Lecture 4
The linear programming problem

- variants of the linear programming problem
- LP feasibility problem
- examples and some general applications
- linear-fractional programming

Variants of the linear programming problem

general form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\ & g_i^T x = h_i, \quad i = 1, \ldots, p \end{array}$$

in matrix notation:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Gx = h \end{array}$$

where

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \in \mathbf{R}^{m \times n}, \qquad G = \begin{bmatrix} g_1^T \\ g_2^T \\ \vdots \\ g_p^T \end{bmatrix} \in \mathbf{R}^{p \times n}$$

inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

in matrix notation:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & g_i^T x = h_i, \quad i = 1, \ldots, m \\ & x \geq 0 \end{array}$$

in matrix notation:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx = h \\ & x \geq 0 \end{array}$$

Reduction of general LP to inequality/standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\ & g_i^T x = h_i, \quad i = 1, \ldots, p \end{array}$$

reduction to inequality form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\ & g_i^T x \geq h_i, \quad i = 1, \ldots, p \\ & g_i^T x \leq h_i, \quad i = 1, \ldots, p \end{array}$$

in matrix notation (where A has rows a_i^T, G has rows g_i^T)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{bmatrix} A \\ -G \\ G \end{bmatrix} x \leq \begin{bmatrix} b \\ -h \\ h \end{bmatrix} \end{array}$$

reduction to standard form:

$$\begin{array}{ll} \text{minimize} & c^T x^+ - c^T x^- \\ \text{subject to} & a_i^T x^+ - a_i^T x^- + s_i = b_i, \quad i = 1, \ldots, m \\ & g_i^T x^+ - g_i^T x^- = h_i, \quad i = 1, \ldots, p \\ & x^+, x^-, s \geq 0 \end{array}$$

- variables x^+, x^-, s
- recover x as $x = x^+ - x^-$
- $s \in \mathbf{R}^m$ is called a *slack* variable

in matrix notation:

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{G}\tilde{x} = \tilde{h} \\ & \tilde{x} \geq 0 \end{array}$$

where

$$\tilde{x} = \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix}, \qquad \tilde{c} = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}, \qquad \tilde{G} = \begin{bmatrix} A & -A & I \\ G & -G & 0 \end{bmatrix}, \qquad \tilde{h} = \begin{bmatrix} b \\ h \end{bmatrix}$$

LP feasibility problem

feasibility problem: find x that satisfies $a_i^T x \leq b_i$, $i = 1, \ldots, m$

solution via LP (with variables t, x)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x \leq b_i + t, \quad i = 1, \ldots, m \end{array}$$

- variables t, x
- if minimizer $x^\star$, $t^\star$ satisfies $t^\star \leq 0$, then $x^\star$ satisfies the inequalities

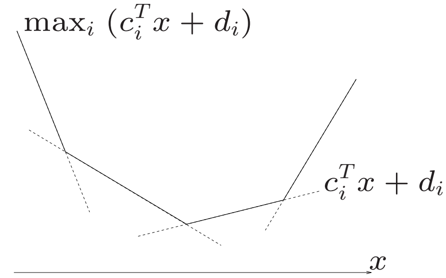
LP in matrix notation:

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \leq \tilde{b} \end{array}$$

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \qquad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} A & -\mathbf{1} \end{bmatrix}, \qquad \tilde{b} = b$$

Piecewise-linear minimization

minimize $\quad \max_{i=1,\ldots,m}(c_i^T x + d_i)$



equivalent LP (with variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$):

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & c_i^T x + d_i \leq t, \quad i = 1, \ldots, m \end{array}$$

in matrix notation:

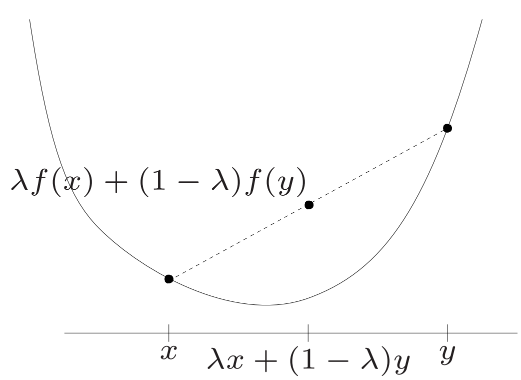
$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \leq \tilde{b} \end{array}$$

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \qquad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} C & -\mathbf{1} \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} -d \end{bmatrix}$$

Convex functions

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if for $0 \leq \lambda \leq 1$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

