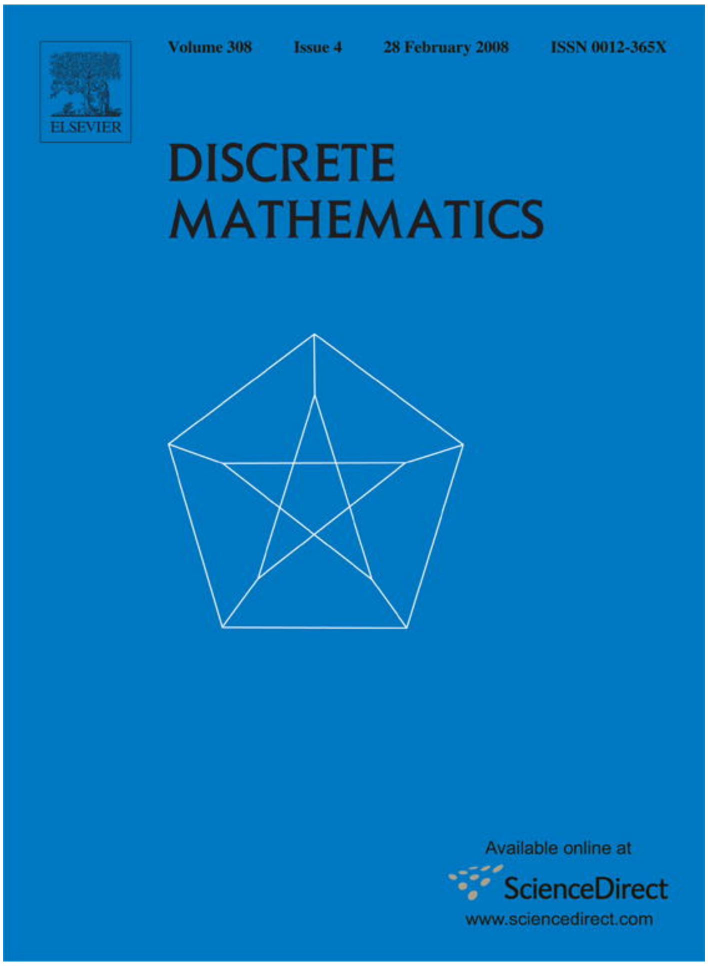

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On the domination number of the generalized Petersen graphs

Arash Behzad[a], Mehdi Behzad[b], Cheryl E. Praeger[c]

[a] *University of California, Los Angeles, USA*
[b] *Shahid Beheshti University, Iran*
[c] *University of Western Australia, Australia*

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Abstract

Graph domination numbers and algorithms for finding them have been investigated for numerous classes of graphs, usually for graphs that have some kind of tree-like structure. By contrast, we study an infinite family of regular graphs, the generalized Petersen graphs $G(n)$. We give two procedures that between them produce both upper and lower bounds for the (ordinary) domination number of $G(n)$, and we conjecture that our upper bound $\lceil 3n/5 \rceil$ is the exact domination number. To our knowledge this is one of the first classes of regular graphs for which such a procedure has been used to estimate the domination number.

Keywords: Domination in graphs; Generalized Petersen graphs

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1. Introduction

For a graph $G = (V, E)$, with vertex set V and edge set E, a subset $S \subseteq V$ is said to *dominate* V if each vertex of $V \backslash S$ is joined by an edge to some vertex of S. The set V itself has this property and, for a finite graph G, the minimum cardinality of subsets S that dominate V is called the (ordinary) *domination number* of G, and is denoted $\gamma(G)$. Domination numbers for graphs and associated concepts have been studied for many years and there is an extensive literature on the subject, see [5]. In general, determining the domination number (and most of its variations) is an NP-complete problem. In fact the book [5] contains a chapter, entitled "Domination, complexity and algorithms", devoted to this broad subject. According to that reference, the first person who proved the NP-completeness of determining the domination number was Johnson [4]. Among the many variations of the domination number, we mention the total domination number which was also proved to be NP-complete, even for bipartite graphs, see [6]. Consequently, most progress has been achieved by focussing on particular classes of graphs, typically graphs that possess a tree-like structure of some sort (chordal graphs, dually chordal graphs, etc.), and developing elimination procedures that enable the use of induction. Classes of regular graphs usually do not have such a structure and rarely appear in investigations concerning domination parameters.

In this paper we consider an infinite family of regular graphs, a sub-family of the generalized Petersen graphs. We present two algorithms which between them lead to the determination of upper and lower bounds on the domination numbers of these graphs, and we believe that our upper bound may be the exact value.

E-mail addresses: abehzad@ee.ucla.edu (A. Behzad), mbehzad@sharif.ac.ir (M. Behzad), praeger@maths.uwa.edu.au (C.E. Praeger).

