Encoding for Input-Symmetric Degraded Broadcast Channels

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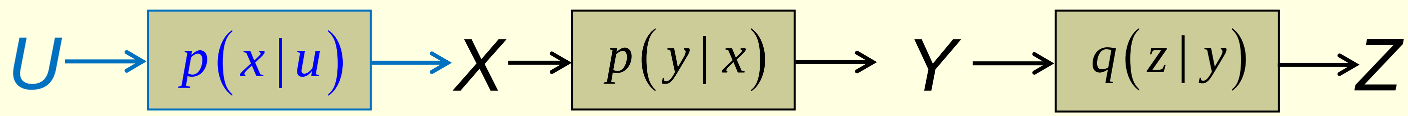
Outline

- The big question
- *F** function and the region $C^*_{\boldsymbol{q}}$
- Input-Symmetric DBCs
- Payoff: Permutation Encoding Approach

Degraded Broadcast Channels

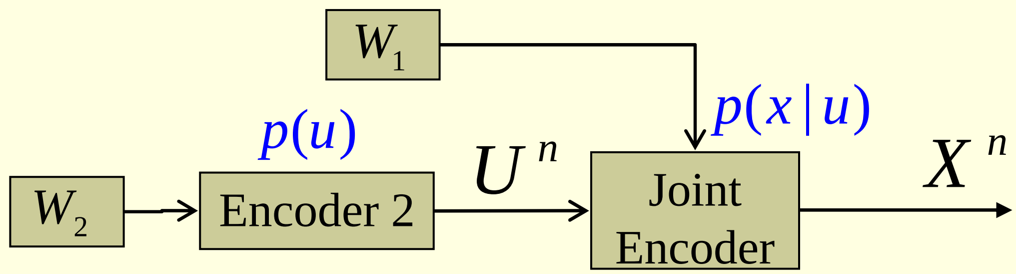
- Capacity Region [Cover72][Bergmans73][Gallager74]



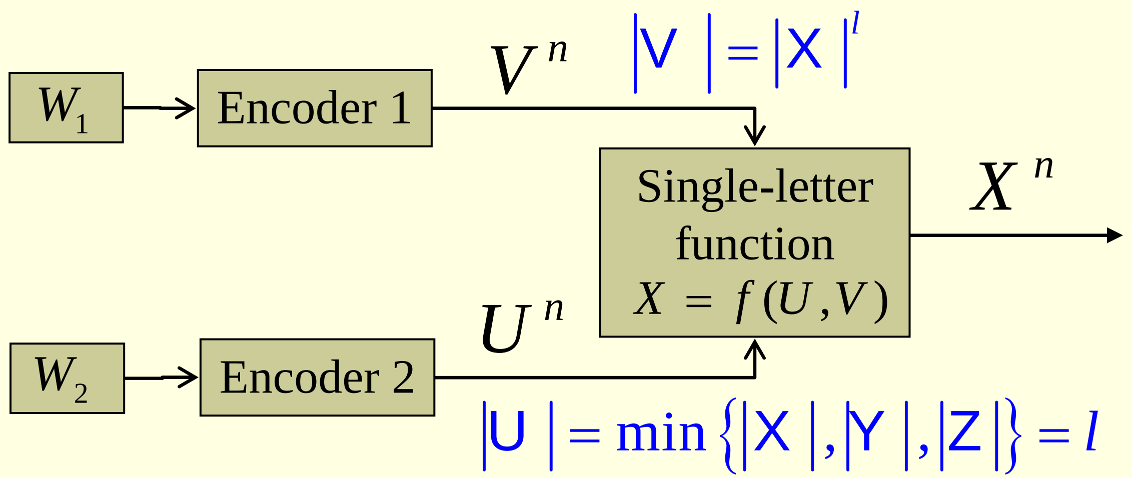
  - The capacity region is the convex hull of the closure of all rate pairs (R_1, R_2) satisfying
$$R_1 \leq I(X;Y|U),$$
$$R_2 \leq I(U;Z),$$
for some joint distribution $p(u)p(x|u)p(y,z|x)$, where the auxiliary random variable *U* has cardinality bounded by $|\mathcal{U}| \leq \min\{|\mathcal{X}|,|\mathcal{Y}|,|\mathcal{Z}|\}$.

