Lecture 16
Large-scale linear programming

- cutting-plane method
- Benders decomposition
- delayed column generation
- Dantzig-Wolfe decomposition

Cutting-plane method

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

$A \in \mathbf{R}^{m \times n}$, $m \gg n$

general idea: solve sequence of *relaxations*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i \in I \subseteq \{1, \ldots, m\} \end{array}$$

- gives a lower bound on the optimal value of the original problem
- if x solves the relaxed LP and $Ax \leq b$, then x solves the original LP
- if x solves the relaxed LP and $a_j^T x > b_j$ for some j, then we add j to I and solve the new relaxed LP

key to an efficient implementation: find a violated inequality without testing all inequalities

Robust linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (\bar{a}_i + B_i y)^T x \leq b_i \text{ for all } y \in \mathcal{P} \text{ and } i = 1, \ldots, m \end{array}$$

where $B_i \in \mathbf{R}^{n \times p}$, $\mathcal{P} = \{y \mid Cy \leq d\}$

assume $\mathcal{P}$ is bounded with extreme points y^k, $k = 1, \ldots, K$; problem is equivalent to

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (\bar{a}_i + B_i y^k)^T x \leq b_i \text{ for } k = 1, \ldots, K \text{ and } i = 1, \ldots, m \end{array}$$

- a linear program in x
- a huge number of inequalities (except for very small p, or special choice of C, d)

key step in cutting-plane method: given x, find y^k and i for which the inequality

$$(\bar{a}_i + B_i y^k)^T x \leq b_i$$

is violated (and do this without enumerating all y^k)

solution: solve the LPs (with variable y)

$$\begin{array}{ll} \text{maximize} & x^T B_i y \\ \text{subject to} & Cy \leq d \end{array}$$

let $y^\star$ be an optimal extreme point

- if $x^T B_i y^\star > b_i - \bar{a}_i^T x$, then $y^\star$ defines a violated inequality
- otherwise, $(\bar{a}_i + B_i y)^T x \leq b_i$ for all $y \in \mathcal{P}$

very fast if dimension of C is small

summary (for simplicity we add bounds $l \le x \le u$)

set $\mathcal{N}_i = \emptyset$, $i = 1, \dots, m$

1. let x be the optimal solution of the relaxed LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (\bar{a}_i + B_i y)^T x \le b_i \text{ for } y \in \mathcal{N}_i \text{ and } i = 1, \dots, m \\ & l \le x \le u \end{array}$$

2. for $i = 1, \dots, m$,
   - let y be the solution of the LP

$$\begin{array}{ll} \text{maximize} & x^T B_i y \\ \text{subject to} & Cy \le d \end{array}$$

   - if $x^T B_i y > b_i - \bar{a}_i^T x$, then set $\mathcal{N}_i := \mathcal{N}_i \cup \{y\}$ and go to step 1

   if no violated inequality is found, return x

Benders decomposition

$$\begin{array}{ll} \text{minimize} & c_1^T x_1 + c_2^T x_2 + d^T y \\ \text{subject to} & A_1 x_1 + B_1 y \le b_1 \\ & A_2 x_2 + B_2 y \le b_2 \end{array}$$

- $A_i \in \mathbf{R}^{m_i \times n_i}$, $B_i \in \mathbf{R}^{m_i \times p}$
- variables $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$, $y \in \mathbf{R}^p$, $p \ll n_1, n_2$

equivalent to

$$\begin{array}{ll} \text{minimize} & t_1 + t_2 + d^T y \\ \text{subject to} & p_1^\star(y) \le t_1, \qquad p_2^\star(y) \le t_2 \end{array}$$

(variables t_1, t_2, y), where

$$\begin{aligned} p_1^\star(y) &= \min\{c_1^T x \mid A_1 x + B_1 y \le b_1\} \\ p_2^\star(y) &= \min\{c_2^T x \mid A_2 x + B_2 y \le b_2\} \end{aligned}$$

using LP duality, can reformulate as:

$$\begin{array}{ll} \text{minimize} & t_1 + t_2 + d^T y \\ \text{subject to} & z^T B_1 y \leq t_1 + b_1^T z \text{ for all } z \in \mathcal{P}_1 \\ & z^T B_2 y \leq t_2 + b_2^T z \text{ for all } z \in \mathcal{P}_2 \end{array}$$