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For each odd integer $n = 2k + 1 \geqslant 3$, the *generalized Petersen graph* $G(n)$ is the graph with vertex set $\mathscr{O} \cup \mathscr{I}$, where $\mathscr{O} = \{O_i | 1 \leqslant i \leqslant n\}$ and $\mathscr{I} = \{I_i | 1 \leqslant i \leqslant n\}$, and edge set $E_1 \cup E_2 \cup E_3$, where $E_1 = \{O_i O_{i+1} | 1 \leqslant i \leqslant n\}$, $E_2 = \{I_i I_{i+k} | 1 \leqslant i \leqslant n\}$, and $E_3 = \{O_i I_i | 1 \leqslant i \leqslant n\}$. Here the subscripts are to be read as integers modulo n. Descriptions of the graph $G(n)$ (or $P(2k + 1, k)$ as it is also denoted) and some of its properties may be found in [7], where it was introduced for the first time. An idea of the structure of these graphs may be obtained from the diagram for $G(15)$ given in Fig. 4, and the three smallest members of the family may be found in Fig. 3. These graphs are quite symmetrical, admitting the automorphism $O_i \mapsto O_{i+1}$, $I_i \mapsto I_{i+1}$ $(1 \leqslant i \leqslant n)$ which generates a subgroup of automorphisms with vertex orbits $\mathscr{O}$, $\mathscr{I}$ and edge orbits E_1, E_2, E_3. Moreover it follows from a result of Frucht et al. [3] that $G(n)$ is vertex-transitive only in the cases $n = 3$ (the 3-sided prism) and $n = 5$ (the Petersen graph) shown in Fig. 3. Our main result is the following.

Theorem 1.1. *For each odd integer* $n \geqslant 3$, $\gamma(G(n)) \leqslant \lceil 3n/5 \rceil$, *and moreover*

$$\gamma(G(n)) \leqslant \gamma(G(n + 2)) \leqslant \gamma(G(n)) + 2.$$

Remark 1.2. (a) We note that this result improves an upper bound derivable from [5], namely [5, Theorem 2.7, p. 48] and [5, Theorem 2.12, p. 50] imply that, for odd $n \geqslant 3$,

$$\left\lceil \frac{n}{2} \right\rceil \leqslant \gamma(G(n)) \leqslant \left\lceil \frac{3n}{4} \right\rceil.$$

(b) We conjecture that for each odd integer $n \geqslant 3$, $\gamma(G(n)) = \lceil 3n/5 \rceil$. To prove this conjecture, one might try to find all odd integers $n \geqslant 3$ such that $\gamma(G(n + 2)) - \gamma(G(n)) = i$, where $i = 0, 1$, or 2.

(c) Our derivation of the upper bound is accompanied by the construction of a dominating subset of size $\lceil 3n/5 \rceil$. Concerning graph theoretic parameters such as γ, we believe that, in applications it is often more important to have a method for constructing an appropriate subset, close to the optimum size, than having the exact value of the parameter without a construction method. We hope that the techniques underpinning our methods may be applicable to some other classes of graphs, and might help in finding bounds for domination parameters and other parameters of such graphs.

In Section 2 we present our two algorithms, the Integration Algorithm that constructs from the generalized Petersen graph $G(n)$ a copy of the smaller graph $G(n - 2)$, and the Disintegration Algorithm that constructs, from $G(n)$ as input, a copy of $G(n + 2)$. We use these algorithms to derive the two inequalities relating the domination numbers of $G(n)$ and $G(n+2)$ given in Theorem 1.1. Finally, we construct a small dominating set for the graph $G(n)$ and thereby obtain the upper bound $\lceil 3n/5 \rceil$ for $\gamma(G(n))$, completing the proof of Theorem 1.1. Notions not defined in this paper may be found in standard texts such as [2].

2. Estimating γ for $G(n)$

We describe and prove the two algorithms. The first shows how to construct from $G(n)$ a smaller generalized Petersen graph.

