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Optimal Cross-Layer Coding for the Block Fading Channel

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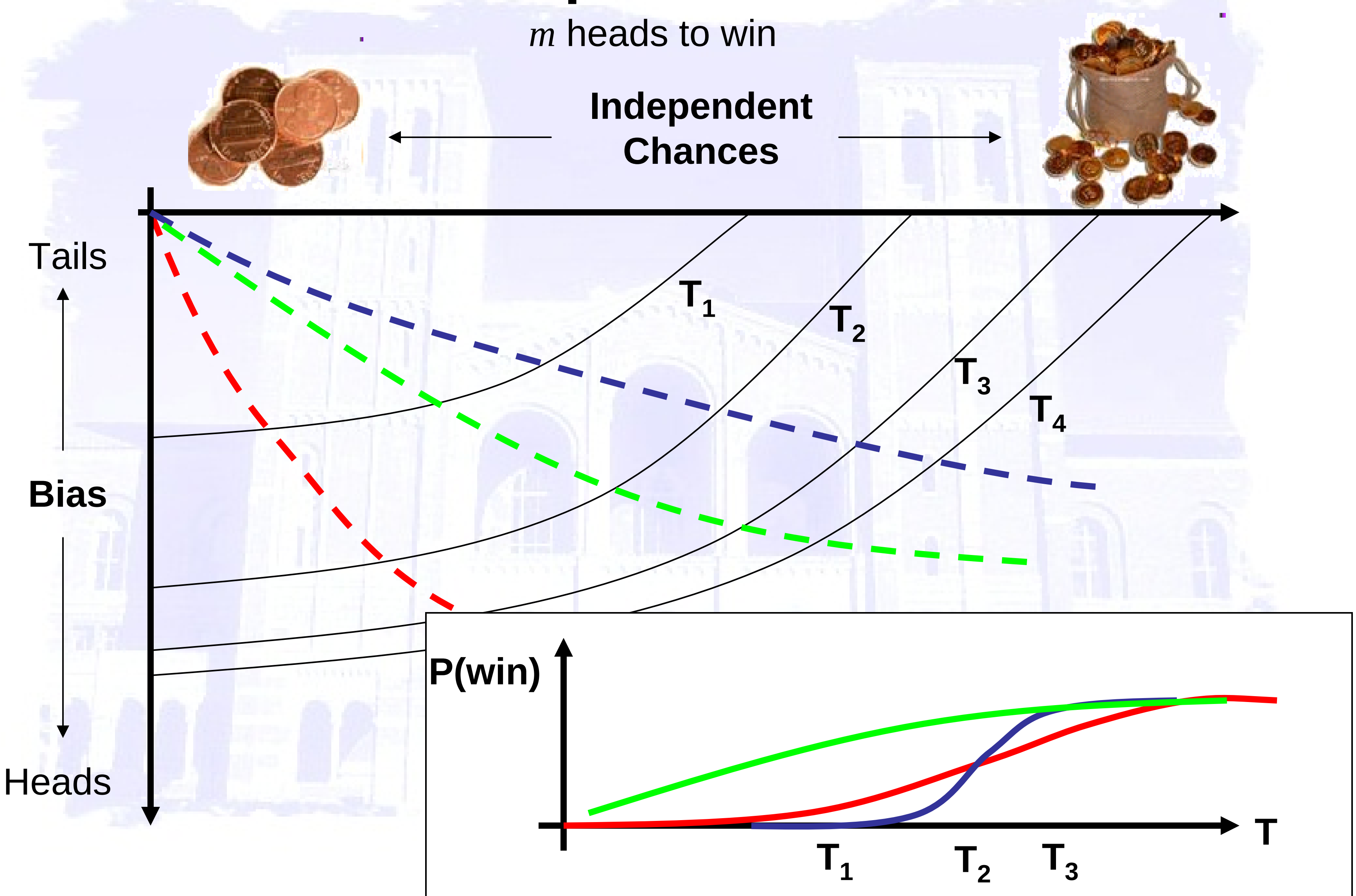
A Simple Game



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Motivation

- Rateless Codes
  - Packet level erasure codes
  - Developed for wired networks
  - Adopted in 3GPP
- Wireless vs. Wired
  - Control over probability of erased packets
  - Significant interaction between physical- and application-layer codes
  - A fundamental tradeoff

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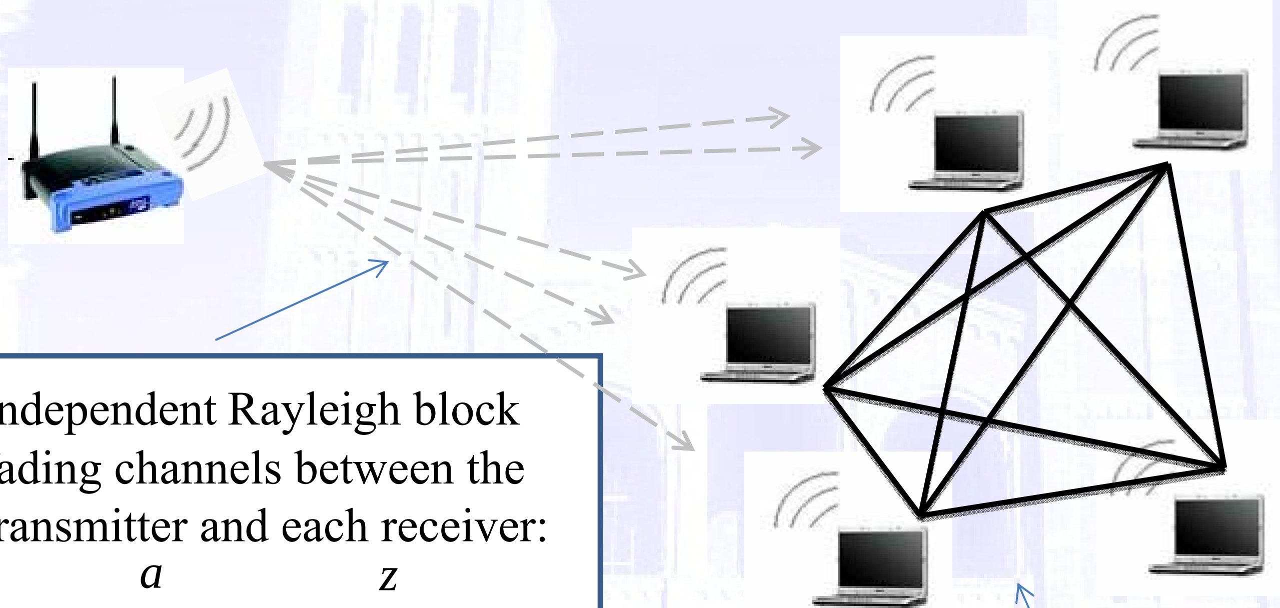
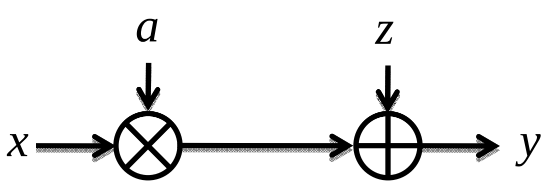
Physical System Model