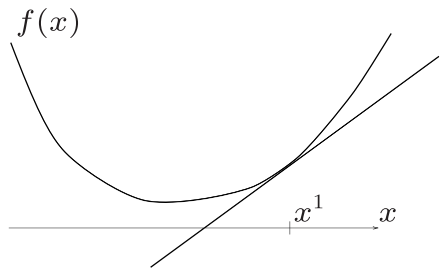


Piecewise-linear approximation

assume $f : \mathbf{R}^n \to \mathbf{R}$ differentiable and convex

- 1st-order approximation at x^1 is a *global lower bound* on f:

$$f(x) \geq f(x^1) + \nabla f(x^1)^T (x - x^1)$$



- evaluating f, ∇f at several x^i yields a *piecewise-linear* lower bound:

$$f(x) \geq \max_{i=1,\ldots,K} \left(f(x^i) + \nabla f(x^i)^T (x - x^i) \right)$$

Convex optimization problem

$$\text{minimize} \quad f_0(x)$$

(f_i convex and differentiable)

LP approximation (choose points x^j, $j = 1, \ldots, K$):

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x^j) + \nabla f_0(x^j)^T (x - x^j) \leq t, \quad j = 1, \ldots, K \end{array}$$

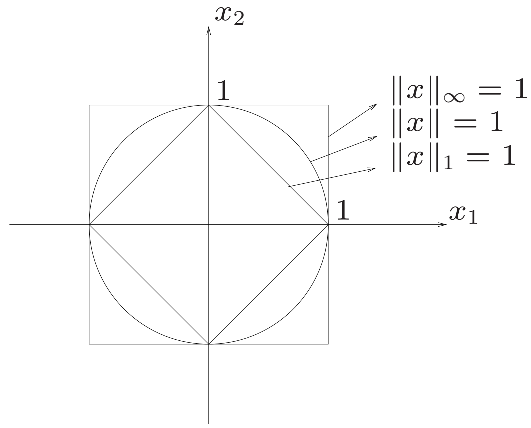
(variables x, t)

- yields lower bound on optimal value
- can be extended to nondifferentiable convex functions
- more sophisticated variation: cutting-plane algorithm (solves convex optimization problem via sequence of LP approximations)

Norms

norms on $\mathbf{R}^n$:

- Euclidean norm $\|x\|$ (or $\|x\|_2$) $= \sqrt{x_1^2 + \cdots + x_n^2}$
- ℓ_1-norm: $\|x\|_1 = |x_1| + \cdots + |x_n|$
- ℓ_∞- (or Chebyshev-) norm: $\|x\|_\infty = \max_i |x_i|$



Norm approximation problems

$$\text{minimize} \quad \|Ax - b\|_p$$

- $x \in \mathbf{R}^n$ is variable; $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$ are problem data
- $p = 1, 2, \infty$
- $r = Ax - b$ is called *residual*
- $r_i = a_i^T x - b_i$ is ith residual (a_i^T is ith row of A)
- usually overdetermined, *i.e.*, $b \notin \mathcal{R}(A)$ (*e.g.*, $m > n$, A full rank)

interpretations:

- approximate or fit b with linear combination of columns of A
- b is corrupted measurement of Ax; find 'least inconsistent' value of x for given measurements

examples:

- $\|r\| = \sqrt{r^T r}$: least-squares or ℓ_2-approximation (a.k.a. regression)

- $\|r\| = \max_i |r_i|$: Chebyshev, ℓ_∞, or minimax approximation

- $\|r\| = \sum_i |r_i|$: absolute-sum or ℓ_1-approximation

solution:

- ℓ_2: closed form expression

$$x_{\rm opt} = (A^T A)^{-1} A^T b$$

  (assume $\mathbf{rank}(A) = n$)
- ℓ_1, ℓ_∞: no closed form expression, but readily solved via LP

ℓ_1-approximation via LP

ℓ_1-approximation problem

$$\text{minimize} \quad \|Ax - b\|_1$$

write as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m y_i \\ \text{subject to} & -y \le Ax - b \le y \end{array}$$

an LP with variables y, x:

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \le \tilde{b} \end{array}$$

with

$$\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \tilde{c} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} b \\ -b \end{bmatrix}$$

ℓ_∞-approximation via LP

ℓ_∞-approximation problem

$$\text{minimize} \quad \|Ax - b\|_\infty$$

write as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -t\mathbf{1} \preceq Ax - b \preceq t\mathbf{1} \end{array}$$

an LP with variables t, x:

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \preceq \tilde{b} \end{array}$$