Algorithm 2.1. Integration Algorithm.

```
Input: the graph $G(n) = (\mathscr{O} \cup \mathscr{I}, E_1 \cup E_2 \cup E_3)$ with $n = 2k + 1 \geqslant 7$.
Output: a graph $G''$ with $2(n - 2)$ vertices.
Step 1. Choose $i$ such that $1 \leqslant i \leqslant k$, remove the four pairs of vertices
        $\{O_i, O_{i+1}\}$, $\{I_i, I_{i+1}\}$, $\{O_{i+k}, O_{i+k+1}\}$ and $\{I_{i+k}, I_{i+k+1}\}$,
        along with their 15 incident edges, and denote the resulting graph by $G'$.
Step 2. Add four new vertices $O'_i, I'_i, O'_{i+k-1}, I'_{i+k-1}$, and define the graph $G''$ to have vertex set $V(G'') = V(G') \cup \{O'_i, I'_i, O'_{i+k-1}, I'_{i+k-1}\}$ and edge set
        $E(G'') = E(G') \cup \{O_{i-1}O'_i, O'_iO_{i+2}, O'_iI'_i, I'_iI'_{i+k-1}, I'_iI_{i+k+2},$
            $I_{i-1}I'_{i+k-1}, O_{i+k+2}O'_{i+k-1}, O'_{i+k-1}O_{i+k-1}, O'_{i+k-1}I'_{i+k-1}\}$.
Return $G''$.
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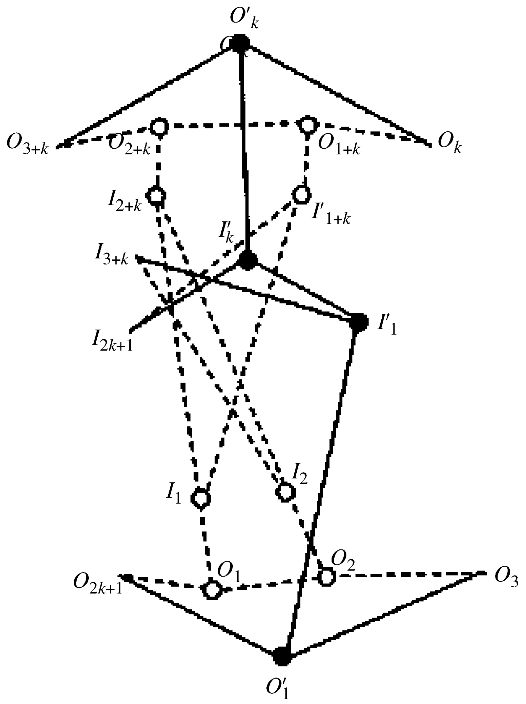

Fig. 1. Integration Algorithm 2.1 for $i = 1$, $n \geqslant 7$.

See Fig. 1 for an example with $i = 1$. Note that, if $i = 1$ in the algorithm then the returned graph G'' contains the edge $I_{2k+1}I'_k$.

Lemma 2.2 (*Integration Lemma*). *For each odd integer $n \geqslant 7$, the graph G'' returned by Algorithm* 2.1 *is isomorphic to $G(n-2)$.*

Proof. Clearly $|V(G'')| = 2(n-2)$ and $|E(G'')| = 3(n-2)$. Relabel the vertices of G'' as follows. For the chosen index i in Step 2, set

$$U_i := O'_i, \quad U_{i+k-1} := O'_{i+k-1} \quad \text{and} \quad W_i := I'_i, \quad W_{i+k-1} := I'_{i+k-1},$$

for each j such that $1 \leqslant j < i$, set

$$U_j := O_j \quad \text{and} \quad W_j := I_j,$$

for each j such that $i + 2 \leqslant j < i + k$, set

$$U_{j-1} := O_j \quad \text{and} \quad W_{j-1} := I_j,$$

and for each j such that $i + k + 2 \leqslant j \leqslant 2k + 1 = n$, set

$$U_{j-2} := O_j \quad \text{and} \quad W_{j-2} := I_j.$$

In this way we obtain the sets $\mathscr{U} = \{U_j \,|\, 1 \leqslant j \leqslant n-2\}$ and $\mathscr{W} = \{W_j \mid 1 \leqslant j \leqslant n-2\}$ such that $V(G'') = \mathscr{U} \cup \mathscr{W}$. Since $V(G(n-2))$ was defined in Section 1 to be $\mathscr{O} \cup \mathscr{I}$ with $|\mathscr{O}| = |\mathscr{I}| = n-2$, and since the bijection $f : \mathscr{O} \cup \mathscr{I} \to \mathscr{U} \cup \mathscr{W}$, defined by $f(O_j) = U_j$ and $f(I_j) = W_j$ for $1 \leqslant j \leqslant n-2$, preserves adjacency and nonadjacency, we have the required result. $\square$

The second algorithm constructs from $G(n)$ a larger generalized Petersen graph.

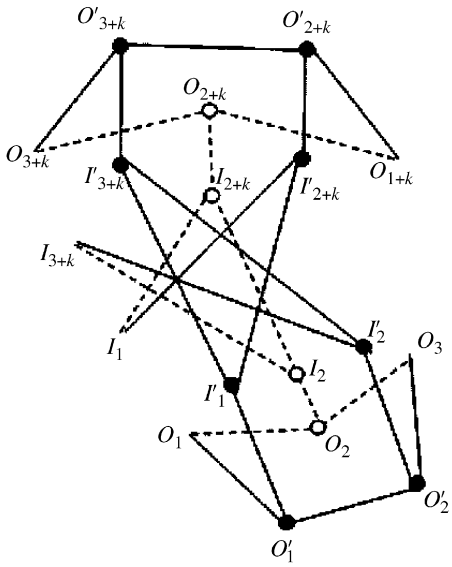

Fig. 2. Disintegration Algorithm 2.3 for $i = 2$, $n \geqslant 5$.

Algorithm 2.3. Disintegration Algorithm.
Input: *the graph* $G(n) = (\mathscr{O} \cup \mathscr{I}, E_1 \cup E_2 \cup E_3)$ *with* $n = 2k + 1 \geqslant 5$.
