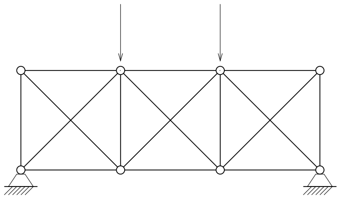
Lecture 5
Structural optimization

- minimum weight truss design
- truss topology design
- limit analysis
- design with minimum number of bars

Truss



- m bars with lengths l_i and cross-sectional areas x_i
- N nodes; nodes $1, \ldots, n$ are free, nodes $n+1, \ldots, N$ are anchored
- external load: forces $f_i \in \mathbf{R}^2$ at nodes $i = 1, \ldots, n$

design problems:

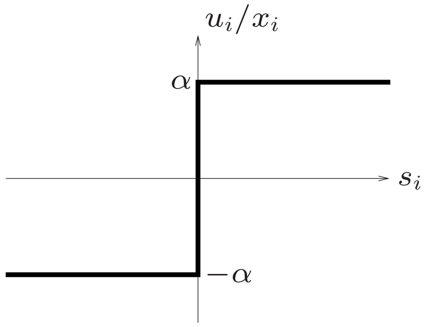
- given the topology ($i.e.$, location of bars and nodes), find the lightest truss that can carry a given load (vars: bar sizes x_k, cost: total weight)
- same problem, where cost $\propto$ #bars used
- find best topology
- find lightest truss that can carry several given loads

analysis problem: for a given truss, what is the largest load it can carry?

Material characteristics

- $u_i \in \mathbf{R}$ is force in bar i ($u_i > 0$: tension, $u_i < 0$: compression)
- $s_i \in \mathbf{R}$ is deformation of bar i ($s_i > 0$: lengthening, $s_i < 0$: shortening)

we assume the material is *rigid/perfectly plastic*:

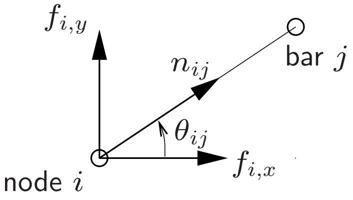


$$\begin{array}{ll} s_i = 0 & \text{if } -\alpha < u_i/x_i < \alpha \\ u_i/x_i = \alpha & \text{if } s_i > 0 \\ u_i/x_i = -\alpha & \text{if } s_i < 0 \end{array}$$

(α is a material constant)

Minimum weight truss for given load

force equilibrium for (free) node i: $\sum_{j=1}^m u_j \begin{bmatrix} n_{ij,x} \\ n_{ij,y} \end{bmatrix} + \begin{bmatrix} f_{i,x} \\ f_{i,y} \end{bmatrix} = 0$



n_{ij} depends on topology:

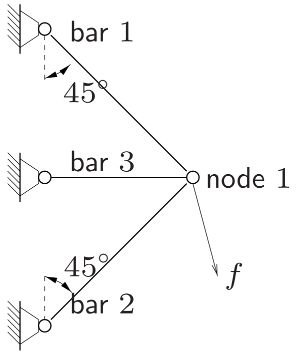
- $n_{ij} = 0$ if bar j is not connected to node i
- $n_{ij} = (\cos\theta_{ij}, \sin\theta_{ij})$ otherwise

minimum weight truss design via LP:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m l_i x_i \\ \text{subject to} & \sum_{j=1}^m u_j n_{ij} + f_i = 0, \quad i = 1, \ldots, n \\ & -\alpha x_j \le u_j \le \alpha x_j, \quad j = 1, \ldots, m \end{array}$$

(variables x_j, u_j)

example



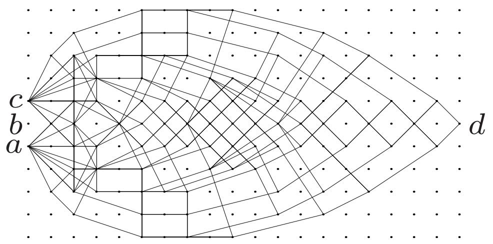
$$\begin{array}{ll} \text{mimimize} & l_1x_1 + l_2x_2 + l_3x_3 \\ \text{subject to} & -u_1/\sqrt{2} - u_2/\sqrt{2} - u_3 + f_x = 0 \\ & u_1/\sqrt{2} - u_2/\sqrt{2} + f_y = 0 \\ & -\alpha x_1 \leq u_1 \leq \alpha x_1 \\ & -\alpha x_2 \leq u_2 \leq \alpha x_2 \\ & -\alpha x_3 \leq u_3 \leq \alpha x_3 \end{array}$$

Truss topology design

- grid of nodes; bars between any pair of nodes
- design minimum weight truss: $u_i = 0$ for most bars
- optimal topology: only use bars with $u_i \neq 0$

example:

- 20×11 grid, *i.e.*, 220 (potential) nodes, 24,090 (potential) bars
- nodes a, b, c are fixed; unit vertical force at node d
- optimal topology has 289 bars



Multiple loading scenarios

minimum weight truss that can carry M possible loads f_i^1, . . . , f_i^M:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m l_i x_i \\ \text{subject to} & \sum_{j=1}^m u_j^k n_{ij} + f_i^k = 0, \quad i = 1, \ldots, n, \quad k = 1, \ldots, M \\ & -\alpha x_j \leq u_j^k \leq \alpha x_j, \quad j = 1, \ldots, m, \quad k = 1, \ldots, M \end{array}$$

(variables x_j, u_j^1, . . . , u_j^M)

adds *robustness*: truss can carry any load

$$f_i = \lambda_1 f_i^1 + \cdots + \lambda_M f_i^M$$

with $\lambda_k \geq 0$, $\sum_k \lambda_k \leq 1$

Limit analysis

- truss with given geometry (including given cross-sectional areas x_i)
- load f_i is given up to a constant multiple: $f_i = \gamma g_i$, with given $g_i \in \mathbf{R}^2$ and $\gamma > 0$

find largest load that the truss can carry:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & \sum_{j=1}^m u_j n_{ij} + \gamma g_i = 0, \quad i = 1, \ldots, n \\ & -\alpha x_j \leq u_j \leq \alpha x_j, \quad j = 1, \ldots, m \end{array}$$

an LP in γ, u_j

maximum allowable γ is called the *safety factor*

Design with smallest number of bars

integer LP formulation (assume wlog $x_i \leq 1$):

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^m z_j \\ \text{subject to} & \sum_{j=1}^m u_j n_{ij} + f_i = 0, \quad i = 1, \ldots, n \\ & -\alpha x_j \leq u_j \leq \alpha x_j, \quad j = 1, \ldots, m \\ & x_j \leq z_j, \quad j = 1, \ldots, m \\ & z_j \in \{0, 1\}, \quad j = 1, \ldots, m \end{array}$$

- variables z_j, x_j, u_j
- extremely hard to solve; we may have to enumerate all 2^m possible values of z

heuristic: replace $z_j \in \{0, 1\}$ by $0 \leq z_j \leq 1$

- yields an LP; at the optimum many (but not all) z_j's will be 0 or 1
- called *LP relaxation* of the integer LP