Independent Rayleigh block fading channels between the transmitter and each receiver:



$z \sim N(0, \sigma^2)$

$a \sim \text{Rayleigh}(\alpha)$

F independent fades per packet

N Receivers can exchange decoded packets freely (N-fold selection diversity).

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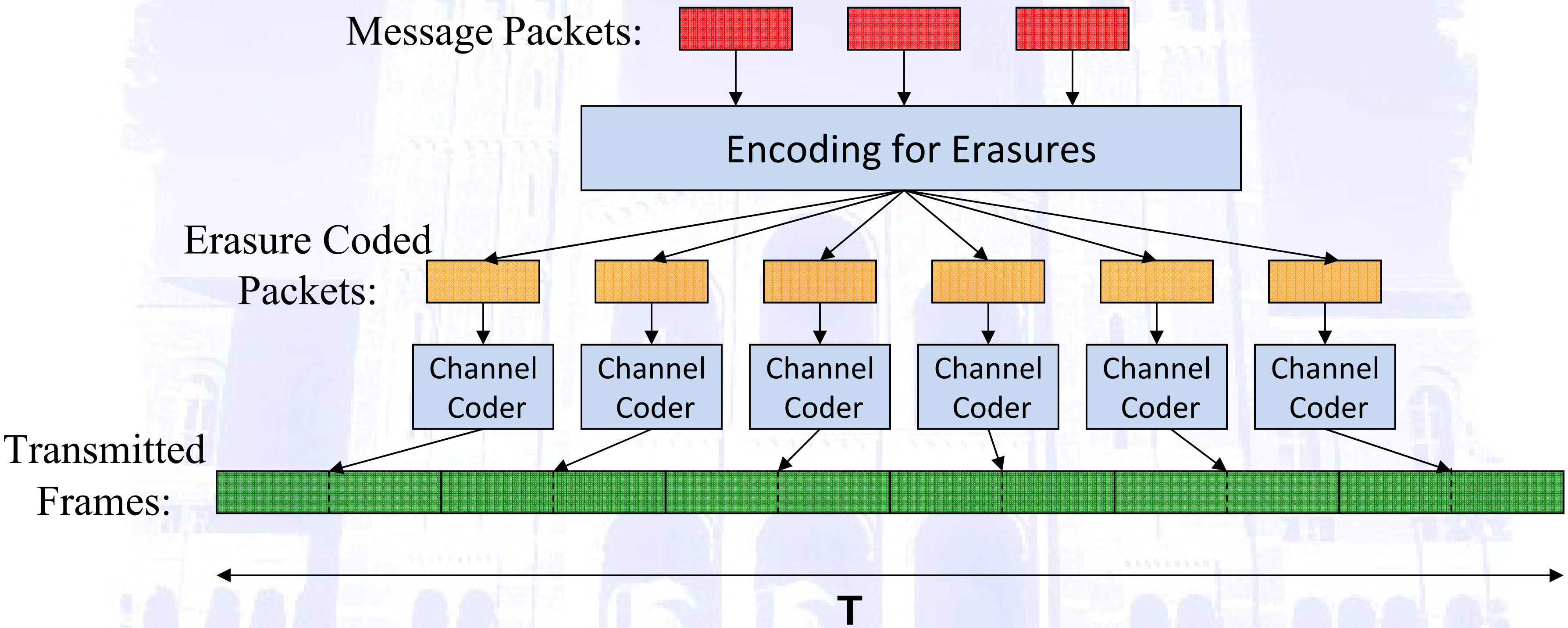
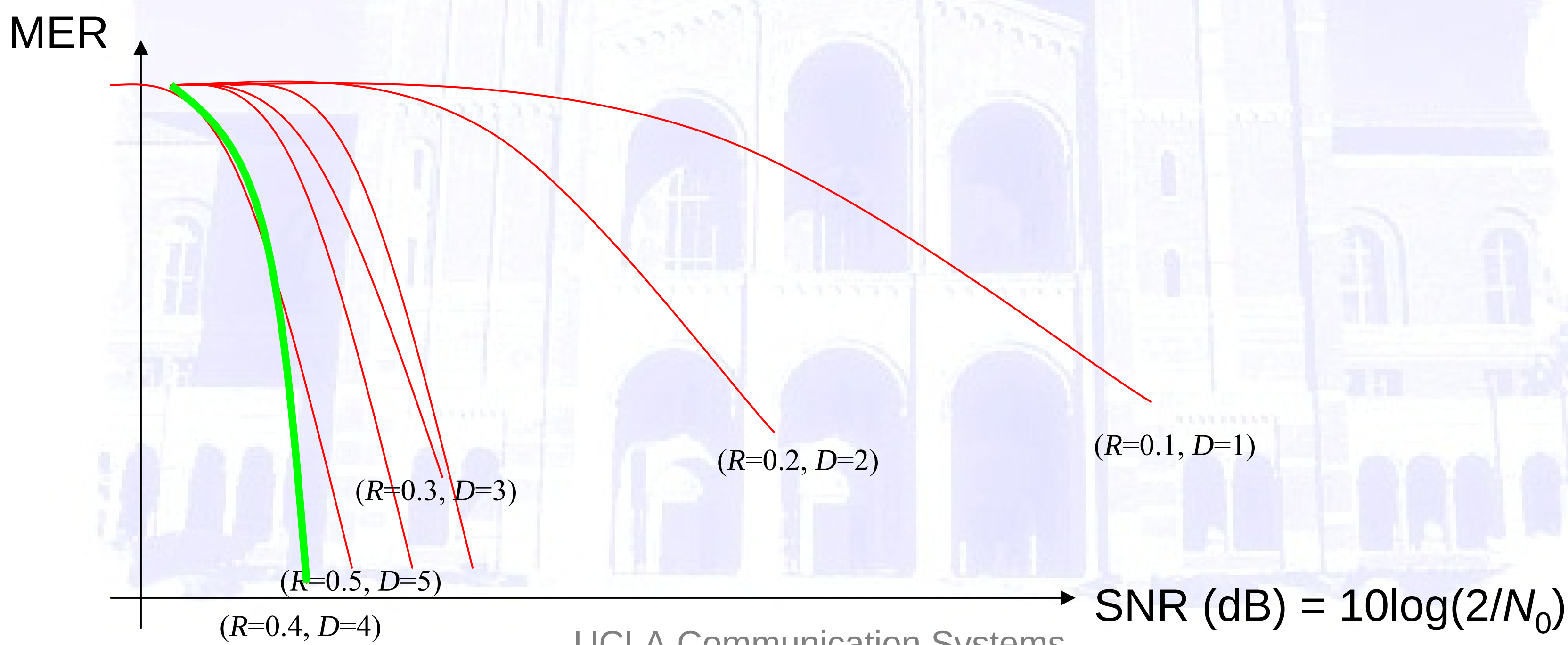
The Cross-Layer Architecture