Output: *a graph* G'' *with* $2(n + 2)$ *vertices*.

Step 1. Choose i such that $2 \leqslant i \leqslant k+1$, remove the four vertices O_i, I_i, O_{i+k}, and I_{i+k}, along with their nine incident edges, and denote the resulting graph by G'.

Step 2. Add eight new vertices

$$V'' := \{O'_{i-1}, O'_i, I'_{i-1}, I'_i, O'_{i+k}, O'_{i+k+1}, I'_{i+k}, I'_{i+k+1}\},$$

and define the graph G'' to have vertex set $V(G'') = V(G') \cup V''$ and edge set

$$\begin{aligned} E(G'') = E(G') \cup \{&O_{i-1}O'_{i-1},\ O'_{i-1}O'_i, O'_iO_{i+1}, O'_{i-1}I'_{i-1}, O'_iI'_i,\\ &O_{i+k-1}O'_{i+k}, O'_{i+k}O'_{i+k+1}, O'_{i+k+1}O_{i+k+1}, O'_{i+k}I'_{i+k},\\ &O'_{i+k+1}I'_{i+k+1}, I'_{i-1}I'_{i+k}, I'_{i+k}I_{i-1}, I'_{i+k+1}I'_{i-1},\\ &I'_{i+k+1}I'_i, I'_iI_{i+k+1}\}. \end{aligned}$$

(See Fig. 2 for an example with $i = 2$.) Return G''.

Lemma 2.4 (*Disintegration Lemma*). *For each odd integer* $n \geqslant 5$, *the graph* G'' *returned by Algorithm* 2.3 *is isomorphic to* $G(n + 2)$.

Proof. Clearly $|V(G'')| = 2(n + 2)$ and $|E(G'')| = 3(n + 2)$. Relabel the vertices of G'' as follows. For the chosen index i in Step 2, set

$$U_i := O'_{i-1}, \quad U_{i+1} := O'_i, \quad U_{i+k+1} := O'_{i+k} \quad \text{and} \quad U_{i+k+2} := O'_{i+k+1},$$

and similarly define W_i, W_{i+1}, W_{i+k+1} and W_{i+k+2}. For each j such that $1 \leqslant j < i$, set

$$U_j := O_j \quad \text{and} \quad W_j := I_j,$$

for each j such that $i + 1 \leqslant j \leqslant i + k - 1$, set

$$U_{j+1} := O_j \quad \text{and} \quad W_{j+1} := I_j,$$

and for each j such that $i+k+1 \leqslant j \leqslant 2k+1=n$, set

$$U_{j+2} := O_j \quad \text{and} \quad W_{j+2} := I_j.$$

In this way we obtain the sets $\mathcal{U}=\{U_j \mid 1 \leqslant j \leqslant n+2\}$ and $\mathcal{W}=\{W_j \mid 1 \leqslant j \leqslant n+2\}$ such that $V(G'')=\mathcal{U} \cup \mathcal{W}$. Now the graph $G(n+2)$ as defined in Section 1 has vertex set $\mathcal{O} \cup \mathcal{I}$ with $|\mathcal{O}|=|\mathcal{I}|=n+2$, and the bijection $f: \mathcal{O} \cup \mathcal{I} \rightarrow \mathcal{U} \cup \mathcal{W}$, defined by $f(O_j)=U_j$ and $f(I_j)=W_j$ for $1 \leqslant j \leqslant n+2$, preserves adjacency and nonadjacency, and hence is an isomorphism. □

As before, the assertion "*a set* $X \subseteq V(G)$ *dominates a set* $Y \subseteq V(G)$", means that the elements of Y are dominated by the elements of X, that is to say, each element of Y is either in X or adjacent to some element of X. We say that a vertex is *covered* by a subset S of vertices if it is either a member of S or adjacent to a member of S. We show next that $\gamma(G(n))$ is a non-decreasing function of n.

Lemma 2.5. *Let n be an odd integer such that $n=2k+1 \geqslant 5$. Then*

$$\gamma(G(n)) \leqslant \gamma(G(n+2)).$$