Finding simple single-letter functions.

- Joint Encoding



- Independent Encoding



Known cases of simple single-letter functions

- Broadcast Gaussian channel [Bergmans74]
- Broadcast binary-symmetric channel [Wyner73] [Witsenhausen74]
- Broadcast Z channel [Xie08]
- Discrete additive degraded broadcast channels [Benzel79]
- Our approach is inspired by [Witsenhausen74] and [Witsenhausen & Wyner 75],which are also seminal to [Benzel79] and [Liu&Ulukus07].

Introduce s and $F^*(\boldsymbol{q},s)$ to optimize (R_1, R_2)

- Given an input distribution $X \sim \boldsymbol{q}$,

$$
\begin{aligned}
R_1 &\le I(X;Y|U) \\
&= H(Y|U) - H(Y|X,U) \\
&= H(Y|U) - H(Y|X) \\
&= s - H(Y|X),
\end{aligned}
$$

$$
\begin{aligned}
R_2 &\le I(U;Z) \\
&= H(Z) - H(Z|U) \\
&= H(Z) - F^*(\boldsymbol{q}, s).
\end{aligned}
$$

Defining F^* and C^*_q

- Definition of F^*

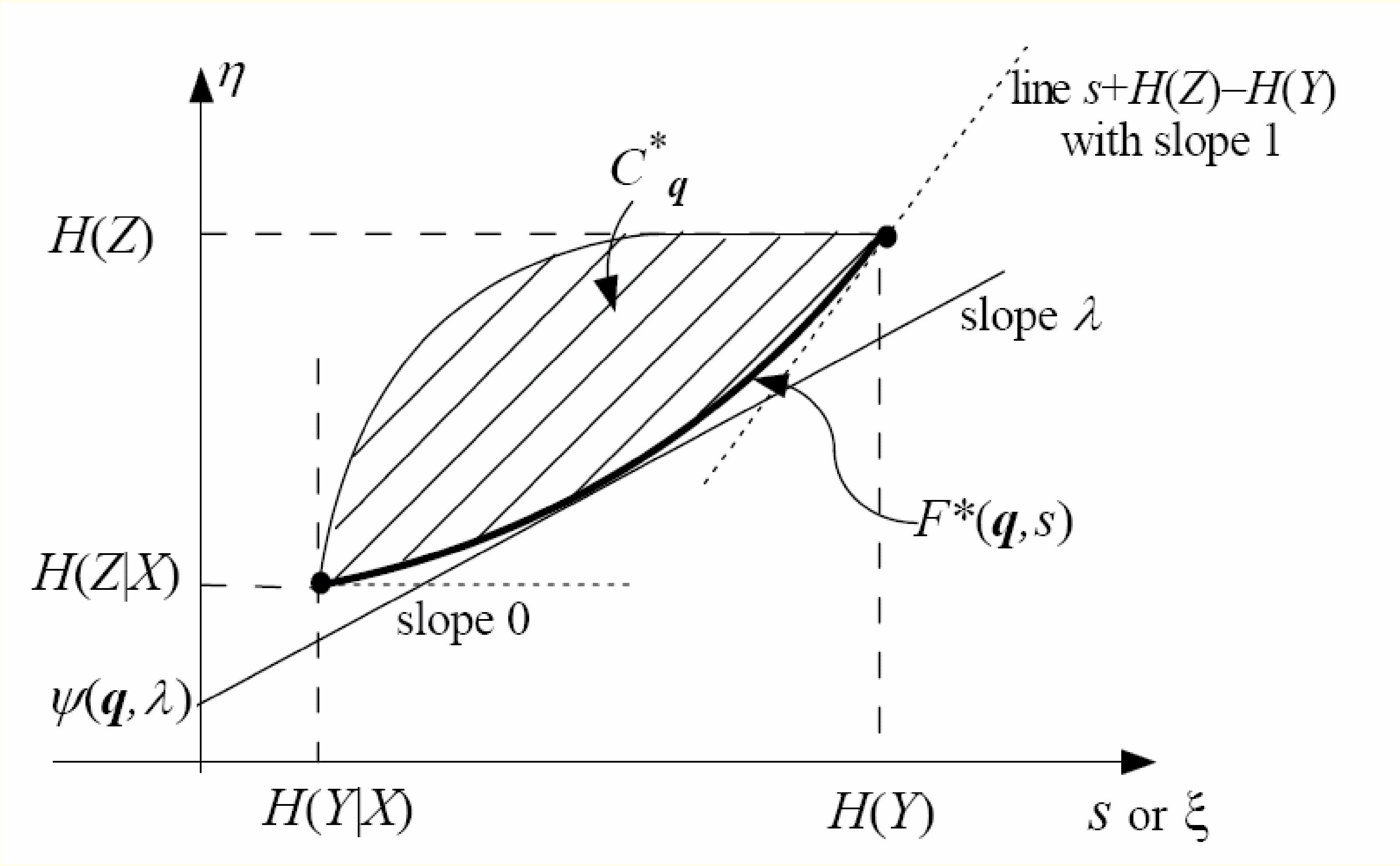
$$F^*(\boldsymbol{q}, s) = \min\ H(Z|U), \quad \text{s.t.}\ H(Y|U) = s, \quad X \sim \boldsymbol{q}.$$