The Choice of a Transmission Strategy

- For a fixed T, each tradeoff of R and D corresponds to a transmission strategy
- The choice of a transmission strategy significantly affects performance



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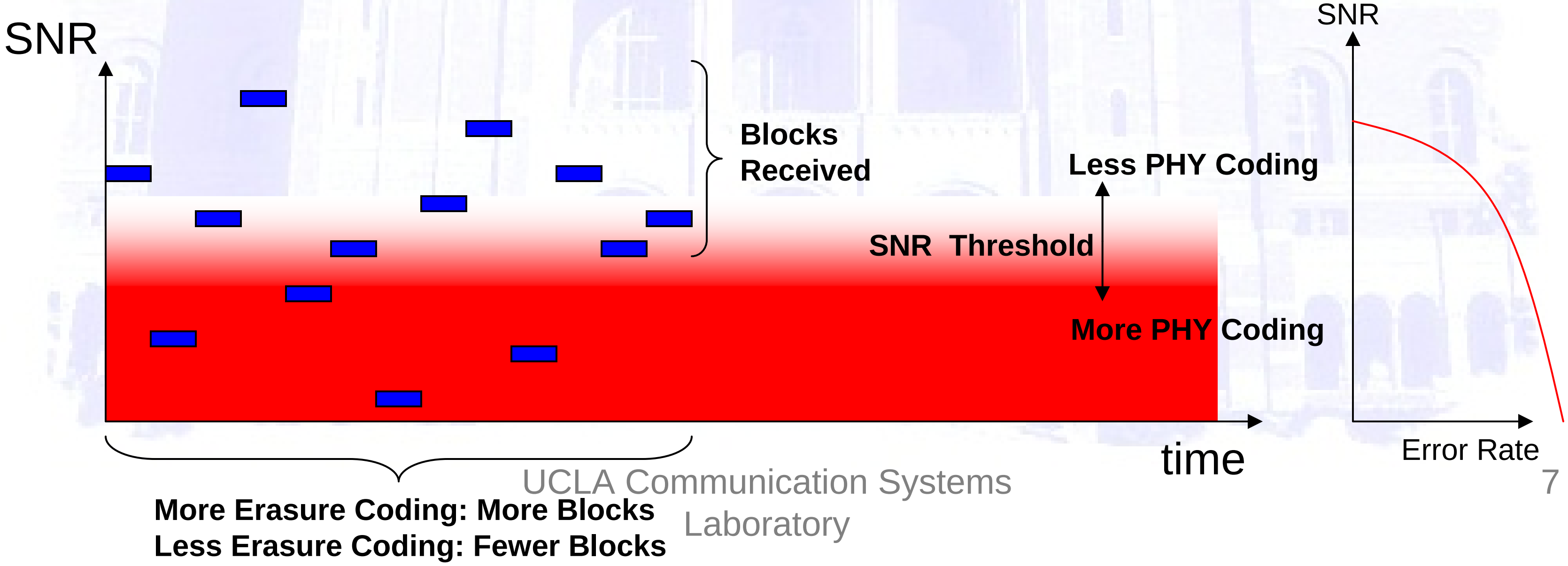
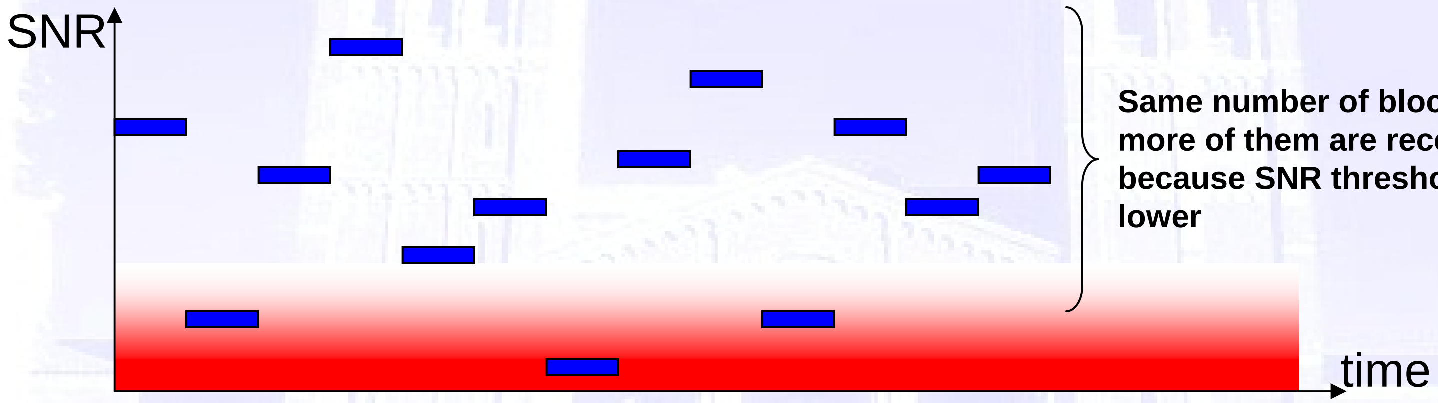
Back to the Game Interpretation

- PHY Coding
  - Shifts "Block Error" SNR Threshold
- Erasure Coding
  - Changes number of transmitted packets
- Must receive a minimum number of packets to recover message
  - What is the best tradeoff?