where $\mathcal{P}_i = \{z \mid A_i^T z + c_i = 0,\ z \geq 0\}$

- variables $y \in \mathbf{R}^p$, t_1, $t_2 \in \mathbf{R}$
- we'll assume for simplicity that $\mathcal{P}_1$ and $\mathcal{P}_2$ are bounded: so we need only consider z that are extreme points of $\mathcal{P}_i$
- (in general) a huge number of constraints

key step in cutting-plane method: given x, t_i, find a $z \in \mathcal{P}_i$ for which

$$z^T B_i y \leq t_i + b_i^T z$$

is violated (and do this without enumerating all extreme points of $\mathcal{P}_i$)

solution: solve the LP (with variable z)

$$\begin{array}{ll} \text{maximize} & (B_i y - b_i)^T z \\ \text{subject to} & A_i^T z + c_i = 0, \quad z \geq 0 \end{array} \qquad (1)$$

corresponding primal problem:

$$\begin{array}{ll} \text{minimize} & c_i^T x \\ \text{subject to} & A_i x \leq b_i - B_i y \end{array}$$

(variable x, optimal value $p_1^\star(y)$)

let $z^\star$ be an optimal vertex for (1)

- if $(B_i y - b_i)^T z^\star > t_i$, then $z^\star$ defines a violated inequality

$$z^T B_1 y \leq t_1 + b_1^T z \text{ for all } z \in \mathcal{P}_1$$

- otherwise, $z^T B_i^T y \leq t_i + b_i^T z$ for all $z \in \mathcal{P}_i$

summary (for simplicity we add bounds $l \leq y \leq u$): set $\mathcal{N}_1 = \mathcal{N}_2 = \emptyset$

1. let y, t_1, t_2 be the solution of the LP

$$\begin{array}{ll} \text{minimize} & d^T y + t_1 + t_2 \\ \text{subject to} & z^T B_1 y \leq t_1 + b_1^T z \text{ for all } z \in N_1 \\ & z^T B_2 y \leq t_2 + b_2^T z \text{ for all } z \in \mathcal{N}_2 \\ & l \leq y \leq u \end{array}$$

   ($t_i = -\infty$ if $\mathcal{N}_i = \emptyset$)

2. let x_1, x_2 be the solutions of the two LPs

$$\begin{array}{ll} \text{minimize} & c_i^T x_i \\ \text{subject to} & A_i x \leq b_i - B_i y \end{array}$$

   and let z_1, z_2 be the corresponding dual solutions

   - if $z_1^T B_1 y > t_1 + b_1^T z_1$, set $\mathcal{N}_1 := \mathcal{N}_1 \cap \{z_1\}$;
   - if $z_2^T B_2 y > t_2 + b_2^T z_2$, set $\mathcal{N}_2 := \mathcal{N}_2 \cap \{z_2\}$;
   - if no violated inequality is found, terminate; otherwise go to step 1

Chebyshev approximation

$$\text{minimize} \quad \|Ax - b\|_\infty$$

where

$$A = \left[\begin{array}{ccc} A_1 & B_1 & 0 \\ 0 & B_2 & A_2 \end{array}\right]$$

and $A_i \in \mathbf{R}^{m_i \times n_i}$, $B_i \in \mathbf{R}^{m_i \times p}$, $p \ll n_1, n_2$

can formulate as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \|A_1 x_1 + B_1 y - b_1\|_\infty \leq t \\ & \|A_2 x_2 + B_2 y - b_2\|_\infty \leq t \end{array}$$

- variables x_1, x_2, y, t
- decouples in two feasibility problems if y, t are fixed

from LP duality: given y, t, there exists an x_i s.t.

$$\|A_i x_i + B_i y - b_i\|_\infty \le t$$

if and only if $(b_i - B_i y)^T z \le t$ for all $z \in \mathcal{P}_i$, where

$$\mathcal{P}_i = \{z \mid A_i^T z = 0, \|z\|_1 \le 1\}$$

hence, can reformulate problem as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & z^T(b_1 - B_1 y) \le t \text{ for all } z \in \mathcal{P}_1 \\ & z^T(b_2 - B_2 y) \le t \text{ for all } z \in \mathcal{P}_2 \end{array}$$