- F^* is increasing in s for any fixed $\boldsymbol{q}$.
- F^* is jointly convex in $(\boldsymbol{q}, s)$.
- Definition of $C^*_{\boldsymbol{q}}$

$$C^*_{\boldsymbol{q}} = \left\{(s,\eta) \mid s = H(Y|U), \eta = H(Z|U), X \sim \boldsymbol{q}\right\}.$$

- $C^*_{\boldsymbol{q}}$ is a convex set.

F^* is the lower (optimal) boundary of C^*_q



Input Symmetry

- The *n*-input *m*-output channel $X \to Y$ with probability transition matrix T_{YX} is input-symmetric if

$$\Sigma_{T_{YX}} = \left\{ G \in \Phi_n \mid \exists H \in \Phi_m \text{ s.t. } T_{YX} G = H T_{YX} \right\}$$

is transitive, i.e., each element in {1,…,n} can be mapped to each element in {1,…,n} by some permutation matrices in $\Sigma_{T_{YX}}$. [Witsenhause&Wyner75]

Example of a G (Binary Erasure Channel)

- Binary-Erasure Channel

$$T_{YX} = \begin{bmatrix} 1-p_1 & 0 \\ p_1 & p_1 \\ 0 & 1-p_1 \end{bmatrix}$$

Example of a G (Binary Erasure Channel)

- Binary-Erasure Channel permuted on the right by G
- (G permutes the columns)

$$T_{YX}G = \begin{bmatrix} 1-p_1 & 0 \\ p_1 & p_1 \\ 0 & 1-p_1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{G}$$

1 2 3

$$= \begin{bmatrix} 0 & 1-p_1 \\ p_1 & p_1 \\ 1-p_1 & 0 \end{bmatrix}$$

Example of a G (Binary Erasure Channel)

- Binary-Erasure Channel permuted on the left by H

$$HT_{YX} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{H} \begin{bmatrix} 1-p_1 & 0 \\ p_1 & p_1 \\ 0 & 1-p_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1-p_1 \\ p_1 & p_1 \\ 1-p_1 & 0 \end{bmatrix}$$

$$= T_{YX} G$$

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- For the degraded broadcast channel $X \to Y \to Z$ with probability transition matrices T_{YX} and T_{ZX} we define the DBC to be input-symmetric if

$$\Sigma_{T_{YX}, T_{ZX}} = \Sigma_{T_{YX}} \cap \Sigma_{T_{ZX}}$$

is transitive.