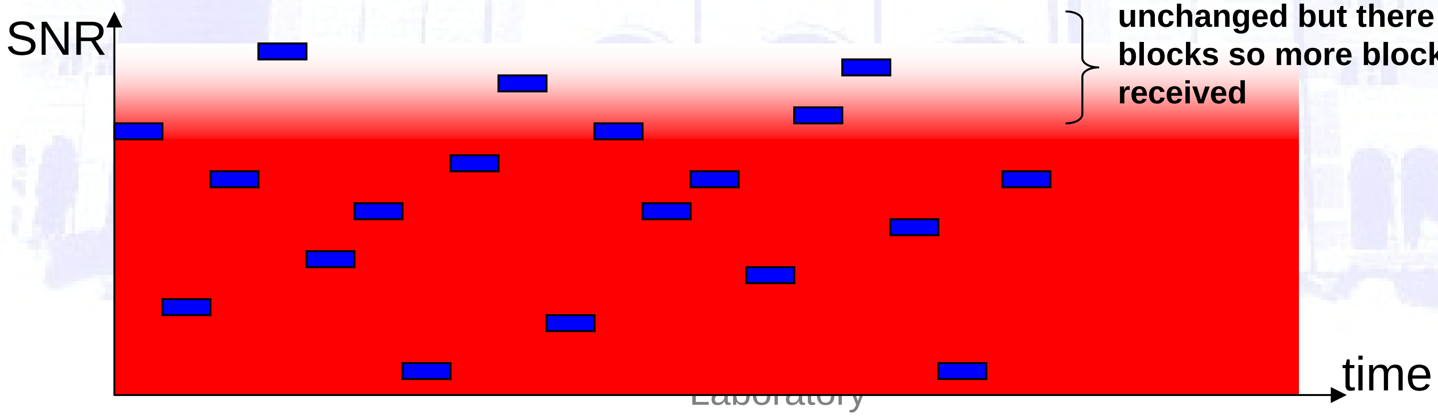
Two Extremes

- Mostly PHY Coding



Same number of blocks but more of them are received because SNR threshold is lower

- Mostly Erasure Coding



SNR threshold stays unchanged but there are more blocks so more blocks are received

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A Nonconvex Optimization Problem

- We develop and solve the following optimization problem:

$$\text{minimize} \quad \lambda^{-1} \qquad \text{Operational SNR}$$

$$\text{Subject to:} \quad \sum_{i=0}^{\hat{m}-1} \binom{RT}{i} \left(1 - p_e^N\right)^i \left(p_e^N\right)^{RT-i} \le k_1 \qquad \text{MER constraint}$$

$$p_e = \Pr\left[cFR > \sum_{i=1}^{F} \log\left(1 + \gamma_i\right) \right] \qquad \text{Channel effects}$$

$$\gamma_i \sim \text{Exponential}\left(\lambda\right)$$

$$\lambda = \sigma^2 / 2\alpha^2$$

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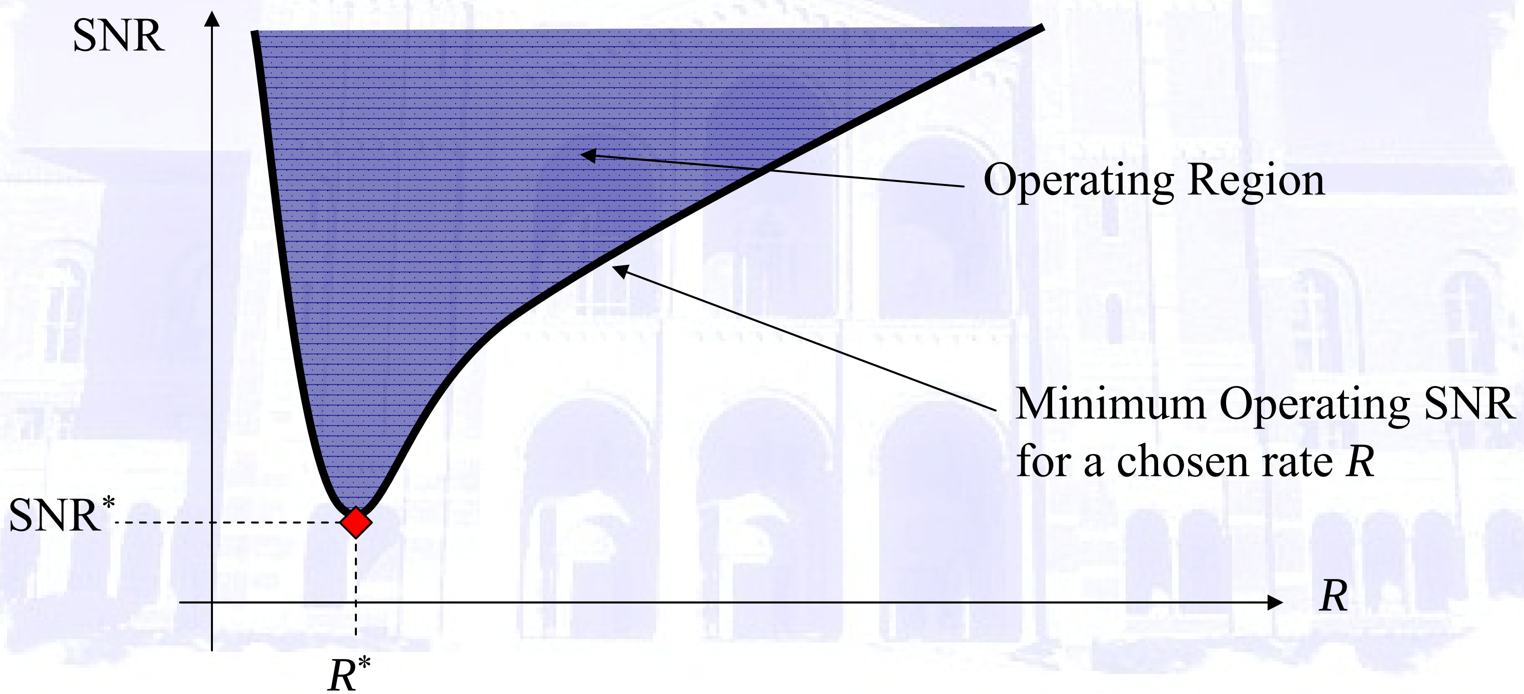
The Optimal R*

- Solving the optimization problem yields a feasible operating region.
- We choose the transmission strategy giving the widest operating range.