Proof. To keep the notation in line with that of Algorithm 2.1, we assume that $n \geqslant 7$, and prove that $\gamma(G(n-2)) \leqslant \gamma(G(n))$. Let $G=G(n)$, and let $S \subseteq V(G)$ be a dominating set for $V(G)$ of minimum cardinality. We claim that at least one element of $\mathcal{I}$, say I_1, must lie in S. If this is not the case then S is a subset of $\mathcal{O}$. Since S dominates $\mathcal{I}$ and since each element of $\mathcal{I}$ is adjacent to exactly one element of $\mathcal{O}$, we must have $S=\mathcal{O}$ of cardinality n. However, $G(n)$ has a dominating set of cardinality less than n (for example, $(\mathcal{I} \backslash \{I_{k+1}, I_{k+2}\}) \cup \{O_{k+1}\}$) contradicting the minimality of S. Thus, we may assume that $I_1 \in S$.

Let G'' be the graph returned by Algorithm 2.1 with the index $i=1$. By Lemma 2.2, $G'' \cong G(n-2)$. We will identify $V(G(n-2))$ with $V(G'')$ so that $V(G(n-2))=(\mathcal{O} \cup \mathcal{I} \backslash T) \cup T'$, where $T'=\{O_1', I_1', O_k', I_k'\}$ and

$$T=\{O_1, O_2, I_1, I_2, O_{k+1}, O_{k+2}, I_{k+1}, I_{k+2}\}.$$

Let G' be the subgraph of G spanned by $V(G) \backslash T$, so that G' is also a subgraph of $G(n-2)$. Then the subset $S' := S \cap V(G')$ dominates all vertices in $V(G')$, except possibly vertices in $R := \{O_3, O_{2k+1}, O_k, O_{k+3}, I_{k+3}, I_{2k+1}\}$. If at least one of the members of the pair O_1, O_2 (or I_1, I_2), lies in S, then add the vertex O_1' (or I_1') of $G(n-2)$ to S'; similarly if at least one of the members of the pair O_{k+1}, O_{k+2} (or I_{k+1}, I_{k+2}), lies in S, then add the vertex O_k' (or I_k') to S'; this produces finally a subset $S^* \subseteq V(G(n-2))$. Since the number $|S^* \backslash S'|$ of vertices added to S' is at most $|S \cap T|$, we have $|S^*| \leqslant |S|$. Also, from the definition of S it follows that $I_1' \in S^*$. Suppose that S^* dominates $V(G(n-2))$. Then $\gamma(G(n-2)) \leqslant |S^*| \leqslant |S| = \gamma(G(n))$, and the theorem is proved.

Thus, it remains for us to prove that S^* dominates $V(G(n-2))$, and to prove this we only need to show that S^* dominates

$$T' \cup R = \{O_1', I_1', O_k', I_k', O_3, O_k, O_{k+3}, O_{2k+1}, I_{k+3}, I_{2k+1}\}.$$

First, since $I_1' \in S^*$, it follows that O_1', I_1', I_{k+3} and I_k' are all covered by S^*. Next we note that, for each row of Table 1, if the vertex in column 1 lies in S^*, then all the vertices in column 2 are covered by S^*.

Consider I_{2k+1}. The vertices adjacent in $G(n-2)$ to I_{2k+1} are I_k, I_k' and O_{2k+1}. (Recall that we are identifying $V(G(n-2))$ with $V(G'')$.) If $I_{k+1} \in S$ then $I_k' \in S^*$, and by Table 1, I_{2k+1} is covered by S^*. On the other hand, if $I_{k+1} \notin S$, then since I_{2k+1} is covered by S in $G(n)$, at least one of I_{2k+1}, I_k, O_{2k+1} lies in S and hence in S^*, and in this case I_{2k+1} is also covered by S^*. An analogous argument proves that O_{2k+1} is covered by S^*.

Table 1
