Lecture 2
Linear inequalities

- vectors
- inner products and norms
- linear equalities and hyperplanes
- linear inequalities and halfspaces
- polyhedra

Vectors

(column) vector $x \in \mathbf{R}^n$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- $x_i \in \mathbf{R}$: ith *component* or *element* of x
- also written as $x = (x_1, x_2, \ldots, x_n)$

some special vectors:

- $x = 0$ (*zero vector*): $x_i = 0$, $i = 1, \ldots, n$
- $x = \mathbf{1}$: $x_i = 1$, $i = 1, \ldots, n$
- $x = e_i$ (ith *basis vector* or ith *unit vector*): $x_i = 1$, $x_k = 0$ for $k \neq i$

(n follows from context)

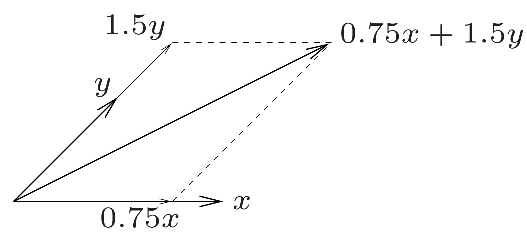
Vector operations

multiplying a vector $x \in \mathbf{R}^n$ with a scalar $\alpha \in \mathbf{R}$:

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

adding and subtracting two vectors x, $y \in \mathbf{R}^n$:

$$x + y = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}, \qquad x - y = \begin{bmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{bmatrix}$$



Inner product

x, $y \in \mathbf{R}^n$

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^T y$$

important properties

- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, x \rangle \geq 0$
- $\langle x, x \rangle = 0 \Longleftrightarrow x = 0$

linear function: $f : \mathbf{R}^n \to \mathbf{R}$ is linear, $i.e.$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),$$

if and only if $f(x) = \langle a, x \rangle$ for some a

Euclidean norm

for $x \in \mathbf{R}^n$ we define the (Euclidean) norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

$\|x\|$ measures *length* of vector (from origin)

important properties:

- $\|\alpha x\| = |\alpha| \|x\|$ (homogeneity)
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
- $\|x\| \geq 0$ (nonnegativity)
- $\|x\| = 0 \iff x = 0$ (definiteness)

distance between vectors: $\mathbf{dist}(x, y) = \|x - y\|$

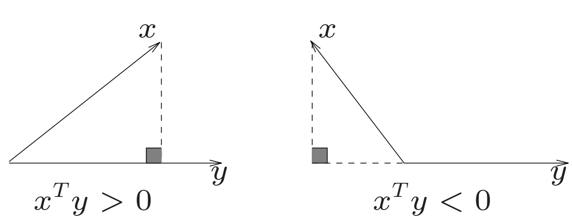
Inner products and angles

angle between vectors in $\mathbf{R}^n$:

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$

$i.e.$, $x^T y = \|x\| \|y\| \cos\theta$

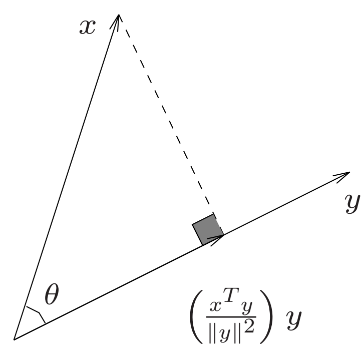
- x and y *aligned*: $\theta = 0$; $x^T y = \|x\| \|y\|$
- x and y *opposed*: $\theta = \pi$; $x^T y = -\|x\| \|y\|$
- x and y *orthogonal*: $\theta = \pi/2$ or $-\pi/2$; $x^T y = 0$ (denoted $x \perp y$)
- $x^T y > 0$ means $\angle(x, y)$ is acute; $x^T y < 0$ means $\angle(x, y)$ is obtuse



