

# Lecture 17

## Integer linear programming

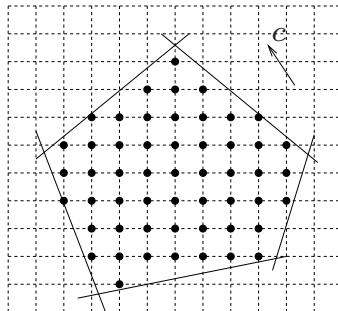
- integer linear programming, 0-1 linear programming
- a few basic facts
- branch-and-bound

17-1

### Definition

**integer linear program (ILP)**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b, \quad Gx = d \\ & x \in \mathbf{Z}^n \end{array}$$



**mixed integer linear program:** only some of the variables are integer

**0-1 (Boolean) linear program** variables take values 0 or 1

## Example: facility location problem

- $n$  potential facility locations,  $m$  clients
- $c_i, i = 1, \dots, n$ : cost of opening a facility at location  $i$
- $d_{ij}, i = 1, \dots, m, j = 1, \dots, n$ : cost of serving client  $i$  from location  $j$

determine optimal location:

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^n c_j y_j + \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, m \\ & x_{ij} \leq y_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n \\ & x_{ij}, y_j \in \{0, 1\} \end{aligned}$$

- $y_j = 1$  if location  $j$  is selected
- $x_{ij} = 1$  if location  $j$  serves client  $i$

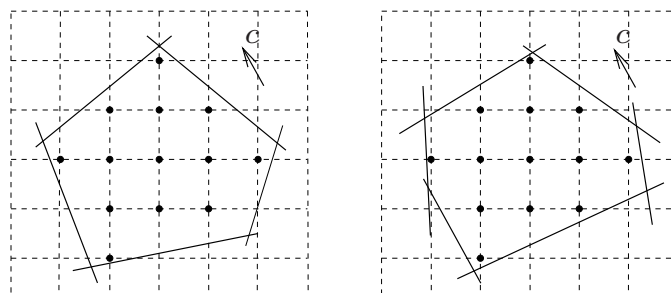
a 0-1 LP

## Linear programming relaxation

the LP obtained by deleting the constraints  $x \in \mathbf{Z}^n$  (or  $x \in \{0, 1\}^n$ ) is called the LP *relaxation*

- provides a lower bound on the optimal value of the integer LP
- if the solution of the relaxation has integer components, then it also solves the integer LP

equivalent ILP formulations of the same problem can have different relaxations

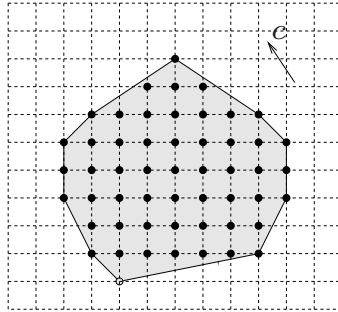


## Strong formulations

the *convex hull* of the feasible set  $\mathcal{S}$  of an ILP is:

$$\mathbf{conv} \mathcal{S} = \left\{ \sum_{i=1}^K \lambda_i x^i \mid x^i \in \mathcal{S}, \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

(the smallest polyhedron containing  $\mathcal{S}$ )



for any  $c$ , the solution of the ILP also solves the relaxation

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && x \in \mathbf{conv} \mathcal{S} \end{aligned}$$

## Branch-and-bound algorithm

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && x \in \mathcal{P} \end{aligned}$$

where  $\mathcal{P}$  is a finite set

**general idea:**

- decompose in smaller problems

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && x \in \mathcal{P}_i \end{aligned}$$

where  $\mathcal{P}_i \subset \mathcal{P}$ ,  $i = 1, \dots, K$

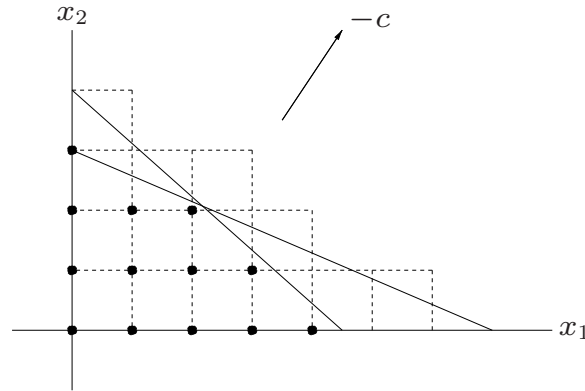
- to solve subproblem: decompose recursively in smaller problems
- use lower bounds from LP relaxation to identify subproblems that don't lead to a solution