Notes for Lemma 2.5

Element of S^*	Vertices of $T' \cup R$ covered by S^*
I_k'	O_k', I_{2k+1}, I_1'
O_1'	I_1', O_{2k+1}, O_3
O_k'	I_k', O_{k+3}, O_k

Next consider O_{k+3}. By Table 1, if $O'_k \in S^*$ then O_{k+3} is covered by S^*, so suppose that $O'_k \notin S^*$. Then neither of O_{k+1}, O_{k+2} is in S. Since O_{k+3} is covered by S in $G(n)$, this means that at least one of $O_{k+3}, I_{k+3}, O_{k+4}$ lies in S and hence in S^*, and in this case O_{k+3} is also covered by S^*. An analogous argument proves that O_k is covered by S^*, except in the degenerate case where $k = 3$ and $O_2 \in S$, while $I_3, O_3, O_4, O_5 \notin S$. Also, an analogous argument proves that O_3 is covered by S^*, except in the degenerate case where $k = 3$ and $O_4 \in S \cap T$, while $I_3, O_1, O_2, O_3 \notin S$. In the case $k = 3$ one of these arguments works if $O_2 \in S$ and the other if $O_2 \notin S$, so we conclude in this case that $O_3 = O_k$ is also covered by S^*.

Finally, consider O'_k. If $O'_k \in S^*$ there is nothing to prove, so assume that $O'_k \notin S^*$. Then, by the definition of S^*, $O_{k+1}, O_{k+2} \notin S$. If $I'_k \in S^*$ then, as $O'_k I'_k \in E(G'')$, again O'_k is covered by S^*. So assume also that $I'_k \notin S^*$. Then, by the definition of S^*, neither of I_{k+1}, I_{k+2} lies in S, and since O_{k+2} must be covered by S in $G(n)$, it follows that O_{k+3} must lie in S. By the definition of $E(G'')$ in Algorithm 2.1, O'_k is adjacent to O_{k+3}, and hence O'_k is covered by S^*. This completes the proof. □

Next, we find an upper bound for $\gamma(G(n+2))$ in terms of $\gamma(G(n))$.

Lemma 2.6. *Let n be an odd integer such that $n = 2k+1 \geqslant 3$. Then $\gamma(G(n+2)) \leqslant \gamma(G(n)) + 2$.*

Proof. Let $G = G(n)$, and $S \subseteq V(G)$ be a dominating set with minimum cardinality for $V(G)$. Arguing as in the first paragraph of the proof of Lemma 2.5, we may assume that $I_2 \in S$. By Lemma 2.4, $G(n+2)$ is isomorphic to the graph G'' returned by Algorithm 2.3 with the index $i = 2$ at Step 1. Moreover, we may assume that the graph G' constructed in Step 1 of Algorithm 2.3 is the subgraph of G spanned by $V(G)\backslash T$ where $T = \{O_2, I_2, O_{k+2}, I_{k+2}\}$. The subset $S' := S \cap V(G')$ dominates all vertices in $V(G')$, except possibly vertices in $R := \{O_1, O_3, O_{k+1}, O_{k+3}, I_1, I_{k+3}\}$.

We will show that $V(G'')$ contains a subset S'' such that $S' \subseteq S''$, S'' dominates $V(G'')$, and $|S''| \leqslant |S| + 2$. This will complete the proof, since then $\gamma(G(n+2)) = \gamma(G'') \leqslant |S''| \leqslant |S| + 2 = \gamma(G(n)) + 2$. To produce such a set S'', we add to the set S' the appropriate number of vertices of $V(G'')$ so that the set

$$T' := \{O'_1, O'_2, I'_1, I'_2, O'_{k+2}, O'_{k+3}, I'_{k+2}, I'_{k+3}\} \cup R$$

