EE236A (Fall 2007-08)

Lecture 6
FIR filter design

- FIR filters
- linear phase filter design
- magnitude filter design
- equalizer design

FIR filters

finite impulse response (FIR) filter:

$$y(t)=\sum_{\tau=0}^{n-1} h_\tau u(t-\tau), \quad t\in \mathbf{Z}$$

- $u:\mathbf{Z}\to\mathbf{R}$ is *input signal*; $y:\mathbf{Z}\to\mathbf{R}$ is *output signal*
- $h_i\in\mathbf{R}$ are called *filter coefficients*; n is filter *order* or *length*

filter frequency response: $H:\mathbf{R}\to\mathbf{C}$

$$\begin{aligned} H(\omega) &= h_0+h_1e^{-j\omega}+\cdots+h_{n-1}e^{-j(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t\cos t\omega - j\sum_{t=0}^{n-1} h_t \sin t\omega \qquad (j=\sqrt{-1}) \end{aligned}$$

periodic, conjugate symmetric, so only need to know/specify for $0\le\omega\le\pi$

FIR filter design problem: choose h so H and h satisfy/optimize specs

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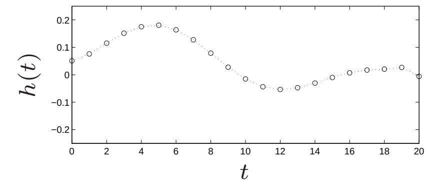
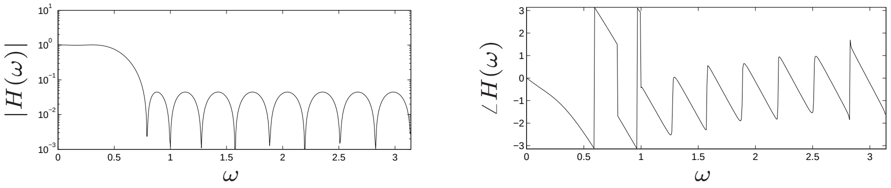
example: (lowpass) FIR filter, order $n=21$

impulse response h:



frequency response magnitude $|H(\omega)|$ and phase $\angle H(\omega)$:



Linear phase filters

suppose $n=2N+1$ is odd and impulse response is symmetric about midpoint:

$$h_t=h_{n-1-t}, \quad t=0,\ldots,n-1$$

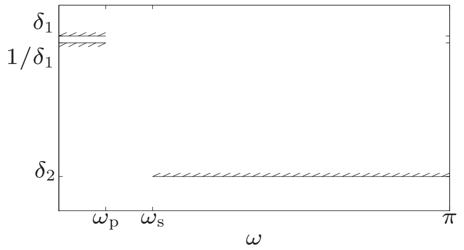
then

$$\begin{aligned} H(\omega) &= h_0+h_1e^{-j\omega}+\cdots+h_{n-1}e^{-j(n-1)\omega} \\ &= e^{-jN\omega}\left(2h_0\cos N\omega+2h_1\cos(N-1)\omega+\cdots+h_N\right) \\ &= e^{-jN\omega}\widetilde{H}(\omega) \end{aligned}$$

- term $e^{-jN\omega}$ represents N-sample delay
- $\widetilde{H}(\omega)$ is **real**
- $|H(\omega)|=|\widetilde{H}(\omega)|$

called **linear phase** filter ($\angle H(\omega)$ is linear except for jumps of $\pm\pi$)

Lowpass filter specifications



specifications:

- maximum *passband ripple* ($\pm 20\log_{10}\delta_1$ in dB):

$$1/\delta_1 \le |H(\omega)| \le \delta_1, \quad 0 \le \omega \le \omega_\mathrm{p}$$

- minimum *stopband attenuation* ($-20\log_{10}\delta_2$ in dB):

$$|H(\omega)| \le \delta_2, \quad \omega_\mathrm{s} \le \omega \le \pi$$

Linear phase lowpass filter design

- sample frequency ($\omega_k = k\pi/K$, $k = 1,\dots,K$)
- can assume wlog $\widetilde{H}(0) > 0$, so ripple spec is

$$1/\delta_1 \le \widetilde{H}(\omega_k) \le \delta_1$$

design for **maximum stopband attenuation:**

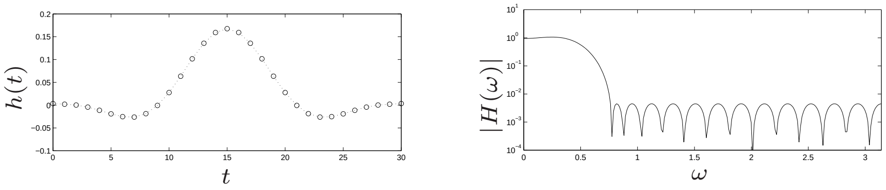
$$\begin{array}{ll} \text{minimize} & \delta_2 \\ \text{subject to} & 1/\delta_1 \le \widetilde{H}(\omega_k) \le \delta_1, \quad 0 \le \omega_k \le \omega_\mathrm{p} \\ & -\delta_2 \le \widetilde{H}(\omega_k) \le \delta_2, \quad \omega_\mathrm{s} \le \omega_k \le \pi \end{array}$$