Cauchy-Schwarz inequality:

$$|x^T y| \leq \|x\| \|y\|$$

projection of x on y



projection is given by

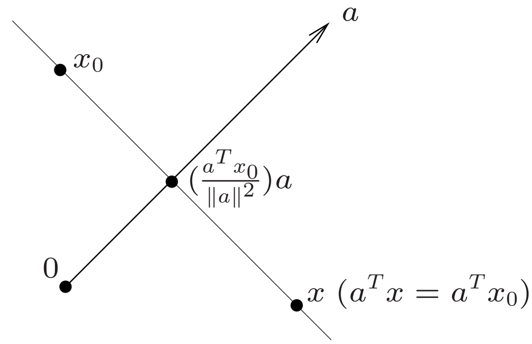
$$\left(\frac{x^T y}{\|y\|^2}\right) y$$

Hyperplanes

hyperplane in $\mathbf{R}^n$:

$$\{x \mid a^T x = b\} \quad (a \neq 0)$$

- solution set of one linear equation $a_1 x_1 + \cdots + a_n x_n = b$ with at least one $a_i \neq 0$
- set of vectors that make a constant inner product with vector $a = (a_1, \ldots, a_n)$ (the *normal* vector)



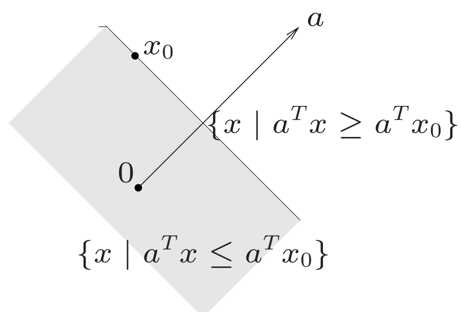
in $\mathbf{R}^2$: a line, in $\mathbf{R}^3$: a plane, . . .

Halfspaces

(closed) halfspace in $\mathbf{R}^n$:

$$\{x \mid a^T x \leq b\} \quad (a \neq 0)$$

- solution set of one linear inequality $a_1 x_1 + \cdots + a_n x_n \leq b$ with at least one $a_i \neq 0$
- $a = (a_1, \ldots, a_n)$ is the (outward) *normal*



- $\{x \mid a^T x < b\}$ is called an *open* halfspace

Affine sets

solution set of a set of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

intersection of m hyperplanes with normal vectors $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ (w.l.o.g., all $a_i \neq 0$)

in matrix notation:

$$Ax = b$$

with

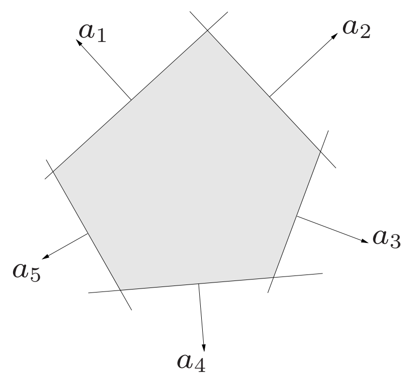
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Polyhedra

solution set of system of linear inequalities

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

intersection of m halfspaces, with normal vectors $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ (w.l.o.g., all $a_i \neq 0$)



matrix notation

$$Ax \leq b$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$Ax \leq b$ stands for *componentwise* inequality, *i.e.*, for y, $z \in \mathbf{R}^n$,

$$y \leq z \qquad \Longleftrightarrow \qquad y_1 \leq z_1, \quad \ldots, \quad y_n \leq z_n$$

Examples of polyhedra

- a hyperplane $\{x \mid a^T x = b\}$:

$$a^T x \leq b, \qquad a^T x \geq b$$

- solution set of system of linear equations/inequalities

$$a_i^T x \leq b_i, \quad i = 1, \ldots, m, \qquad c_i^T x = d_i, \quad i = 1, \ldots, p$$

- a slab $\{x \mid b_1 \leq a^T x \leq b_2\}$
- the probability simplex $\{x \in \mathbf{R}^n \mid \mathbf{1}^T x = 1, \;\; x_i \geq 0, \; i = 1, \ldots, n\}$
- (hyper)rectangle $\{x \in \mathbf{R}^n \mid l \leq x \leq u\}$ where $l < u$