

# Lecture 6

## FIR filter design

- FIR filters
- linear phase filter design
- magnitude filter design
- equalizer design

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### FIR filters

**finite impulse response (FIR) filter:**

$$y(t) = \sum_{\tau=0}^{n-1} h_{\tau} u(t - \tau), \quad t \in \mathbf{Z}$$

- $u : \mathbf{Z} \rightarrow \mathbf{R}$  is *input signal*;  $y : \mathbf{Z} \rightarrow \mathbf{R}$  is *output signal*
- $h_i \in \mathbf{R}$  are called *filter coefficients*;  $n$  is *filter order* or *length*

**filter frequency response:  $H : \mathbf{R} \rightarrow \mathbf{C}$**

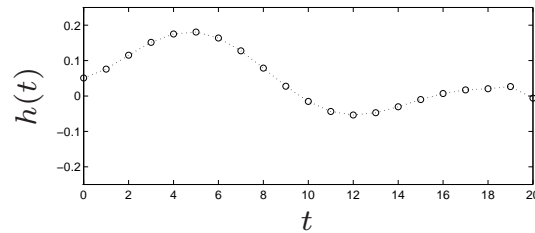
$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \dots + h_{n-1} e^{-j(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega - j \sum_{t=0}^{n-1} h_t \sin t\omega \quad (j = \sqrt{-1}) \end{aligned}$$

periodic, conjugate symmetric, so only need to know/specify for  $0 \leq \omega \leq \pi$

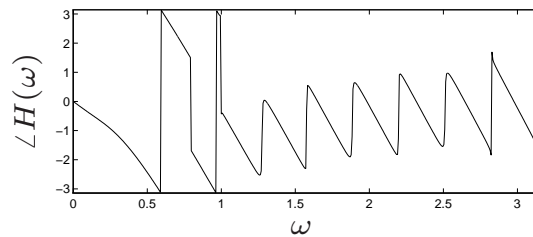
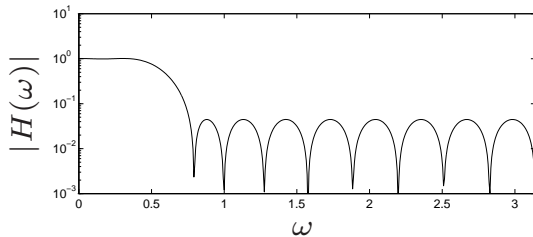
**FIR filter design problem:** choose  $h$  so  $H$  and  $h$  satisfy/optimize specs

**example:** (lowpass) FIR filter, order  $n = 21$

impulse response  $h$ :



frequency response magnitude  $|H(\omega)|$  and phase  $\angle H(\omega)$ :



## Linear phase filters

suppose  $n = 2N + 1$  is odd and impulse response is symmetric about midpoint:

$$h_t = h_{n-1-t}, \quad t = 0, \dots, n-1$$

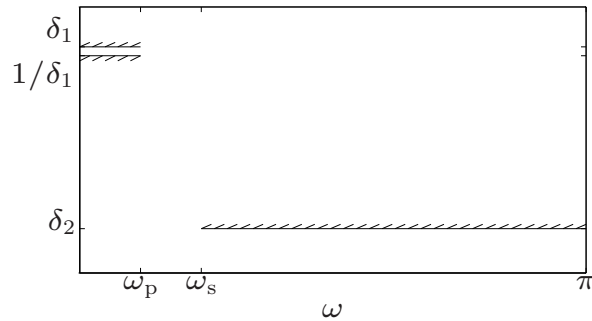
then

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \dots + h_{n-1} e^{-j(n-1)\omega} \\ &= e^{-jN\omega} (2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \dots + h_N) \\ &= e^{-jN\omega} \tilde{H}(\omega) \end{aligned}$$

- term  $e^{-jN\omega}$  represents  $N$ -sample delay
- $\tilde{H}(\omega)$  is **real**
- $|H(\omega)| = |\tilde{H}(\omega)|$

called **linear phase filter** ( $\angle H(\omega)$  is linear except for jumps of  $\pm\pi$ )

## Lowpass filter specifications



### specifications:

- maximum *passband ripple* ( $\pm 20 \log_{10} \delta_1$  in dB):

$$1/\delta_1 \leq |H(\omega)| \leq \delta_1, \quad 0 \leq \omega \leq \omega_p$$

- minimum *stopband attenuation* ( $-20 \log_{10} \delta_2$  in dB):

$$|H(\omega)| \leq \delta_2, \quad \omega_s \leq \omega \leq \pi$$

## Linear phase lowpass filter design

- sample frequency ( $\omega_k = k\pi/K, k = 1, \dots, K$ )
- can assume  $\text{wlog } \tilde{H}(0) > 0$ , so ripple spec is

$$1/\delta_1 \leq \tilde{H}(\omega_k) \leq \delta_1$$

design for **maximum stopband attenuation**:

