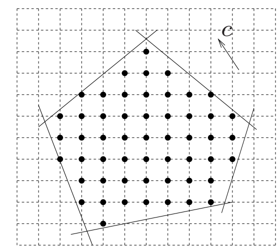
Lecture 17
Integer linear programming

- integer linear programming, 0-1 linear programming
- a few basic facts
- branch-and-bound

Definition

integer linear program (ILP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b, \quad Gx = d \\ & x \in \mathbf{Z}^n \end{array}$$



mixed integer linear program: only some of the variables are integer

0-1 (Boolean) linear program variables take values 0 or 1

Example: facility location problem

- n potential facility locations, m clients
- c_i, $i = 1, \ldots, n$: cost of opening a facility at location i
- d_{ij}, $i = 1 \ldots, m$, $j = 1, \ldots, n$: cost of serving client i from location j

determine optimal location:

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^n c_j y_j + \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} \\ \text{subject to} & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \ldots, m \\ & x_{ij} \leq y_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \\ & x_{ij}, y_j \in \{0, 1\} \end{array}$$

- $y_j = 1$ if location j is selected
- $x_{ij} = 1$ if location j serves client i

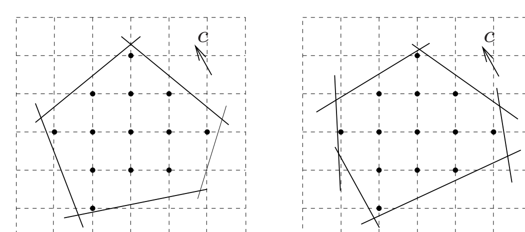
a 0-1 LP

Linear programming relaxation

the LP obtained by deleting the constraints $x \in \mathbf{Z}^n$ (or $x \in \{0,1\}^n$) is called the LP *relaxation*

- provides a lower bound on the optimal value of the integer LP
- if the solution of the relaxation has integer components, then it also solves the integer LP

equivalent ILP formulations of the same problem can have different relaxations

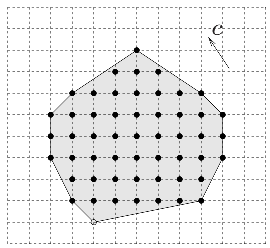


Strong formulations

the *convex hull* of the feasible set $\mathcal{S}$ of an ILP is:

$$\textbf{conv}\,\mathcal{S} = \left\{ \sum_{i=1}^{K} \lambda_i x^i \;\middle|\; x^i \in \mathcal{S}, \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

(the smallest polyhedron containing $\mathcal{S}$)



for any c, the solution of the ILP also solves the relaxation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \textbf{conv}\,\mathcal{S} \end{array}$$

Branch-and-bound algorithm

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P} \end{array}$$

where $\mathcal{P}$ is a finite set

general idea:

- decompose in smaller problems

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}_i \end{array}$$

  where $\mathcal{P}_i \subset \mathcal{P}$, $i = 1, \ldots, K$

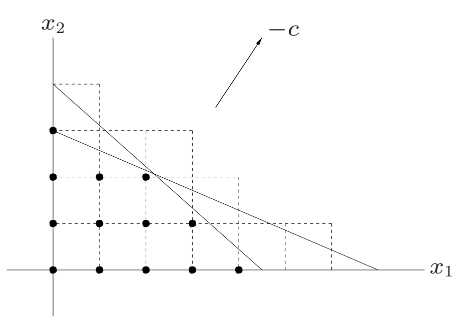
- to solve subproblem: decompose recursively in smaller problems
- use lower bounds from LP relaxation to identify subproblems that don't lead to a solution

example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P} \end{array}$$

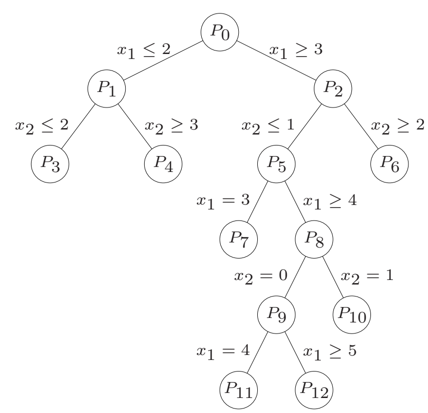
where $c = (-2, -3)$, and

$$\mathcal{P} = \left\{ x \in \mathbf{Z}_+^2 \;\middle|\; \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1, \quad \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \right\}$$



optimal point: $(2, 2)$

tree of subproblems and results of LP relaxations:



	$x^\star$	$p^\star$
P_0	$(2.17, 2.07)$	-10.56
P_1	$(2.00, 2.14)$	-10.43
P_2	$(3.00, 1.33)$	-10.00
P_3	$(2.00, 2.00)$	-10.00
P_4	$(0.00, 3.00)$	-9.00
P_5	$(3.38, 1.00)$	-9.75
P_6		$+\infty$
P_7	$(3.00, 1.00)$	-9.00
P_8	$(4.00, 0.44)$	-9.33
P_9	$(4.50, 0.00)$	-9.00
P_{10}		$+\infty$
P_{11}	$(4.00, 0.00)$	-8.00
P_{12}		$+\infty$

conclusions from subproblems:

- P_2: the optimal value of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \geq 3 \end{array}$$

  is greater than or equal to -10.00

- P_3: the solution of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \leq 2, \quad x_2 \leq 2 \end{array}$$

  is $(2, 2)$

- P_6: the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \leq 3, \quad x_2 \geq 2 \end{array}$$

  is infeasible

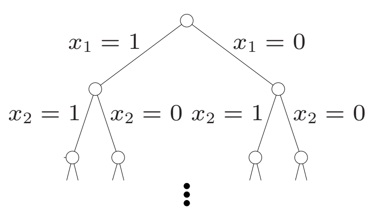
suppose we enumerate the subproblems in the order

$$P_0, \quad P_1, \quad P_2, \quad P_3, \quad \ldots$$

then after solving subproblem P_4 we can conclude that $(2, 2)$ is optimal

branch-and-bound for 0-1 linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b, \quad x \in \{0,1\}^n \end{array}$$



can solve by enumerating all 2^n possible x; every node represents a problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x_i = 0, \quad i \in I_1, \qquad x_i = 1, \quad i \in I_2 \\ & x_i \in \{0,1\}, \quad i \in I_3 \end{array}$$

where I_1, I_2, I_3 partition $\{1, \ldots, n\}$

branch-and-bound method

set $U = +\infty$, mark all nodes in the tree as active

1. select an active node k, and solve the corresponding LP relaxation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x_i = 0, \ i \in I_1^k \\ & x_i = 1, \ i \in I_2^k \\ & 0 \leq x_i \leq 1, \ i \in I_3^k \end{array}$$

   let $\hat{x}$ be the solution of the relaxation
2. if $c^T \hat{x} \geq U$, mark all nodes in the subtree with root k as inactive
3. if all components of $\hat{x}$ are 0 or 1, mark all nodes in the subtree with root k as inactive; if moreover $c^T \hat{x} < U$, then set $U := c^T \hat{x}$ and save $\hat{x}$ as the best feasible point found so far
4. otherwise, mark node k as inactive
5. go to step 1