Examples

- Broadcast Binary-Symmetric Channel

$$T_{YX} = \begin{bmatrix} 1-p_1 & p_1 \\ p_1 & 1-p_1 \end{bmatrix} \qquad T_{ZX} = \begin{bmatrix} 1-p_2 & p_2 \\ p_2 & 1-p_2 \end{bmatrix}$$

$$\Sigma_{T_{YX},T_{ZX}} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \text{ is transitive.}$$

Examples

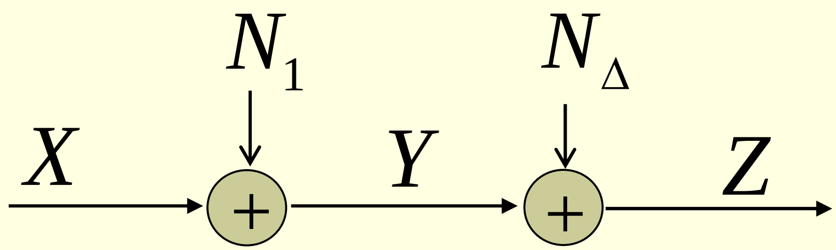
- Broadcast Binary-Erasure Channel

$$T_{YX} = \begin{bmatrix} 1-p_1 & 0 \\ p_1 & p_1 \\ 0 & 1-p_1 \end{bmatrix} \quad T_{ZX} = \begin{bmatrix} 1-p_2 & 0 \\ p_2 & p_2 \\ 0 & 1-p_2 \end{bmatrix}$$

$$\Sigma_{T_{YX},T_{ZX}} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \text{ is transitive.}$$

Examples

- Group-Additive Degraded Broadcast Channel



$$N_1 \sim (\alpha_1, \cdots \alpha_n), \quad N_2 \sim N_1 \oplus N_\Delta \sim (\beta_1, \cdots, \beta_n)$$