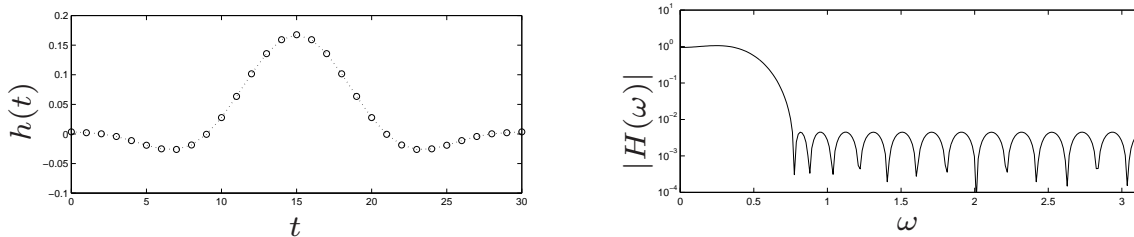
$$\begin{aligned} &\text{minimize} && \delta_2 \\ &\text{subject to} && 1/\delta_1 \leq \tilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_p \\ &&& -\delta_2 \leq \tilde{H}(\omega_k) \leq \delta_2, \quad \omega_s \leq \omega_k \leq \pi \end{aligned}$$

- passband ripple  $\delta_1$  is given
- an LP in variables  $h, \delta_2$
- known (and used) since 1960's
- can add other constraints, *e.g.*,  $|h_i| \leq \alpha$

## example

- linear phase filter,  $n = 31$
- passband  $[0, 0.12\pi]$ ; stopband  $[0.24\pi, \pi]$
- max ripple  $\delta_1 = 1.059$  ( $\pm 0.5\text{dB}$ )
- design for maximum stopband attenuation

impulse response  $h$  and frequency response magnitude  $|H(\omega)|$



## Some variations

$$\tilde{H}(\omega) = 2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \cdots + h_N$$

**minimize passband ripple** (given  $\delta_2, \omega_s, \omega_p, N$ )

$$\begin{aligned} &\text{minimize } \delta_1 \\ &\text{subject to } 1/\delta_1 \leq \tilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_p \\ &\quad \quad \quad -\delta_2 \leq \tilde{H}(\omega_k) \leq \delta_2, \quad \omega_s \leq \omega_k \leq \pi \end{aligned}$$

**minimize transition bandwidth** (given  $\delta_1, \delta_2, \omega_p, N$ )

$$\begin{aligned} &\text{minimize } \omega_s \\ &\text{subject to } 1/\delta_1 \leq \tilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_p \\ &\quad \quad \quad -\delta_2 \leq \tilde{H}(\omega_k) \leq \delta_2, \quad \omega_s \leq \omega_k \leq \pi \end{aligned}$$

**minimize filter order** (given  $\delta_1, \delta_2, \omega_s, \omega_p$ )

$$\begin{aligned} &\text{minimize} && N \\ &\text{subject to} && 1/\delta_1 \leq \tilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_p \\ &&& -\delta_2 \leq \tilde{H}(\omega_k) \leq \delta_2, \quad \omega_s \leq \omega_k \leq \pi \end{aligned}$$

- can be solved using bisection
- each iteration is an LP feasibility problem

## Filter magnitude specifications

transfer function *magnitude spec* has form

$$L(\omega) \leq |H(\omega)| \leq U(\omega), \quad \omega \in [0, \pi]$$

where  $L, U : \mathbf{R} \rightarrow \mathbf{R}_+$  are given and

$$H(\omega) = \sum_{t=0}^{n-1} h_t \cos t\omega - j \sum_{t=0}^{n-1} h_t \sin t\omega$$

- arises in many applications, *e.g.*, audio, spectrum shaping
- not equivalent to a set of linear inequalities in  $h$  (lower bound is not even convex)
- can change variables and convert to set of linear inequalities

## Autocorrelation coefficients

*autocorrelation coefficients* associated with impulse response

$h = (h_0, \dots, h_{n-1}) \in \mathbf{R}^n$  are

$$r_t = \sum_{\tau=0}^{n-1-t} h_{\tau} h_{\tau+t} \quad (\text{with } h_k = 0 \text{ for } k < 0 \text{ or } k \geq n)$$

