i\right)^2$$

- solution: $x = (A^T A)^{-1} A^T b$ if A has full column rank
- reliable and efficient algorithms and software
- easy to recognize in applications
- flexibility is increased by adding weights, quadratic regularization terms

Linear programming

$$
\begin{array}{ll}
\text{minimize} & c^T x = c_1 x_1 + \cdots + c_n x_n \\
\text{subject to} & a_i^T x + b_i \leq 0, \quad i = 1, \ldots, m
\end{array}
$$

- no analytical formula for solution
- reliable and efficient algorithms and software
- not as easy to recognize as least squares problems
- a few standard techniques are used to convert problems into linear programs

  e.g., handling 1-norms or ∞-norms, piecewise-linear functions

Convex optimization problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- objective and inequality constraint functions are convex: for $0 \leq \theta \leq 1$,

$$f_i(\theta x + (1-\theta) y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

  (see lecture 3)

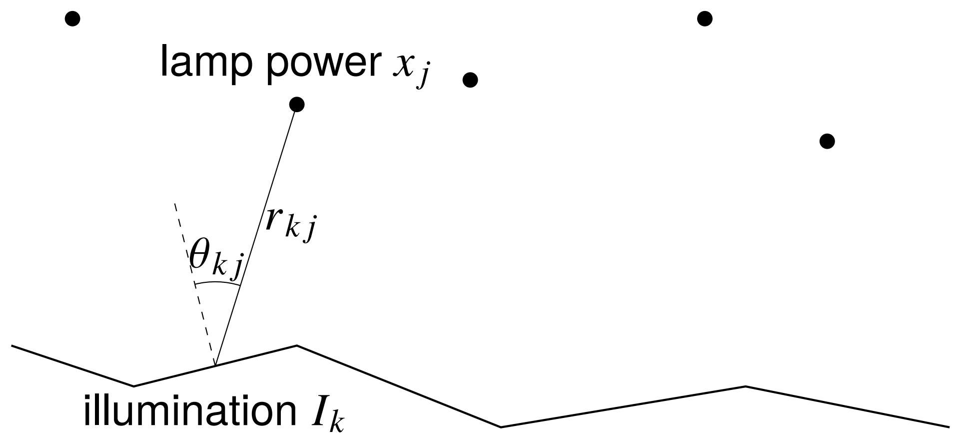
- equality constraints are linear
- includes least squares problems and linear programs as special cases

Using convex optimization

- no analytical formula for solution
- reliable and efficient algorithms
- may be difficult to recognize in applications
- many techniques available for transforming problems into convex form
- surprisingly many problems can be solved via convex optimization
- modeling languages (CVXPY, CVX, ...) greatly simplify interface with solvers

Example

- n lamps illuminate m (small, flat) patches



- intensity I_k at patch k depends linearly on lamp powers x_j:

$$I_k(x) = \sum_{j=1}^{n} a_{kj} x_j, \qquad \text{where } a_{kj} = r_{kj}^{-2} \max\{\cos\theta_{kj}, 0\}$$

Problem: achieve desired illumination I_{des} with bounded lamp powers

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,m} |\log I_k(x) - \log I_{\text{des}}| \\ \text{subject to} & 0 \le x_j \le p_{\max}, \quad j = 1, \ldots, n \end{array}$$

Approximate solutions

1. use uniform power: $x_j = p$ for $j = 1, \ldots, n$, vary p
2. use least squares: solve

$$\text{minimize} \quad \sum_{k=1}^{m} (I_k(x) - I_{\text{des}})^2$$

   and round x_j if $x_j > p_{\max}$ or $x_j < 0$

3. use weighted least squares:

$$\text{minimize} \quad \sum_{k=1}^{m} (I_k(x) - I_{\text{des}})^2 + \sum_{j=1}^{n} w_j (x_j - p_{\max}/2)^2$$

   iteratively adjust weights w_j until $0 \le x_j \le p_{\max}$

4. use linear programming:

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,m} |I_k(x) - I_{\text{des}}| \\ \text{subject to} & 0 \le x_j \le p_{\max}, \quad j = 1, \ldots, n \end{array}$$

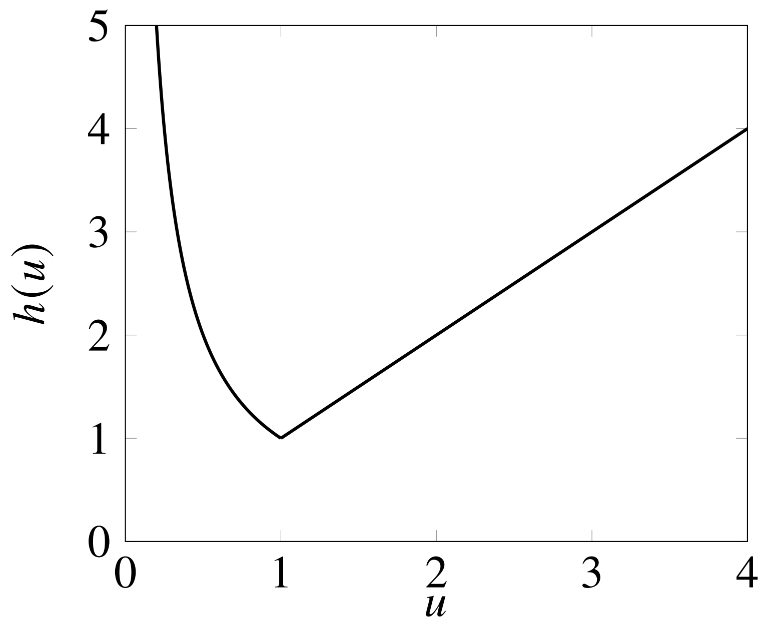
   which can be solved via linear programming

Convex formulation

problem is equivalent to

$$\begin{array}{ll} \text{minimize} & f_0(x) = \max_{k=1,\ldots,m} h(I_k(x)/I_{\text{des}}) \\ \text{subject to} & 0 \le x_j \le p_{\max}, \quad j = 1, \ldots, n \end{array}$$

with $h(u) = \max\{u, 1/u\}$



f_0 is a convex function (see lecture 3)

exact solution obtained with effort $\approx$ modest factor $\times$ least squares effort

Nonconvex optimization

algorithms for general nonconvex optimization

Local optimization (nonlinear programming)

- find a solution that minimizes objective among feasible points near it
- fast algorithms, handle large problems
- often require initial guess
- provide no information about distance to (global) optimum

Global optimization

- find the global solution, with guarantee of optimality
- worst-case complexity grows exponentially with problem size

these algorithms are often based on iteratively solving convex subproblems

Course information

Course material

- textbook available online at web.stanford.edu/˜boyd/cvxbook
- lecture slides, homework assignments on Bruin Learn course website
  bruinlearn.ucla.edu/courses/177014
- slides from previous years available on www.seas.ucla.edu/˜vandenbe/ee236b

Course requirements (see syllabus on the on the course website)

- weekly homework
- computational problems will use the Python package CVXPY (cvxpy.org) or the MATLAB package CVX (cvxr.com)
- open-book final exam (Tuesday, March 19, 11:30am–2:30pm)