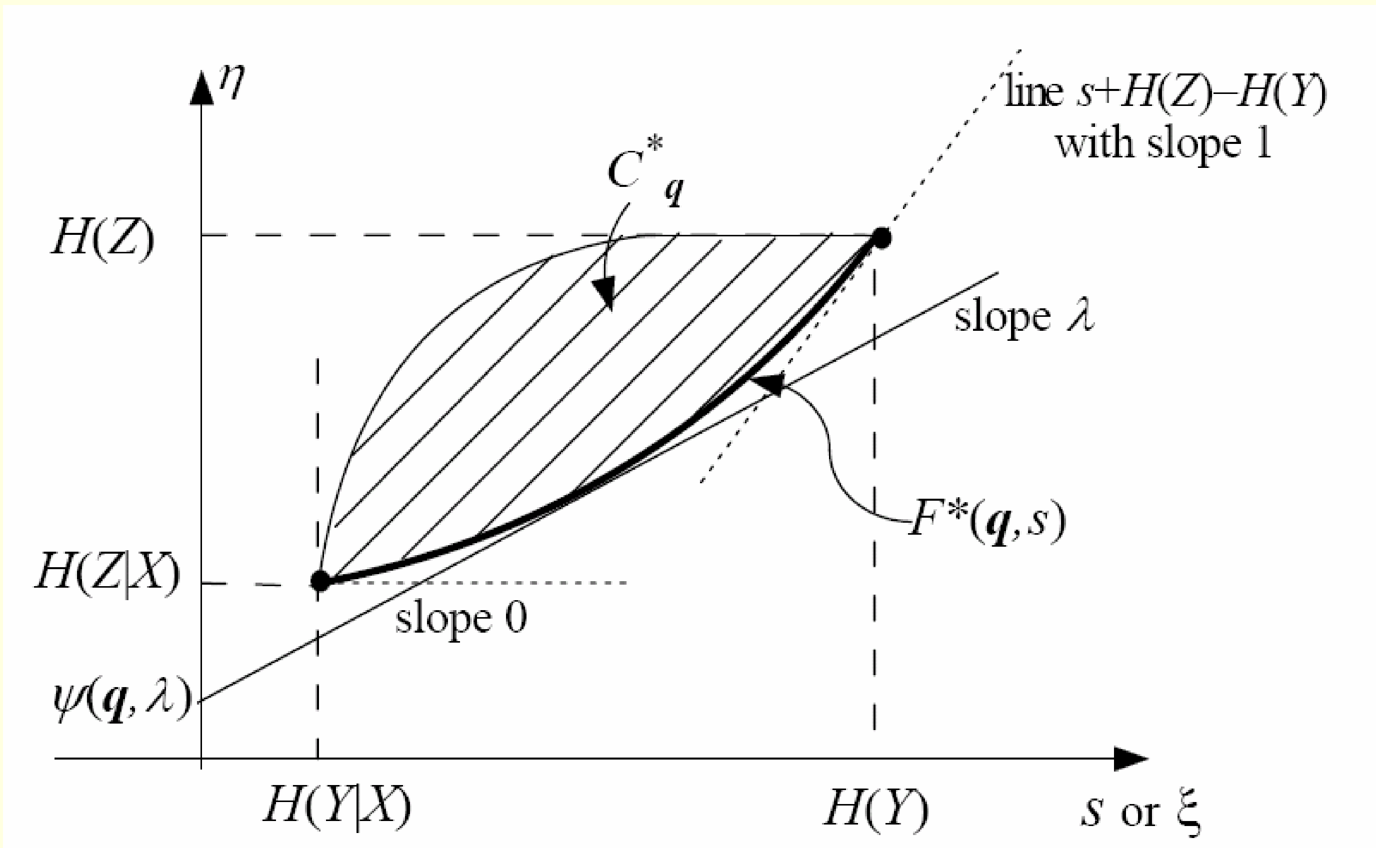
$$G_x(i,j) = \begin{cases} 1 & \text{if } j \oplus x = i \\ 0 & \text{otherwise} \end{cases} \quad \text{for } x, i, j = 1, \cdots, n.$$

$$T_{YX} = \sum_{x=1}^{n} \alpha_x G_x \qquad T_{ZX} = \sum_{x=1}^{n} \beta_x G_x$$

$$\Sigma_{T_{YX}, T_{ZX}} \supseteq \{G_1, \cdots, G_n\} \quad \text{is transitive.}$$

Key Result: C^*_q ™ C^*_u

- For an input-symmetric DBC, a uniform distribution on X can achieve the entire capacity region. A single C^*_q tells all.



Permutation Encoding Approach

- A *k*-input, *n*,*m*-output input-symmetric degraded broadcast channel $X \to Y \to Z$ with transitive set