**example**

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && x \in \mathcal{P} \end{aligned}$$

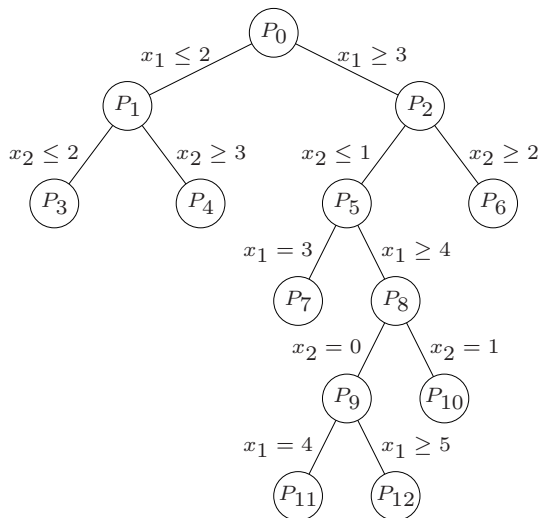
where  $c = (-2, -3)$ , and

$$\mathcal{P} = \left\{ x \in \mathbf{Z}_+^2 \mid \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1, \quad \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \right\}$$



optimal point: (2, 2)

tree of subproblems and results of LP relaxations:



	$x^*$	$p^*$
$P_0$	(2.17, 2.07)	-10.56
$P_1$	(2.00, 2.14)	-10.43
$P_2$	(3.00, 1.33)	-10.00
$P_3$	(2.00, 2.00)	-10.00
$P_4$	(0.00, 3.00)	-9.00
$P_5$	(3.38, 1.00)	-9.75
$P_6$		$+\infty$
$P_7$	(3.00, 1.00)	-9.00
$P_8$	(4.00, 0.44)	-9.33
$P_9$	(4.50, 0.00)	-9.00
$P_{10}$		$+\infty$
$P_{11}$	(4.00, 0.00)	-8.00
$P_{12}$		$+\infty$

conclusions from subproblems:

- $P_2$ : the optimal value of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \geq 3 \end{array}$$

is greater than or equal to  $-10.00$

- $P_3$ : the solution of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \leq 2, \quad x_2 \leq 2 \end{array}$$

is  $(2, 2)$

- $P_6$ : the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \leq 3, \quad x_2 \geq 2 \end{array}$$

is infeasible

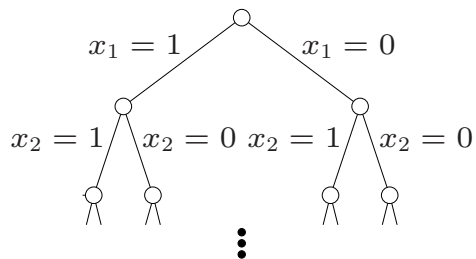
suppose we enumerate the subproblems in the order

$$P_0, \quad P_1, \quad P_2, \quad P_3, \quad \dots$$

then after solving subproblem  $P_4$  we can conclude that  $(2, 2)$  is optimal

## branch-and-bound for 0-1 linear program

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b, \quad x \in \{0, 1\}^n \end{aligned}$$



can solve by enumerating all  $2^n$  possible  $x$ ; every node represents a problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \\ &&& x_i = 0, \quad i \in I_1, \quad x_i = 1, \quad i \in I_2 \\ &&& x_i \in \{0, 1\}, \quad i \in I_3 \end{aligned}$$

where  $I_1, I_2, I_3$  partition  $\{1, \dots, n\}$

## branch-and-bound method

set  $U = +\infty$ , mark all nodes in the tree as active

1. select an active node  $k$ , and solve the corresponding LP relaxation

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \\ &&& x_i = 0, \quad i \in I_1^k \\ &&& x_i = 1, \quad i \in I_2^k \\ &&& 0 \leq x_i \leq 1, \quad i \in I_3^k \end{aligned}$$

let  $\hat{x}$  be the solution of the relaxation

2. if  $c^T \hat{x} \geq U$ , mark all nodes in the subtree with root  $k$  as inactive
3. if all components of  $\hat{x}$  are 0 or 1, mark all nodes in the subtree with root  $k$  as inactive; if moreover  $c^T \hat{x} < U$ , then set  $U := c^T \hat{x}$  and save  $\hat{x}$  as the best feasible point found so far
4. otherwise, mark node  $k$  as inactive
5. go to step 1