is covered. Note that $I_2 \in T \cap S$ and $1 \leqslant |T \cap S| \leqslant 4$.

Case $|T \cap S| = 1$: Here $I_2 \in S$ and $O_2, O_{k+2}, I_{k+2} \notin S$. In this case $S'' := (S\backslash\{I_2\}) \cup \{I'_1, I'_2, O'_{k+2}\}$ covers T'.

Case $|T \cap S| = 2$: If $I_2, O_2 \in S$ and $O_{k+2}, I_{k+2} \notin S$, then $S'' := (S\backslash\{O_2, I_2\}) \cup \{O'_1, O'_2, I'_2, O'_{k+2}\}$ covers T'; if $I_2, O_{k+2} \in S$ and $O_2, I_{k+2} \notin S$, then $S'' := (S\backslash\{O_{k+2}, I_2\}) \cup \{I'_1, I'_2, O'_{k+2}, O'_{k+3}\}$ covers T'; and if $I_2, I_{k+2} \in S$ and $O_2, O_{k+2} \notin S$, then $S'' := (S\backslash\{I_2, I_{k+2}\}) \cup \{O'_1, I'_{k+2}, I'_{k+3}, I_{k+3}\}$ covers T'.

Case $|T \cap S| = 3$: If $I_2, O_2, O_{k+2} \in S$ and $I_{k+2} \notin S$, then $S'' := (S\backslash\{O_2, O_{k+2}, I_2\}) \cup \{O_1, O'_2, O_{k+3}, O'_{k+2}, I'_1\}$ covers T'. The case where $O_{k+2} \notin S$ and $I_2, O_2, I_{k+2} \in S$ follows by a similar argument using symmetry. Finally if $I_2, O_{k+2}, I_{k+2} \in S$ and $O_2 \notin S$, then $S'' := (S\backslash\{I_2, O_{k+2}, I_{k+2}\}) \cup \{I_1, I'_1, I'_2, O'_{k+2}, O'_{k+3}\}$ covers T'.

Case $|T \cap S| = 4$: In this case $S'' := (S\backslash T) \cup \{O_1, O'_2, O'_{k+2}, O_{k+3}, I'_1\}$ covers T'.

In all cases the subset S'' has size at most $|S| + 2$. □

Thus far we have proved that for each odd integer $n \geqslant 5$,

$$\gamma(G(n)) \leqslant \gamma(G(n+2)) \leqslant \gamma(G(n)) + 2.$$

Finally, we find an explicit upper bound for $\gamma(G(n))$, and thereby complete the proof of Theorem 1.1. For the three smallest cases the value of $\gamma(G(n))$ may be found easily by inspection, and minimum cardinality dominating sets are shown for these graphs in Fig. 3. We have $\gamma(G(3)) = 2$, $\gamma(G(5)) = 3$, $\gamma(G(7)) = 5$.

Lemma 2.7. *If n is an odd integer such that $n \geqslant 3$, then $\gamma(G(n)) \leqslant \lceil 3n/5 \rceil$.*

Proof. We observed above that the assertion holds for $n = 3, 5$ and 7. Let $n = 5h$, where $h \geqslant 3$ and h is an odd integer. Consider the sets $V(G(n))$, and $E(G(n))$ defined in Section 1. Let $S_O := \{O_{4+5(t-1)} \mid 1 \leqslant t \leqslant h\}$, and

$$S_I := \{I_{1+5(t-1)} | 1 \leqslant t \leqslant h\} \cup \{I_{2+5(t-1)} | 1 \leqslant t \leqslant h\}.$$

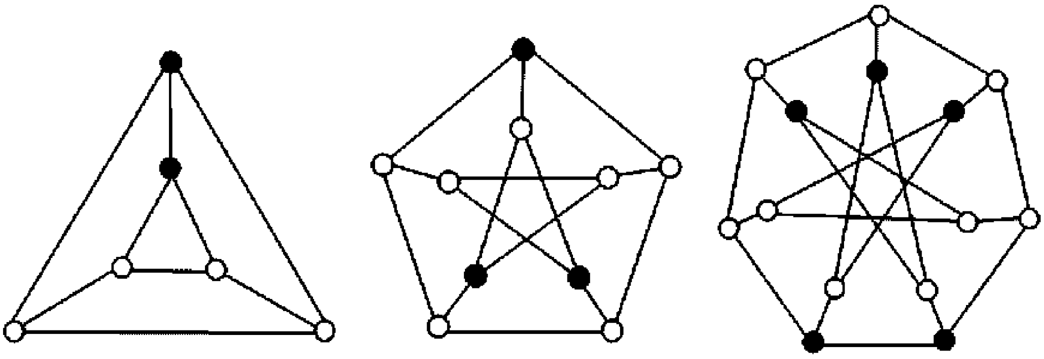

Fig. 3. $\gamma(G(3))=2$, $\gamma(G(5))=3$, $\gamma(G(7))=5$.

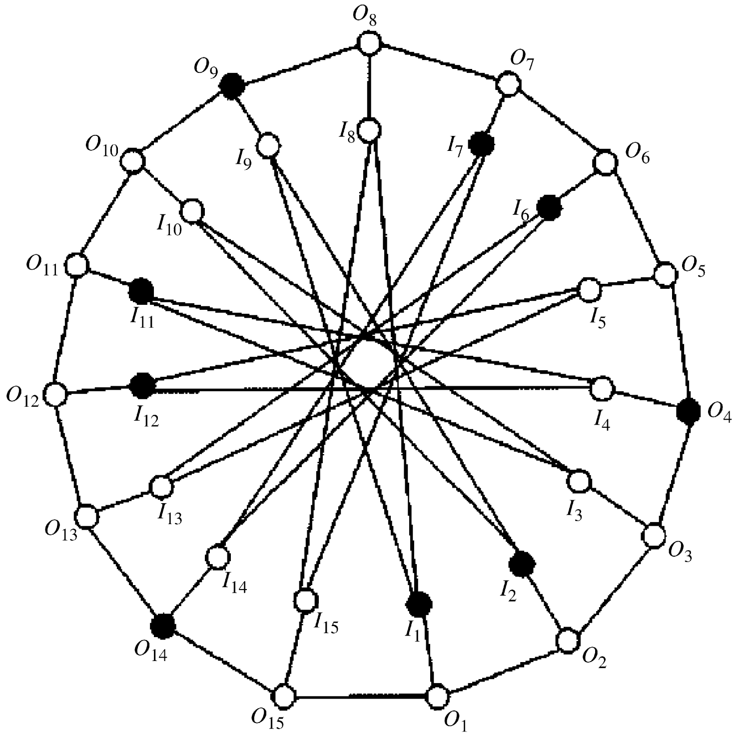

Fig. 4. $\gamma(G(15))\leqslant 9$.