$r_t = r_{-t}$  and  $r_t = 0$  for  $|t| \geq n$ ; hence suffices to specify  $r = (r_0, \dots, r_{n-1})$

Fourier transform of autocorrelation coefficients is

$$R(\omega) = \sum_{\tau} e^{-j\omega\tau} r_{\tau} = r_0 + \sum_{t=1}^{n-1} 2r_t \cos \omega t = |H(\omega)|^2$$

can express magnitude specification as

$$L(\omega)^2 \leq R(\omega) \leq U(\omega)^2, \quad \omega \in [0, \pi]$$

. . . linear inequalities in  $r$

## Spectral factorization

**question:** when is  $r \in \mathbf{R}^n$  the autocorrelation coefficients of some  $h \in \mathbf{R}^n$ ?

**answer** (*spectral factorization theorem*): if and only if  $R(\omega) \geq 0$  for all  $\omega$

- spectral factorization condition is convex in  $r$  (a linear inequality for each  $\omega$ )
- many algorithms for spectral factorization, *i.e.*, finding an  $h$  such that  $R(\omega) = |H(\omega)|^2$

magnitude design via autocorrelation coefficients:

- use  $r$  as variable (instead of  $h$ )
- add spectral factorization condition  $R(\omega) \geq 0$  for all  $\omega$
- optimize over  $r$
- use spectral factorization to recover  $h$

## Magnitude lowpass filter design

maximum stopband attenuation design with variables  $r$  becomes

$$\begin{aligned} & \text{minimize} && \tilde{\delta}_2 \\ & \text{subject to} && 1/\tilde{\delta}_1 \leq R(\omega) \leq \tilde{\delta}_1, \quad \omega \in [0, \omega_p] \\ & && R(\omega) \leq \tilde{\delta}_2, \quad \omega \in [\omega_s, \pi] \\ & && R(\omega) \geq 0, \quad \omega \in [0, \pi] \end{aligned}$$

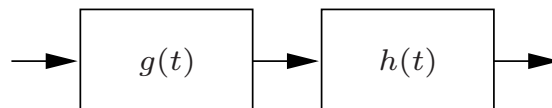
( $\tilde{\delta}_i$  corresponds to  $\delta_i^2$  in original problem)

now discretize frequency:

$$\begin{aligned} & \text{minimize} && \tilde{\delta}_2 \\ & \text{subject to} && 1/\tilde{\delta}_1 \leq R(\omega_k) \leq \tilde{\delta}_1, \quad 0 \leq \omega_k \leq \omega_p \\ & && R(\omega_k) \leq \tilde{\delta}_2, \quad \omega_s \leq \omega_k \leq \pi \\ & && R(\omega_k) \geq 0, \quad 0 \leq \omega_k \leq \pi \end{aligned}$$

... an LP in  $r, \tilde{\delta}_2$

## Equalizer design



**(time-domain) equalization:** given

- $g$  (unequalized impulse response)
- $g_{\text{des}}$  (desired impulse response)

design (FIR equalizer)  $h$  so that  $\tilde{g} = h * g \approx g_{\text{des}}$

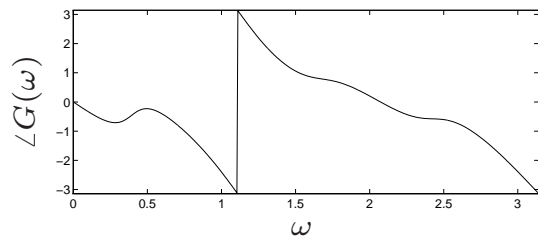
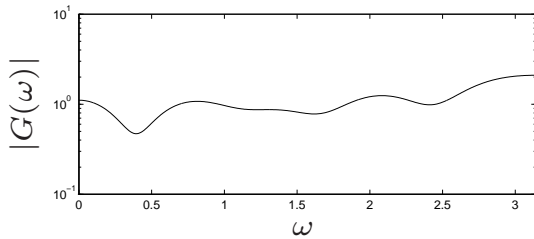
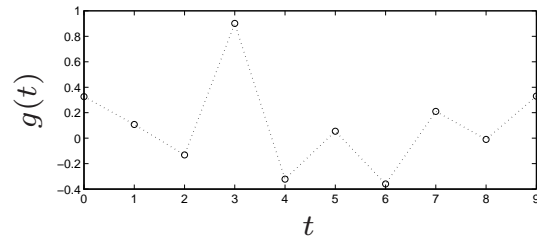
common choice: pure delay  $D$ :  $g_{\text{des}}(t) = \begin{cases} 1 & t = D \\ 0 & t \neq D \end{cases}$

as an LP:

$$\begin{aligned} & \text{minimize} && \max_{t \neq D} |\tilde{g}(t)| \\ & \text{subject to} && \tilde{g}(D) = 1 \end{aligned}$$