$$\Sigma_{T_{YX},T_{ZX}} = \{G_1, \ldots, G_l\}.$$

- Permutation function $g(u,v)$

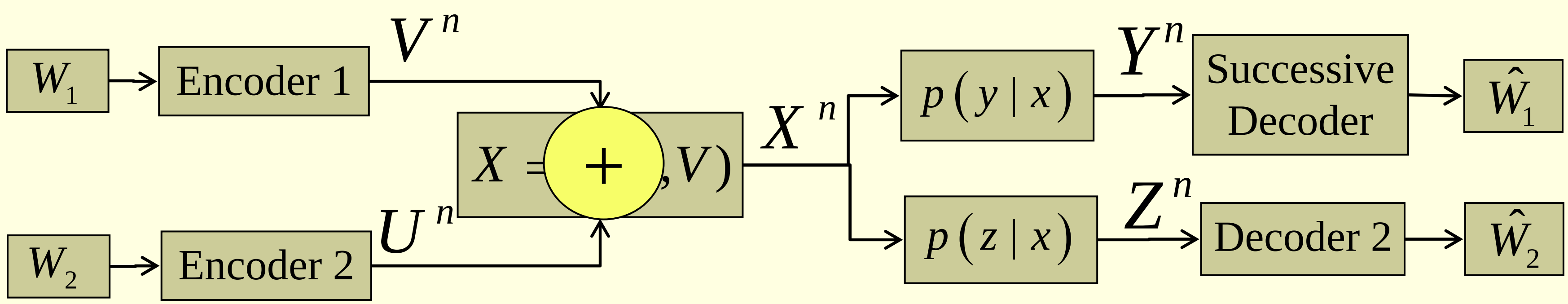
$$g(u,v) = x \text{ iff } G_u(v,x) = 1$$

$$\text{for } u = 1, \ldots, l, \text{ and } v, x = 1, \ldots, k.$$

- The permutation function $g(u,v) = x$ if the permutation matrix G_u maps the v^{th} column to the x^{th} column.

Permutation Encoding Approach

- System Structure



- l-ary R.V. U is uniformly distributed.
- V has distribution $\boldsymbol{p} \in \Delta_k$.
- The achievable region is

$$\overline{\text{co}}\left[\bigcup_{\boldsymbol{p} \in \Delta_k} \left\{ (R_1, R_2) : \begin{array}{l} R_1 \le h_n(T_{YX}\,\boldsymbol{p}) - h_n(T_{YX}\,\boldsymbol{e}_1) \\ R_2 \le h_m(T_{ZX}\,\boldsymbol{u}) - h_m(T_{ZX}\,\boldsymbol{p}) \end{array} \right\} \right].$$

Permutation Encoding Approach

- It achieves the boundary of the capacity region for input-symmetric degraded broadcast channels.
- The achievable region of permutation encoding approach is the capacity region.
- Thus, the capacity region is

$$\overline{\mathrm{co}}\left[\bigcup_{\boldsymbol{p}\in\Delta_k}\left\{(R_1,R_2): \begin{array}{l} R_1 \le h_n(T_{YX}\,\boldsymbol{p}) - h_n(T_{YX}\,\boldsymbol{e}_1) \\ R_2 \le h_m(T_{ZX}\,\boldsymbol{u}) - h_m(T_{ZX}\,\boldsymbol{p}) \end{array}\right\}\right].$$

The rest of the story…

- Optimal simple encoding scheme for discrete OR degraded broadcast channels. [Trans IT September 2008]
- Optimal simple encoding scheme for discrete multiplicative degraded broadcast channels. [accepted ISIT 2009]
- Optimal simple encoding schemes for several classes of degraded broadcast channels. [almost submitted to IEEE Trans. Inform. Theory]

Thank you.

Proof of Converse

- Represent the capacity region with *F**

$$\overline{\text{co}}\left[\bigcup_{p(u)p(x|u)}\left\{(R_1,R_2): \begin{array}{c} R_1 \le I(X;Y|U) \\ R_2 \le I(U;Z) \end{array}\right\}\right]$$

$$=\overline{\text{co}}\left[\bigcup_{p(u)p(x|u)}\left\{(R_1,R_2): \begin{array}{c} R_1 \le H(Y|U)-H(Y|X) \\ R_2 \le H(Z)-H(Z|U) \end{array}\right\}\right]$$

$$=\overline{\text{co}}\left[\bigcup_{X\sim \boldsymbol{q}}\left\{\bigcup_{\substack{p(u)p(x|u) \\ \text{with } X\sim \boldsymbol{q}}}\left\{(R_1,R_2): \begin{array}{c} R_1 \le H(Y|U)-H(Y|X) \\ R_2 \le H(Z)-H(Z|U) \end{array}\right\}\right\}\right]$$

$$=\overline{\text{co}}\left[\bigcup_{X\sim \boldsymbol{q}}\left\{\bigcup_{H(Y|X)\le s\le H(Y)}\left\{(R_1,R_2): \begin{array}{c} R_1 \le s-H(Y|X) \\ R_2 \le H(Z)-F^*(\boldsymbol{q},s) \end{array}\right\}\right\}\right].$$