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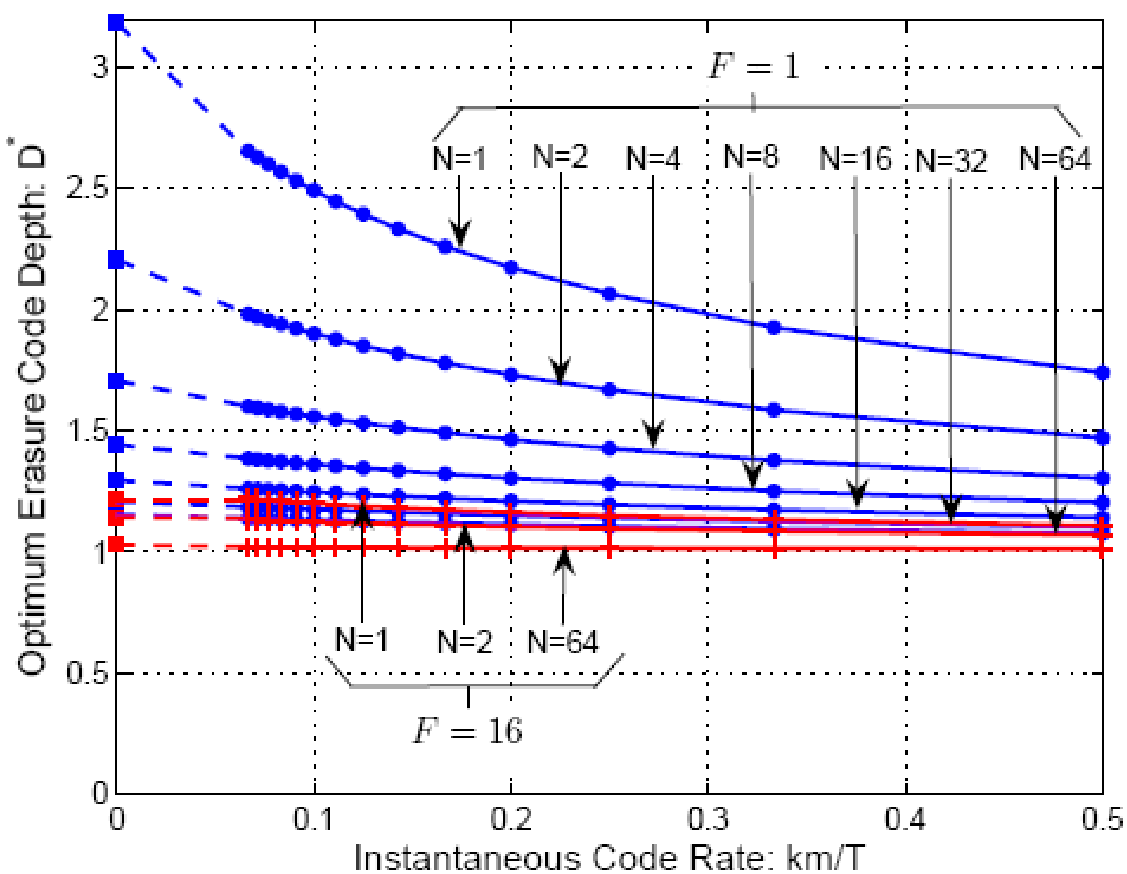
Tractability of the Problem

- Extremely difficult in general form
- Break the problem into three scenarios
  - Slow Fading
  - Fast Fading
  - Low average SNR
- Each case allows for a different simplifying assumption.
- Putting the results together gives a picture that is almost complete.
  - "Guess" what happens in between

Main Results

- Evolution of the optimal transmission strategy



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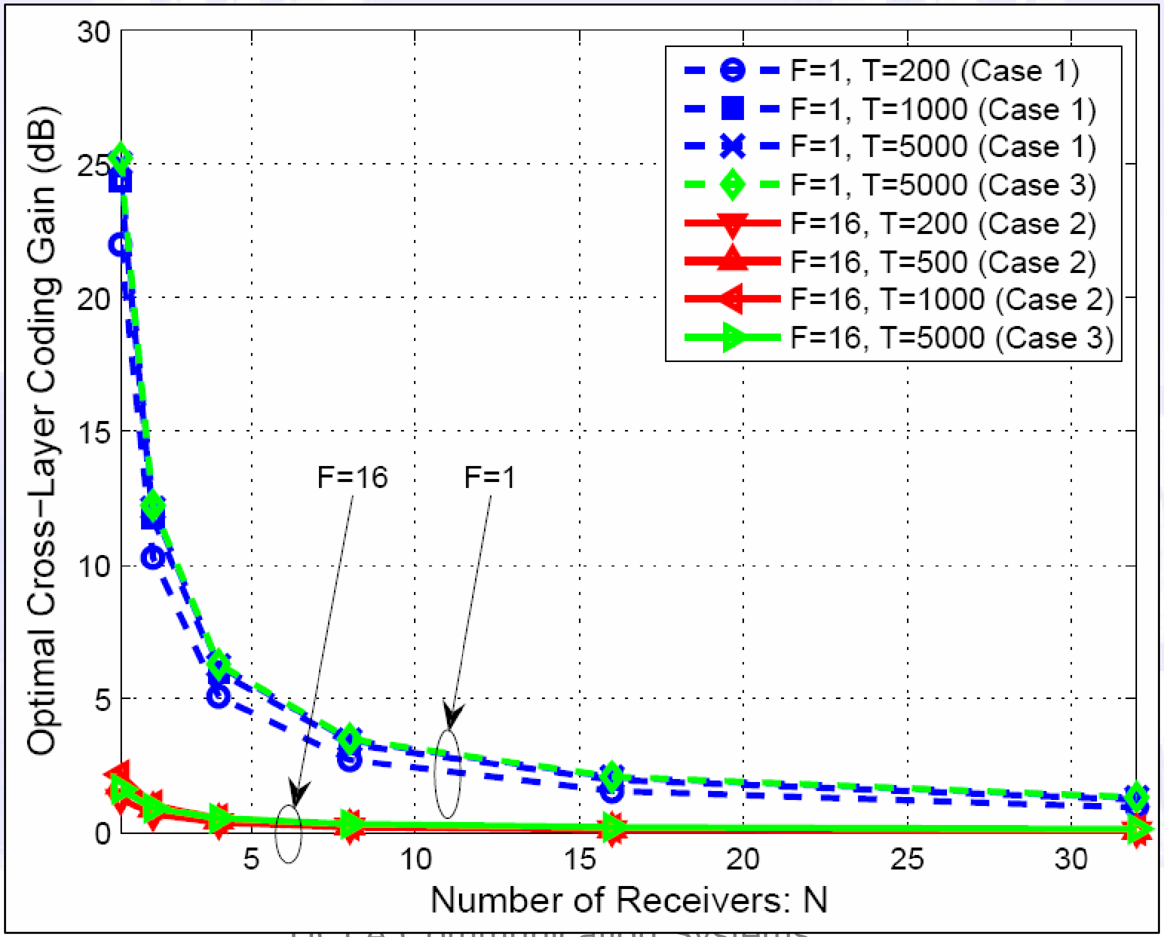
Main Results

