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The *z*-Transform for Analog Designers

In this new series of articles, we look at how analog designers think and how they go about analysis and design. The *z*-transform is an important tool that proves useful in the development of mixed-signal systems, such as $\Delta\Sigma$ modulators, digital phase-locked loops, and equalizers. As analog designers, we have an intuitive understanding of the Fourier and Laplace transforms. In this article, we extend our intuition to the *z*-transform as well.

A Few Basics

For continuous-time (CT) systems, we obtain the transfer function by taking the Laplace transform of the impulse response. The poles and zeros of the transfer function reveal how different frequencies travel through the system and whether the system is stable.

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For our studies in this article, it is helpful to remember that a system that delays a signal by ΔT seconds has a transfer function given by

$$H(s) = \int_0^\infty \delta(t - \Delta T)\, e^{-st}\, dt \qquad (1)$$

$$= e^{-\Delta Ts}. \qquad (2)$$

If $|s| \ll 1$, we have $H(s) \approx 1 - \Delta Ts$; i.e., $H(s)$ contains a zero in the right half plane.

Now, suppose a CT signal is sampled periodically and converted to a discrete-time (DT) signal (Figure 1). Our analog mentality readily considers $w(t)$ as the product of $x(t)$ and a train of impulses:

$$w(t) = x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_{CK}) \qquad (3)$$

$$= \sum_{k=-\infty}^{+\infty} x(kT_{CK})\,\delta(t - kT_{CK}), \qquad (4)$$

where T_{CK} denotes the clock period. But, if $w(t)$ is next subjected to digital processing, we must seek a transform that can be applied to such an environment. In a DT system, we rewrite $x(kT_{CK})$ as $x[k]$.

In our analysis of DT systems, we will often draw intuition from their CT counterparts. Concepts such as impulse response, step response, and frequency response prove valuable in our studies here.

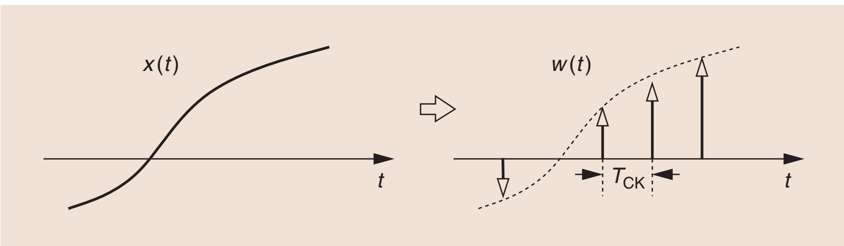

FIGURE 1: The sampling of a CT signal by impulses.

The *z*-Transform

Consider a CT system characterized by an impulse response, $h_1(t)$ in Figure 2(a), and a transfer function, $H_1(s)$, that is the Laplace transform of $h_1(t)$:

$$H_1(s) = \int_0^\infty h_1(t)\, e^{-st}\, dt. \qquad (5)$$

We consider only causal systems, hence, the integration from $t = 0$. This is our first system.

We next wish to study a DT system having a similar impulse response. But let us first sample $h_1(t)$

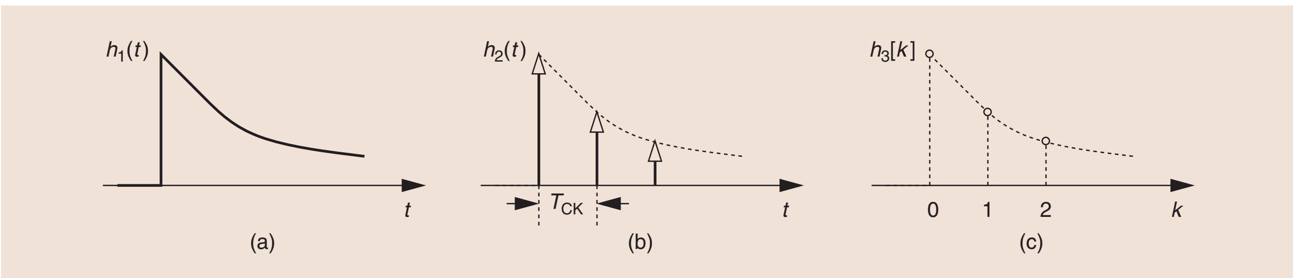

FIGURE 2: (a) An impulse response example, (b) the sampled version of $h_1(t)$, and (c) the DT version of $h_1(t)$.