- passband ripple δ_1 is given
- an LP in variables h, δ_2
- known (and used) since 1960's
- can add other constraints, $e.g.$, $|h_i| \le \alpha$

example

- linear phase filter, $n = 31$
- passband $[0, 0.12\pi]$; stopband $[0.24\pi, \pi]$
- max ripple $\delta_1 = 1.059$ (± 0.5dB)
- design for maximum stopband attenuation

impulse response h and frequency response magnitude $|H(\omega)|$



Some variations

$$\widetilde{H}(\omega) = 2h_0\cos N\omega + 2h_1\cos(N-1)\omega + \cdots + h_N$$

minimize passband ripple (given δ_2, ω_s, ω_p, N)

$$\begin{array}{ll} \text{minimize} & \delta_1 \\ \text{subject to} & 1/\delta_1 \le \widetilde{H}(\omega_k) \le \delta_1, \quad 0 \le \omega_k \le \omega_p \\ & -\delta_2 \le \widetilde{H}(\omega_k) \le \delta_2, \quad \omega_s \le \omega_k \le \pi \end{array}$$

minimize transition bandwidth (given $\delta_1, \delta_2, \omega_p, N$)

$$\begin{array}{ll} \text{minimize} & \omega_s \\ \text{subject to} & 1/\delta_1 \le \widetilde{H}(\omega_k) \le \delta_1, \quad 0 \le \omega_k \le \omega_p \\ & -\delta_2 \le \widetilde{H}(\omega_k) \le \delta_2, \quad \omega_s \le \omega_k \le \pi \end{array}$$

minimize filter order (given δ_1, δ_2, ω_s, ω_p)

$$\begin{array}{ll} \text{minimize} & N \\ \text{subject to} & 1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_p \\ & -\delta_2 \leq \widetilde{H}(\omega_k) \leq \delta_2, \quad \omega_s \leq \omega_k \leq \pi \end{array}$$

- can be solved using bisection
- each iteration is an LP feasibility problem

Filter magnitude specifications

transfer function *magnitude spec* has form

$$L(\omega) \leq |H(\omega)| \leq U(\omega), \quad \omega \in [0, \pi]$$

where $L, U : \mathbf{R} \to \mathbf{R}_+$ are given and

$$H(\omega) = \sum_{t=0}^{n-1} h_t \cos t\omega - j \sum_{t=0}^{n-1} h_t \sin t\omega$$

- arises in many applications, $e.g.$, audio, spectrum shaping
- not equivalent to a set of linear inequalities in h (lower bound is not even convex)
- can change variables and convert to set of linear inequalities

Autocorrelation coefficients

autocorrelation coefficients associated with impulse response $h = (h_0, \ldots, h_{n-1}) \in \mathbf{R}^n$ are

$$r_t = \sum_{\tau=0}^{n-1-t} h_\tau h_{\tau+t} \quad \text{(with } h_k = 0 \text{ for } k < 0 \text{ or } k \geq n\text{)}$$

$r_t = r_{-t}$ and $r_t = 0$ for $|t| \geq n$; hence suffices to specify $r = (r_0, \ldots, r_{n-1})$

Fourier transform of autocorrelation coefficients is

$$R(\omega) = \sum_\tau e^{-j\omega\tau} r_\tau = r_0 + \sum_{t=1}^{n-1} 2r_t \cos \omega t = |H(\omega)|^2$$

can express magnitude specification as

$$L(\omega)^2 \leq R(\omega) \leq U(\omega)^2, \quad \omega \in [0, \pi]$$

. . . linear inequalities in r

Spectral factorization

question: when is $r \in \mathbf{R}^n$ the autocorrelation coefficients of some $h \in \mathbf{R}^n$?

answer (*spectral factorization theorem*): if and only if $R(\omega) \geq 0$ for all ω

- spectral factorization condition is convex in r (a linear inequality for each ω)
- many algorithms for spectral factorization, $i.e.$, finding an h such that $R(\omega) = |H(\omega)|^2$

magnitude design via autocorrelation coefficients:

- use r as variable (instead of h)
- add spectral factorization condition $R(\omega) \geq 0$ for all ω
- optimize over r
- use spectral factorization to recover h

Magnitude lowpass filter design

maximum stopband attenuation design with variables r becomes

$$\begin{array}{ll} \text{minimize} & \tilde{\delta}_2 \\ \text{subject to} & 1/\tilde{\delta}_1 \le R(\omega) \le \tilde{\delta}_1, \quad \omega \in [0, \omega_{\rm p}] \\ & R(\omega) \le \tilde{\delta}_2, \quad \omega \in [\omega_{\rm s}, \pi] \\ & R(\omega) \ge 0, \quad \omega \in [0, \pi] \end{array}$$