Then $S=S_O\cup S_I$ dominates $V(G(n))$. To see this, first let $O_j\in\mathcal{O}$, and $I_j\in\mathcal{I}$. Then there exists a unique i ($1\leqslant i\leqslant 5$), and a unique t ($1\leqslant t\leqslant h$), such that $j=i+5(t-1)$. For $i=1, 2$, $O_{i+5(t-1)}$ is covered by $I_{i+5(t-1)}\in S_I$; $O_{3+5(t-1)}$ and $O_{5+5(t-1)}$ are both covered by $O_{4+5(t-1)}\in S_O$; while $O_{4+5(t-1)}\in S_O$. In addition, $I_{1+5(t-1)}$, $I_{2+5(t-1)}\in S_I$; since $1+5(t+(h-1)/2)=3+5(t-1)+(k+1)$, $I_{3+5(t-1)}$ is adjacent to, and hence covered by $I_{1+5(t+(h-1)/2)}\in S_I$; $I_{4+5(t-1)}$ is covered by $O_{4+5(t-1)}\in S_O$; and finally, since $2+5(t+(h-1)/2)=5+5(t-1)+k$, $I_{5+5(t-1)}$ is adjacent to, and hence covered by $I_{2+5(t+(h-1)/2)}\in S_I$. (As an example, see Fig. 4.) Since $|S|=|S_O\cup S_I|=|S_O|+|S_I|=3h=\lceil 3(5h)/5\rceil$, the theorem is proved for this case.

To complete the proof for the other congruence classes modulo 5, we use Algorithm 2.1. To illustrate this process we first consider $n\leqslant 15$. As Fig. 4 shows, for $n=15$, the graph $G(15)$ is dominated by the set S of cardinality 9 comprising the three pairs I_1, I_2; I_6, I_7; and I_{11}, I_{12}, along with O_4, O_9 and O_{14}. We apply Algorithm 2.1 three times, such that one of the four pairs of vertices removed at Step 1 is as follows: on the first application I_1, I_2; on the second, I_6, I_7; and finally I_{11}, I_{12}—to produce $G(13)$, $G(11)$ and $G(9)$ respectively. Now consider the process described in the proof of Lemma 2.5 to modify a dominating set S for $V(G(n))$ into a dominating set S^* for the graph $G(n-2)$ returned by Algorithm 2.1. Since in each of our three applications of Algorithm 2.1 at least one pair of the elements of S is 'integrated' into a single element of S^*, we have

$$\gamma(G(13)) \leqslant 9-1=8, \quad \gamma(G(11)) \leqslant 8-1=7, \quad \gamma(G(9)) \leqslant 7-1=6.$$

Thus, for odd integers $n \leqslant 15$, we have $\gamma(G(n)) \leqslant \lceil 3n/5 \rceil$.

Next, assume that $n=5h$, where h is odd and $h \geqslant 5$. Then, the dominating set $S=S_O \cup S_I$ for $G(n)$, defined above, contains at least the four pairs I_1, I_2; I_6, I_7; I_{11}, I_{12}; and I_{16}, I_{17}. We apply Algorithm 2.1 four times, such that, on each application, one of the pairs of vertices removed at Step 1 is one of these four pairs—to produce $G(5h-2)$, $G(5h-4)$, $G(5h-6)$ and $G(5h-8)$ respectively. As for the case $n=15$ (in the process described in the proof of Lemma 2.5 to modify a dominating set for $G(n)$ into a dominating set for the graph $G(n-2)$) each time a pair of elements of S is integrated into a single element of the new dominating set. Hence, again for each $i=1, \ldots, 4$, we have

$$\gamma(G(5h-2i)) \leqslant |S|-i=3h-i=\left\lceil \frac{3(5h-2i)}{5} \right\rceil. \quad \square$$

At the end of the proof, if we apply Algorithm 2.1 once more, we produce a dominating set for $G(5h-10)$ which contains $3h-5$ elements. Moreover, since $5h-10$ is divisible by 5, we have

$$\gamma(G(5h-10)) \leqslant \left\lceil \frac{3(5h-10)}{5} \right\rceil = 3h-6.$$

It is this observation that led to our conjecture in Remark 1.2.

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