- Cross-Layer Coding Gain



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Notes

- Our paper, “A Cross-Layer Perspective on Rateless Coding for Wireless Channels”, can be found online at:
  - http://www.ee.ucla.edu/~csl/files/publications/Courtade_ICC2009.pdf
- This research was supported by Rockwell Collins through contract #4502769987.

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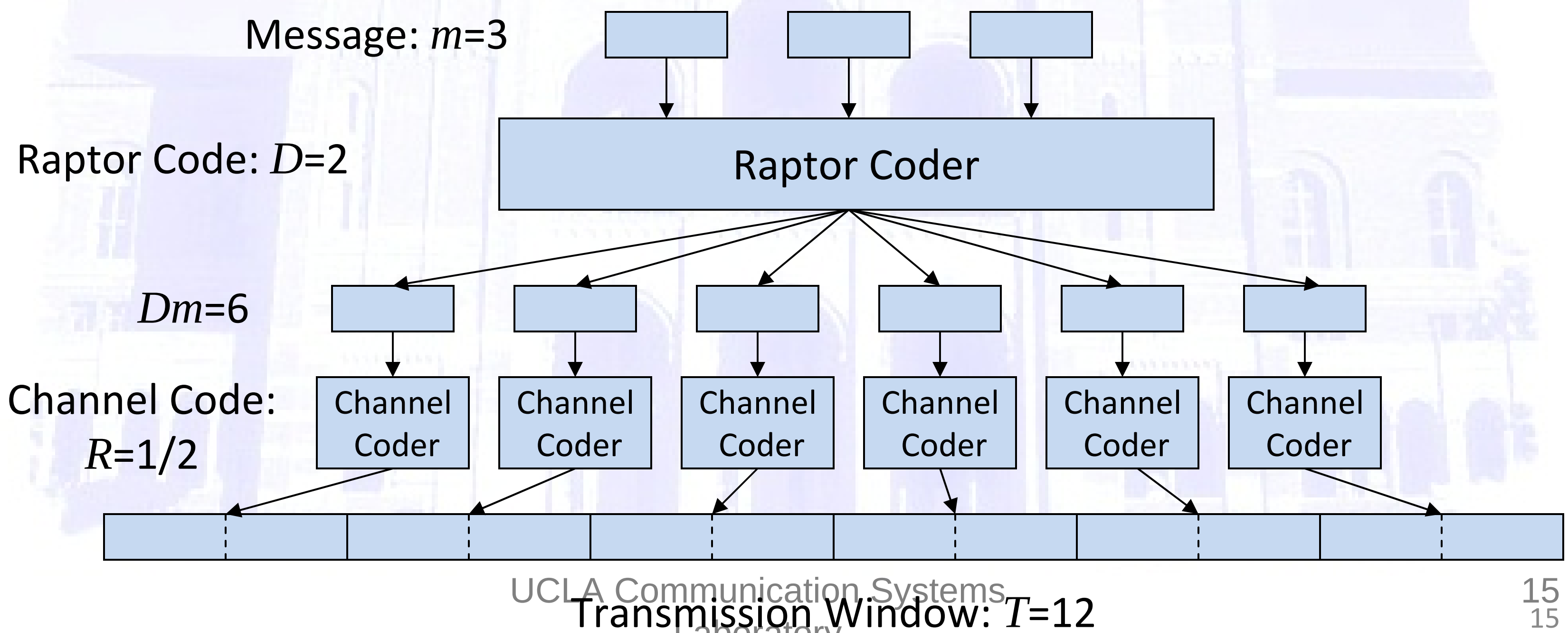
Transmission Model

In order to study the tradeoff between the packet-level and physical layer coding, we fix the amount of time required to communicate a message of m packets. For convenience, we will measure our available transmission time in packets and we will assume that we have T units of time in which to communicate our message.

This defines a tradeoff between our Raptor code depth D and our channel code rate R. Specifically $Dm = TR$. For example:



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Probability of Message Error

Under the assumption of channel independence, the probability that a packet is not received by a network of *N* nodes is:

$$\Pr\left[\text{Network doesn't receive packet}\right] = p_e^N$$

Given a network level erasure code (we'll call it a Raptor code) that can recover a message consisting of *m* packets from $\hat{m} = (1+\delta)m$ packets, the message error rate can be computed as:

$$MER = \sum_{i=0}^{\hat{m}-1} \binom{Dm}{i} \left(1 - p_e^N\right)^i \left(p_e^N\right)^{Dm-i}$$

Where *D* is the Raptor code "depth". Specifically, a message of *m* packets is encoded into *Dm* packets.