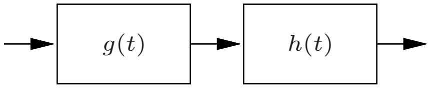
($\tilde{\delta}_i$ corresponds to δ_i^2 in original problem)

now discretize frequency:

$$\begin{array}{ll} \text{minimize} & \tilde{\delta}_2 \\ \text{subject to} & 1/\tilde{\delta}_1 \le R(\omega_k) \le \tilde{\delta}_1, \quad 0 \le \omega_k \le \omega_{\rm p} \\ & R(\omega_k) \le \tilde{\delta}_2, \quad \omega_{\rm s} \le \omega_k \le \pi \\ & R(\omega_k) \ge 0, \quad 0 \le \omega_k \le \pi \end{array}$$

. . . an LP in r, $\tilde{\delta}_2$

Equalizer design



(time-domain) equalization: given

- g (unequalized impulse response)
- $g_{\rm des}$ (desired impulse response)

design (FIR equalizer) h so that $\widetilde{g} = h * g \approx g_{\rm des}$

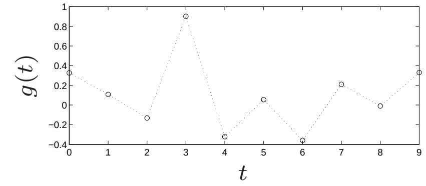
common choice: pure delay D: $g_{\rm des}(t) = \left\{ \begin{array}{ll} 1 & t = D \\ 0 & t \neq D \end{array} \right.$

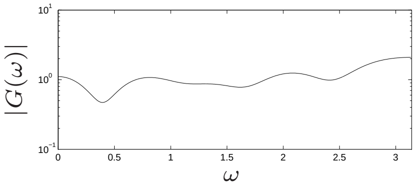
as an LP:

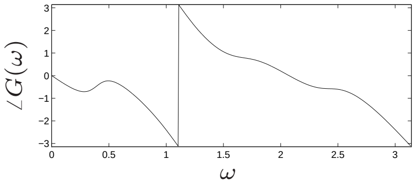
$$\begin{array}{ll} \text{minimize} & \max_{t \neq D} |\tilde{g}(t)| \\ \text{subject to} & \tilde{g}(D) = 1 \end{array}$$

example

unequalized system G is 10th order FIR:



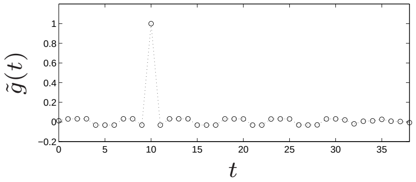




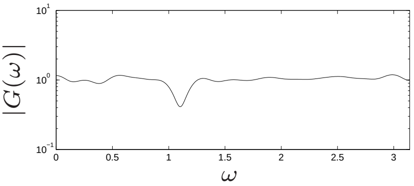
design 30th order FIR equalizer with $\widetilde{G}(\omega) \approx e^{-j10\omega}$

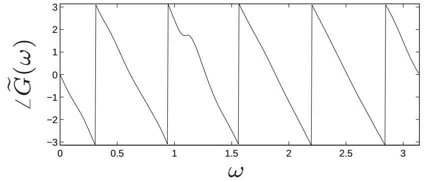
$$\text{minimize} \quad \max_{t \neq 10} |\tilde{g}(t)|$$

equalized system impulse response $\tilde{g}$

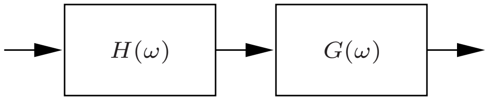


equalized frequency response magnitude $|\widetilde{G}|$ and phase $\angle \widetilde{G}$





Magnitude equalizer design



- given system frequency response $G : [0,\pi] \to \mathbf{C}$
- design FIR equalizer H so that $|G(\omega)H(\omega)| \approx 1$:

$$\text{minimize} \quad \max_{\omega\in[0,\pi]} \big|\, |G(\omega)H(\omega)|^2 - 1 \,\big|$$

use autocorrelation coefficients as variables:

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \big|\, |G(\omega)|^2 R(\omega) - 1 \,\big| \leq \alpha, \quad \omega \in [0,\pi] \\ & R(\omega) \geq 0, \quad \omega \in [0,\pi] \end{array}$$

when discretized, an LP in r, α, . . .

Multi-system magnitude equalization

- given M frequency responses $G_k : [0,\pi] \to \mathbf{C}$
- design FIR equalizer H so that $|G_k(\omega)H(\omega)| \approx$ constant:

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,M} \max_{\omega\in[0,\pi]} \big|\, |G_k(\omega)H(\omega)|^2 - \gamma_k \,\big| \\ \text{subject to} & \gamma_k \geq 1, \quad k = 1,\ldots,M \end{array}$$

use autocorrelation coefficients as variables:

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \big|\, |G_k(\omega)|^2 R(\omega) - \gamma_k \,\big| \leq \alpha, \quad \omega \in [0,\pi], \quad k = 1,\ldots,M \\ & R(\omega) \geq 0, \quad \omega \in [0,\pi] \\ & \gamma_k \geq 1, \quad k = 1,\ldots,M \end{array}$$

. . . when discretized, an LP in γ_k, r, α

example. $M = 2$, $n = 25$, $\gamma_k \geq 1$

unequalized and equalized frequency responses