**example**

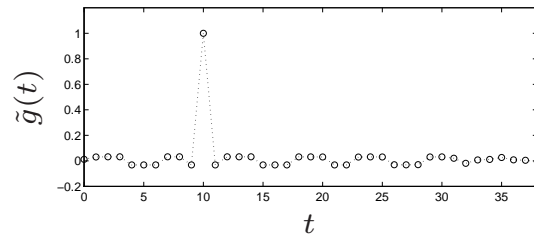
unequalized system  $G$  is 10th order FIR:



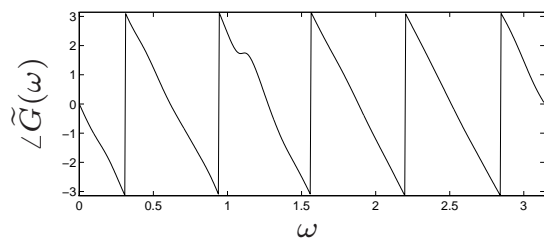
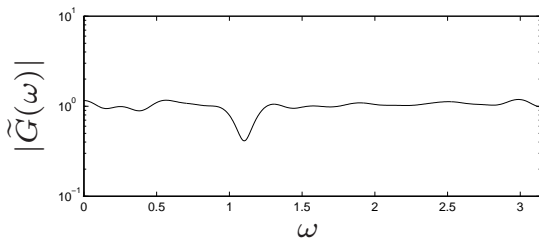
design 30th order FIR equalizer with  $\tilde{G}(\omega) \approx e^{-j10\omega}$

$$\text{minimize } \max_{t \neq 10} |\tilde{g}(t)|$$

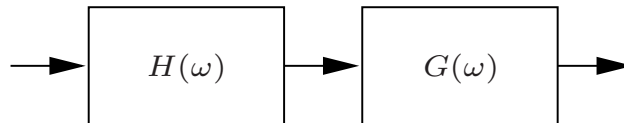
equalized system impulse response  $\tilde{g}$



equalized frequency response magnitude  $|\tilde{G}|$  and phase  $\angle \tilde{G}$



## Magnitude equalizer design



- given system frequency response  $G : [0, \pi] \rightarrow \mathbf{C}$
- design FIR equalizer  $H$  so that  $|G(\omega)H(\omega)| \approx 1$ :

$$\text{minimize } \max_{\omega \in [0, \pi]} \left| |G(\omega)H(\omega)|^2 - 1 \right|$$

use autocorrelation coefficients as variables:

$$\begin{aligned} &\text{minimize } \alpha \\ &\text{subject to } \left| |G(\omega)|^2 R(\omega) - 1 \right| \leq \alpha, \quad \omega \in [0, \pi] \\ &\quad R(\omega) \geq 0, \quad \omega \in [0, \pi] \end{aligned}$$

when discretized, an LP in  $r, \alpha, \dots$

## Multi-system magnitude equalization

- given  $M$  frequency responses  $G_k : [0, \pi] \rightarrow \mathbf{C}$
- design FIR equalizer  $H$  so that  $|G_k(\omega)H(\omega)| \approx \text{constant}$ :

$$\begin{aligned} &\text{minimize } \max_{k=1, \dots, M} \max_{\omega \in [0, \pi]} \left| |G_k(\omega)H(\omega)|^2 - \gamma_k \right| \\ &\text{subject to } \gamma_k \geq 1, \quad k = 1, \dots, M \end{aligned}$$

use autocorrelation coefficients as variables:

$$\begin{aligned} &\text{minimize } \alpha \\ &\text{subject to } \left| |G_k(\omega)|^2 R(\omega) - \gamma_k \right| \leq \alpha, \quad \omega \in [0, \pi], \quad k = 1, \dots, M \\ &\quad R(\omega) \geq 0, \quad \omega \in [0, \pi] \\ &\quad \gamma_k \geq 1, \quad k = 1, \dots, M \end{aligned}$$

$\dots$  when discretized, an LP in  $\gamma_k, r, \alpha$

**example.**  $M = 2$ ,  $n = 25$ ,  $\gamma_k \geq 1$

unequalized and equalized frequency responses