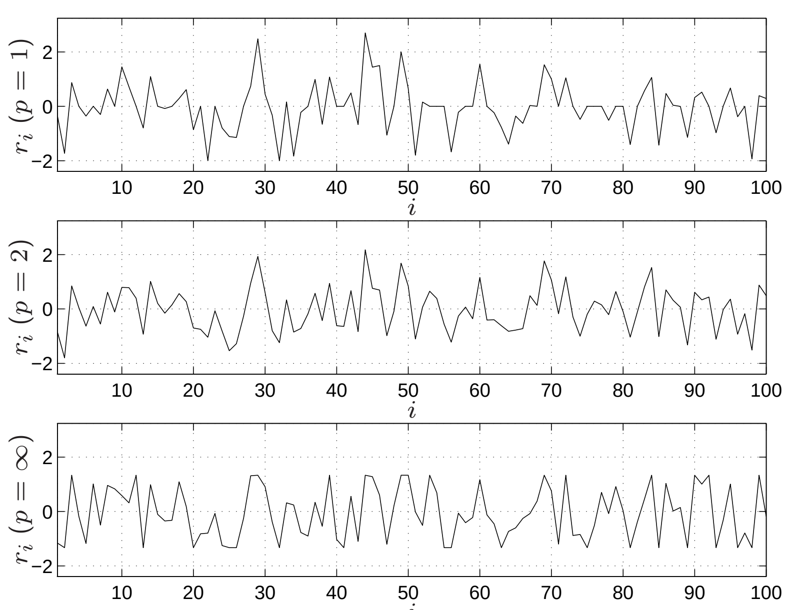
with

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \qquad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} b \\ -b \end{bmatrix}$$

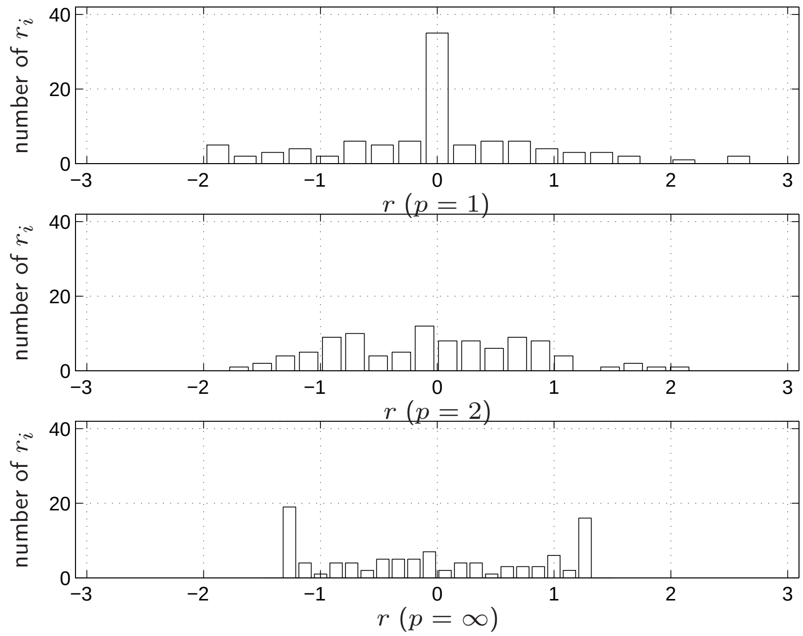
Example

minimize $\|Ax - b\|_p$ for $p = 1,\ 2,\ \infty$ $(A \in \mathbf{R}^{100 \times 30})$

resulting residuals:



histogram of residuals:



- $p = \infty$ gives 'thinnest' distribution; $p = 1$ gives widest distribution
- $p = 1$ most very small (or even zero) r_i

Interpretation: maximum likelihood estimation

m linear measurements $y_1, \ldots, y_m$ of $x \in \mathbf{R}^n$:

$$y_i = a_i^T x + v_i, \quad i = 1, \ldots, m$$

- v_i: measurement noise, IID with density p
- y is a random variable with density $p_x(y) = \prod_{i=1}^m p(y_i - a_i^T x)$

log-likelihood function is defined as

$$\log p_x(y) = \sum_{i=1}^m \log p(y_i - a_i^T x)$$

maximum likelihood (ML) estimate of x is

$$\hat{x} = \underset{x}{\operatorname{argmax}} \sum_{i=1}^m \log p(y_i - a_i^T x)$$

examples

- v_i Gaussian: $p(z) = 1/(\sqrt{2\pi}\sigma)e^{-z^2/2\sigma^2}$

  ML estimate is ℓ_2-estimate $\hat{x} = \operatorname{argmin}_x \|Ax - y\|_2$

- v_i double-sided exponential: $p(z) = (1/2a)e^{-|z|/a}$

  ML estimate is ℓ_1-estimate $\hat{x} = \operatorname{argmin}_x \|Ax - y\|_1$

- v_i is one-sided exponential: $p(z) = \begin{cases} (1/a)e^{-z/a} & z \geq 0 \\ 0 & z < 0 \end{cases}$

  ML estimate is found by solving LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T(y - Ax) \\ \text{subject to} & y - Ax \geq 0 \end{array}$$