- variables t, y
- (in general) a huge number of linear inequalities (one for each extreme point of $\mathcal{P}_1$ and $\mathcal{P}_2$)

key step in cutting-plane method: given y, t, find $z \in \mathcal{P}_i$ for which $z^T(b_i - B_i y) \le t$ is violated

solution: solve LP (with variable z)

$$\begin{array}{ll} \text{maximize} & (b_i - B_i y)^T z \\ \text{subject to} & A_i^T z = 0 \\ & \|z\|_1 \le 1 \end{array} \qquad (1)$$

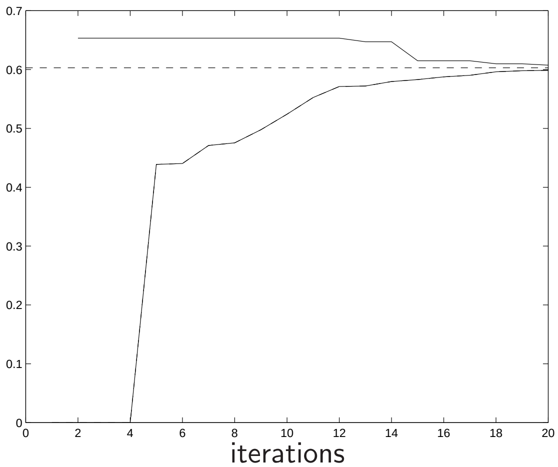
corresponding primal problem (variable x)

$$\text{minimize} \quad \|A_i x - b_i + B_i y\|_\infty$$

let $z^\star$ be an optimal vertex for (1)

- if $(b_i - B_i y)^T z^\star > t$, then $z^\star$ defines a violated inequality in
- otherwise, $z^T(b_i - B_i y) \le t$ for all $z \in \mathcal{P}_i$

example: ($m_1 = m_2 = 100$, $n_1 = n_2 = 80$, $p = 5$)



- lower bound: optimal value of relaxed problem
- upper bound: cost function evaluated at solutions x_1, x_2 from subproblems, solution y from relaxed problem

Delayed column generation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0 \end{array}$$

$A \in \mathbf{R}^{m \times n}$, $m \ll n$

general idea: solve sequence of *restrictions*

$$\begin{array}{ll} \text{minimize} & \sum_{i \in I} c_i x_i \\ \text{subject to} & \sum_{i \in I} x_i A_i = b \\ & x_i \geq 0, \quad i \in I \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A_i^T y \leq c_i, \quad i \in I \end{array}$$

- if x_i ($i \in I$), y are optimal for the restriction, and $A^T y \leq c$, then x (with $x_i = 0$ for $i \notin I$) is optimal for the original LP
- if $A_j^T y > c_j$ for some j, then we add j to I and solve the new restriction

key to an efficient implementation: find a violated dual constraint without testing all inequalities

Dantzig-Wolfe decomposition

$$\begin{array}{ll} \text{minimize} & c_1^T x_1 + c_2^T x_2 \\ \text{subject to} & A_1 x_1 = b_1, \quad A_2 x_2 = b_2 \\ & B_1 x_1 + B_2 x_2 = d \\ & x_1 \geq 0, \;\; x_2 \geq 0 \end{array}$$

$A_i \in \mathbf{R}^{m_i \times n_i}$, $B_i \in \mathbf{R}^{p \times n_i}$, p small

assume $\mathcal{P}_i = \{x \mid A_i x = b_i, \; x \geq 0\}$ $(i = 1, 2)$ is bounded with extreme points x_i^k, $k = 1, \ldots, K_i$

we can reformulate the problem as

$$\begin{array}{ll} \text{minimize} & c_1^T \sum_{k=1}^{K_1} \lambda_k x_1^k + c_2^T \sum_{k=1}^{K_2} \mu_k x_2^k \\ \text{subject to} & B_1 \sum_{k=1}^{K_1} \lambda_k x_1^k + B_2 \sum_{k=1}^{K_2} \mu_k x_2^k = d \\ & \mathbf{1}^T \lambda = 1, \quad \mathbf{1}^T \mu = 1 \\ & \mu \geq 0, \quad \lambda \geq 0 \end{array}$$

few constraints, a huge number of variables λ, μ

Conclusion

- cutting-plane method and column generation are dual techniques
- many variations, *e.g.*, problems with a few coupling variables and a few coupling constraints
- general idea: decompose in smaller problems; use duality to combine the results

implementation:

- simplex method: very well suited for column generation
- interior-point methods: same decomposition techniques work with small modifications (to take into account the fact that interior-point methods provide suboptimal solutions)