Capacity of the Block Fading Channel

We can view the channel model as a set of parallel Gaussian channels, each with an independent SNR drawn from an exponential distribution. Hence, for reliable communication we must satisfy:

$$(1+\varepsilon)R = (1+\varepsilon)\frac{1}{F}\sum_{i=1}^{F} R_i < \frac{1}{F}\sum_{i=1}^{F}\frac{1}{2}\log\left(1+\gamma_i\right)$$

Where: $\gamma_i \sim \exp(\lambda), \quad \lambda = \dfrac{\sigma^2}{2\alpha^2}$

Our probability of erasure, p_e, can then be expressed as:

$$p_e = \Pr\left[F(1+\varepsilon)R > \sum_{i=1}^{F}\frac{1}{2}\log\left(1+SNR_i\right)\right]$$

$$= \Pr\left[cFR > \sum_{i=1}^{F}\log\left(1+SNR_i\right)\right]$$

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The Optimization Problem

We say a communication strategy is optimal if it allows reliable communication at the lowest possible average SNR. Therefore, we want to minimize the operational SNR (or equivalently maximize λ) with respect to our rate *R*. This gives the following optimization problem:

Maximize λ

Subject to: $$\sum_{i=0}^{\hat{m}-1} \binom{RT}{i} \left(1 - p_e^N\right)^i \left(p_e^N\right)^{RT-i} \le k_1$$

$$\Pr\left[cFR > \sum_{i=1}^{F} \log\left(1+\gamma_i\right) \right] = p_e$$

Essentially, we bound the MER at a specified operating point (k_1) and optimize over the physical layer code rate, *R*. There is no need to introduce *D*, because by our problem definition, *D* is a function of *R*.

This is a nonconvex optimization problem that includes combinatorial terms, so it is essentially unsolvable in its current form. Fortunately, we can make some simplifying observations.

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A Gaussian Approximation

If we define P_i to be the i^{th} packet transmitted and let $P_i = 1$ if the frame is received by a node in the network and 0 otherwise, then we can define a random variable P that is equal to the number of frames successfully received.

$$P = P_1 + P_2 + \ldots + P_{Dm}$$

Now, if m is relatively large, then we can invoke the Central Limit Theorem and approximate P by a Gaussian random variable with:

$$\mu(P) = RT(1 - p_e^N)$$

$$\mathrm{var}(P) = RTp_e^N(1 - p_e^N)$$

This allows us to rewrite the constraint on the MER as:

$$\hat{m} - RT\left(1 - p_e^N\right) + k_2\sqrt{RTp_e^N\left(1 - p_e^N\right)} \le 0$$

Where $k_2 = Q^{-1}(k_1)$.

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A Change of Variables

We were able to eliminate the combinatorial terms through the Gaussian approximation, however we are still left with a nonconvex problem. Fortunately, there are not many variables, so we can formulate a geometrical interpretation.

If we introduce the change of variables:

$$X = \sqrt{RT\left(1 - p_e^N\right)}$$

$$Y = \sqrt{p_e^N}$$

Then the constraint becomes:

$$\hat{m} - X^2 + k_2 XY \le 0$$

$$RT\left(1 - Y^2\right) = X^2$$

Or equivalently:

$$Y \le \frac{1}{k_2}\left(X - \frac{\hat{m}}{X}\right)$$ A constraint-induced region

$$\frac{1}{RT}X^2 + Y^2 = 1$$ An ellipse

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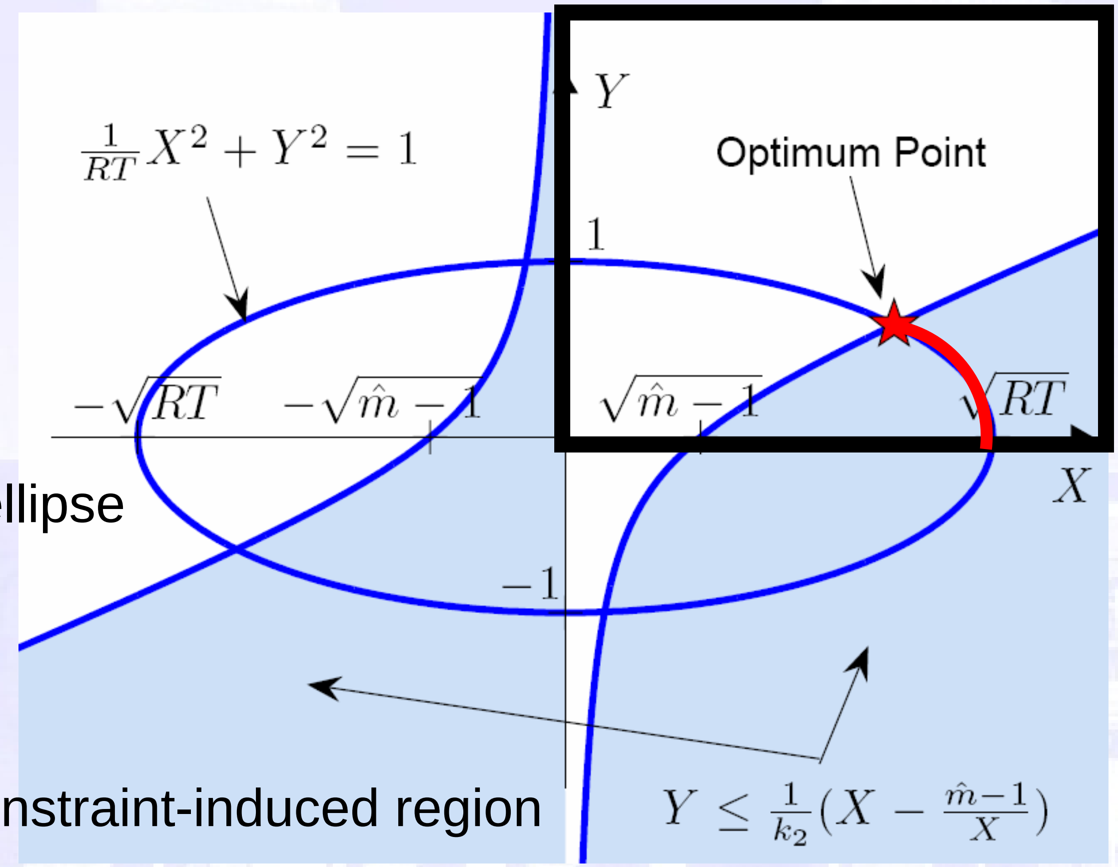
A Geometric Interpretation



An ellipse

A constraint-induced region

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The New Optimization Problem

We have reduced the optimization problem to the following:

Maximize λ

Subject to: $$\Pr\left[cFR > \sum_{i=1}^{F} \log(1+\gamma_i)\right] = p_e^*(R)$$

However, it is quite difficult to solve the above problem because for each *R*, we need to find a λ that achieves equality in the constraint. Then, we need to find the maximum of these λ's. Since the distribution in the constraint has no closed form expression, we need to use various approximations which allow us to solve cases of interest by using standard methods. These cases are:

- Single Fade per Packet
- Many Fades per Packet
- Low Average SNR Scenario.