Proof of Converse

- Uniformly distributed *X* is optimal.

$$\overline{\text{co}}\left[\bigcup_{X\sim\boldsymbol{q}}\left\{\bigcup_{s}\left\{(R_1,R_2): \begin{array}{c} R_1 \le s - H(Y\,|\,X) \\ R_2 \le H(Z) - F^*(\boldsymbol{q},s) \end{array}\right\}\right\}\right]$$

$$\subseteq \overline{\text{co}}\left[\bigcup_{X\sim\boldsymbol{q}}\left\{\bigcup_{s}\left\{(R_1,R_2): \begin{array}{c} R_1 \le s - h_n(T_{YX}\boldsymbol{e}_1) \\ R_2 \le h_m(T_{ZX}\boldsymbol{u}) - F^*(\boldsymbol{u},s) \end{array}\right\}\right\}\right]$$

$$= \overline{\text{co}}\left[\bigcup_{s}\left\{(R_1,R_2): \begin{array}{c} R_1 \le s - h_n(T_{YX}\boldsymbol{e}_1) \\ R_2 \le h_m(T_{ZX}\boldsymbol{u}) - F^*(\boldsymbol{u},s) \end{array}\right\}\right]$$

$$= \overline{\text{co}}\left[\bigcup_{\substack{X\sim\boldsymbol{q}\\ \boldsymbol{q}=\boldsymbol{u}}}\left\{\bigcup_{s}\left\{(R_1,R_2): \begin{array}{c} R_1 \le s - H(Y\,|\,X) \\ R_2 \le H(Z) - F^*(\boldsymbol{q},s) \end{array}\right\}\right\}\right]$$

$$\subseteq \overline{\text{co}}\left[\bigcup_{X\sim\boldsymbol{q}}\left\{\bigcup_{s}\left\{(R_1,R_2): \begin{array}{c} R_1 \le s - H(Y\,|\,X) \\ R_2 \le H(Z) - F^*(\boldsymbol{q},s) \end{array}\right\}\right\}\right].$$

Proof of Converse

- Define $\tilde{F}(s) = \inf h_m(T_{ZX}\,\boldsymbol{p}),$

  $$\text{s.t.}\, h_n(T_{YX}\,\boldsymbol{p}) = s.$$

- We can show $\underline{\text{env}}\tilde{F}(s) \le F^*(\boldsymbol{u}, s).$
- The converse is proved by

$$\overline{\text{co}}\left[\bigcup_{\boldsymbol{p}\in\Delta_k}\left\{(R_1,R_2): \begin{matrix} R_1 \le h_n(T_{YX}\,\boldsymbol{p}) - h_n(T_{YX}\,\boldsymbol{e}_1) \\ R_2 \le h_m(T_{ZX}\,\boldsymbol{u}) - h_m(T_{ZX}\,\boldsymbol{p}) \end{matrix}\right\}\right]$$

$$= \overline{\text{co}}\left[\bigcup_{s}\left\{(R_1,R_2): \begin{matrix} R_1 \le s - h_n(T_{YX}\,\boldsymbol{e}_1) \\ R_2 \le h_m(T_{ZX}\,\boldsymbol{u}) - \underline{\text{env}}\tilde{F}(s) \end{matrix}\right\}\right]$$

$$\supseteq \overline{\text{co}}\left[\bigcup_{s}\left\{(R_1,R_2): \begin{matrix} R_1 \le s - h_n(T_{YX}\,\boldsymbol{e}_1) \\ R_2 \le h_m(T_{ZX}\,\boldsymbol{u}) - F^*(\boldsymbol{u},s) \end{matrix}\right\}\right].$$

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