- v_i are uniform on $[-a, a]$: $p(z) = \begin{cases} 1/(2a) & -a \leq z \leq a \\ 0 & \text{otherwise} \end{cases}$

  ML estimate is any x satisfying $\|Ax - y\|_\infty \leq a$

Linear-fractional programming

$$\begin{array}{ll} \text{minimize} & \dfrac{c^T x + d}{f^T x + g} \\ \text{subject to} & Ax \leq b \\ & f^T x + g \geq 0 \end{array}$$

(assume $a/0 = +\infty$ if $a > 0$, $a/0 = -\infty$ if $a \leq 0$)

- nonlinear objective function
- like LP, can be solved very efficiently

equivalent form with linear objective (vars. x, γ):

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & c^T x + d \leq \gamma(f^T x + g) \\ & f^T x + g \geq 0 \\ & Ax \leq b \end{array}$$

Bisection algorithm for linear-fractional programming

given: interval $[l, u]$ that contains optimal γ
repeat: solve feasibility problem for $\gamma = (u + l)/2$

$$\begin{array}{l} c^T x + d \leq \gamma(f^T x + g) \\ f^T x + g \geq 0 \\ Ax \leq b \end{array}$$

if feasible $u := \gamma$; if infeasible $l := \gamma$
until $u - l \leq \epsilon$

- each iteration is an LP feasibility problem
- accuracy doubles at each iteration
- number of iterations to reach accuracy ϵ starting with initial interval of width $u - l = \epsilon_0$:

$$k = \lceil \log_2(\epsilon_0/\epsilon) \rceil$$

Generalized linear-fractional programming

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,K} \dfrac{c_i^T x + d_i}{f_i^T x + g_i} \\ \text{subject to} & Ax \leq b \\ & f_i^T x + g_i \geq 0, \quad i = 1, \ldots, K \end{array}$$

equivalent formulation:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & Ax \leq b \\ & c_i^T x + d_i \leq \gamma(f_i^T x + g_i), \quad i = 1, \ldots, K \\ & f_i^T x + g_i \geq 0, \quad i = 1, \ldots, K \end{array}$$

- efficiently solved via bisection on γ
- each iteration is an LP feasibility problem

Von Neumann economic growth problem

simple model of an economy: m goods, n economic sectors

- $x_i(t)$: 'activity' of sector i in current period t
- $a_i^T x(t)$: amount of good i consumed in period t
- $b_i^T x(t)$: amount of good i produced in period t

choose $x(t)$ to maximize *growth rate* $\min_i x_i(t+1)/x_i(t)$:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & Ax(t+1) \leq Bx(t), \quad x(t+1) \geq \gamma x(t), \quad x(t) \geq \mathbf{1} \end{array}$$

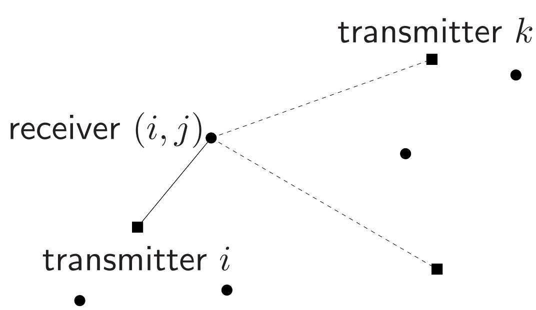
or equivalently (since $a_{ij} \geq 0$):

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & \gamma Ax(t) \leq Bx(t), \quad x(t) \geq \mathbf{1} \end{array}$$

(linear-fractional problem with variables $x(0)$, γ)

Optimal transmitter power allocation

- m transmitters, mn receivers all at same frequency
- transmitter i wants to transmit to n receivers labeled (i, j), $j = 1, \ldots, n$



- A_{ijk} is path gain from transmitter k to receiver (i, j)
- N_{ij} is (self) noise power of receiver (i, j)
- variables: transmitter powers p_k, $k = 1, \ldots, m$

at receiver (i, j):

- signal power: $S_{ij} = A_{iji} p_i$
- noise plus interference power: $I_{ij} = \sum_{k \neq i} A_{ijk} p_k + N_{ij}$
- signal to interference/noise ratio (SINR): S_{ij}/I_{ij}

problem: choose p_i to maximize smallest SINR:

$$\begin{array}{ll} \text{maximize} & \displaystyle \min_{i,j} \frac{A_{iji} p_i}{\sum_{k \neq i} A_{ijk} p_k + N_{ij}} \\ \text{subject to} & 0 \leq p_i \leq p_{\max} \end{array}$$

- a (generalized) linear-fractional program
- special case with analytical solution: $m = 1$, no upper bound on p_i (see exercises)