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Case 1: Single Fade per Packet

In this case, the distribution has a closed form, allowing us to solve the problem easily:

$$p_e^*(R) = \Pr(e^{cR} - 1 > \gamma)$$
$$= 1 - e^{-\lambda(\exp(cR)-1)}$$

$$\Rightarrow \lambda = -\frac{\log(1 - p_e^*(R))}{e^{cR} - 1}$$

We can take derivatives of λ with respect to *R* and solve the problem via standard methods.

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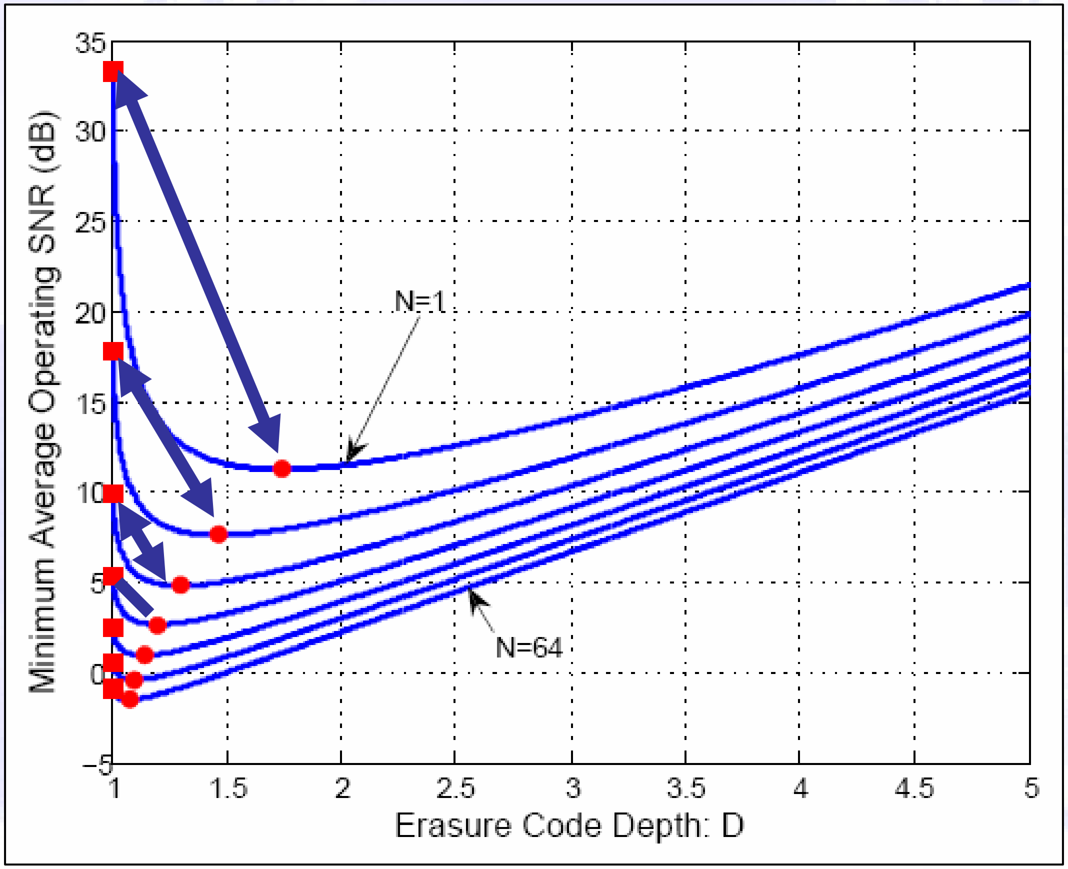
Numerical Results

•Some typical curves of minimum average operating SNR vs. *D* for several *N* for T=200 and m=100.

• The gains achievable by erasure coding diminish as the diversity available from multiple receivers grows.



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Case 2: Many Fades per Packet

In this case, we can approximate the channel capacity by a Gaussian random variable which allows the constraint to be formulated as follows.

$$\Phi^{-1}(p_e^*(R)) = \sqrt{F}\,\frac{cR - \mu(\lambda)}{\sqrt{\mathrm{var}(\lambda)}}$$

We can implicitly differentiate the above equation and solve the problem for the optimal λ via standard methods.

In the above formulation, the mean and variance as a function of λ are computed by:

$$\mu(\lambda) = Ei(\lambda)$$
$$\mathrm{var}(\lambda) = 2LEi(\lambda) - 2\log(\lambda)Ei(\lambda) - Ei^2(\lambda).$$

Where:

$$Ei(\lambda) = e^{\lambda}\int_{\lambda}^{\infty} \frac{1}{t} e^{-t}\,dt$$
$$LEi(\lambda) = e^{\lambda}\int_{\lambda}^{\infty} \frac{\log t}{t} e^{-t}\,dt.$$

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Case 3: Low SNR Scenario

In this case, we take advantage of the approximation $\log(1+x) \sim x$ for small x. Then, the CDF for Gamma random variable can be used, giving:

$$\lambda = \frac{F_{\Gamma}^{-1}(p_e^*(R) \mid F, 1)}{cFR}$$

We can differentiate the above equation and solve the problem for the optimal λ via standard methods.

In the above equation, the inverse of the Gamma CDF was used, where:

$$F_{\Gamma}^{-1}(p \mid F, 1) = \{x : F_{\Gamma}(x \mid F, 1) = p\}.$$

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Explicit Erasures

The p_e*(R) that was previously computed is the total erasure probability because it comes from the message error constraint. It is equal to $p+q-pq$, where p is the probability of error due to the channel and q is the explicit erasure probability. Rearranging terms gives the following optimization problem, which can be solved in the same manner as the previous problem:

Maximize λ

Subject to: $$\Pr\left[cFR > \sum_{i=1}^{F} \log\left(1+\gamma_i\right)\right] = \frac{p_e^*(R)+q}{1-q}$$

The addition of explicit erasures essentially adds a DC offset to the optimal D*. In particular, we require that D* is large enough to overcome the explicit erasures and the additional redundancy is balanced between erasure coding and physical layer coding as in the previous problem.

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