by a train of impulses, as shown in Figure 2(b), and write

$$h_2(t) = \sum_{k=0}^{+\infty} h_1(kT_{CK})\delta(t - kT_{CK}). \quad (6)$$

Taking the Laplace transform, we have for our second system

$$H_2(s) = \sum_{k=0}^{+\infty} h_1(kT_{CK})e^{-kT_{CK}s}. \quad (7)$$

For a DT version of this response, Figure 2(c), we denote $e^{T_{CK}s}$ by the letter z and write the transfer function of our third system as

$$H_3(z) = \sum_{k=0}^{+\infty} h_3[k]z^{-k}. \quad (8)$$

We call $H_3(z)$ the z-transform of $h_3[k]$. Note that z can generally assume a complex value because we have $s = \sigma + j\omega$ and hence

$$z = e^{T_{CK}s} \quad (9)$$
$$= e^{T_{CK}\sigma}e^{T_{CK}j\omega}. \quad (10)$$

In this article, we wish to develop an intuitive understanding of (8).

The Unit Delay

Most of our intuition will draw upon the z-transform of the unit-delay system, one in which the input is delayed by one clock cycle. Equation (2) yielded a transfer function equal to $e^{-\Delta Ts}$ for the CT delay stage. The DT equivalent is $e^{-T_{CK}s}$ and hence

$$H(z) = z^{-1}. \quad (11)$$

Figure 3(a) depicts the impulse response and Figure 3(b) the step response.

Let us examine the unit-delay stage for a slowly varying input. In this case, the input and output waveforms appear nearly the same (Figure 4), suggesting that $z^{-1} \to 1$ at low frequencies. This result is also corroborated by the CT delay circuit characterized by (2), where $e^{-\Delta Ts}$ reduces to about 1 if s is very small. These conditions hold if the input frequency is much less than $f_{CK} = 1/T_{CK}$.

A stage having $H(z) = z$ advances the signal by one clock cycle, as in Figure 5(a). This can be seen by noting that the cascade of z^{-1} and z [Figure 5(b)] is equivalent to $H(z) = 1$. These points prove useful in interpreting more complex transfer functions.

Low-Pass Filters

Before constructing a DT low-pass filter (LPF), we return to the more familiar CT counterpart and apply an impulse to it [see Figure 6(a)]. We observe that the attenuation of high-frequency components "broadens" the impulse as it emerges at the output. How can we create a similar effect in the DT case? We can simply introduce one more impulse at the output, as shown in Figure 6(b). The output pulsewidth is now broader than that of the input. With the impulse response given by $\delta[k] + \delta[k-1]$, the transfer function assumes the form $1 + z^{-1}$, as in Figure 6(c). That is, a system that adds a delayed copy of the signal to itself acts as an LPF.

We intuitively see that a slowly varying random signal and its delayed replica appear nearly the same (Figure 4) and hence add constructively most of the time. For a fast-varying random signal, on the other hand, there are more instances at which the two cancel

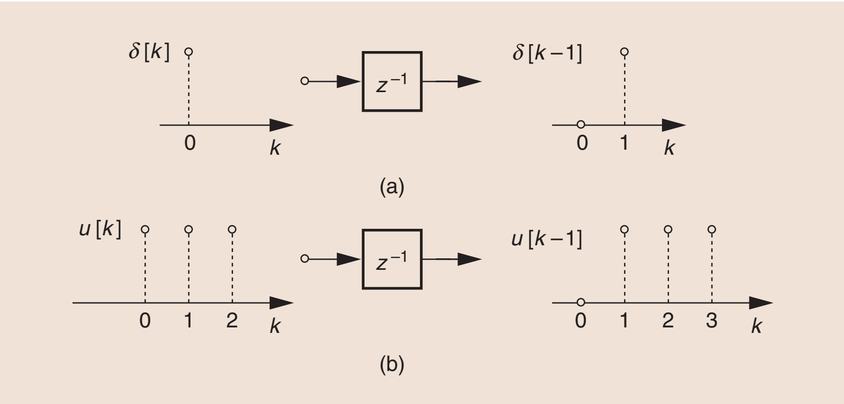

FIGURE 3: The unit-delay stage: the (a) impulse response, and (b) step response.

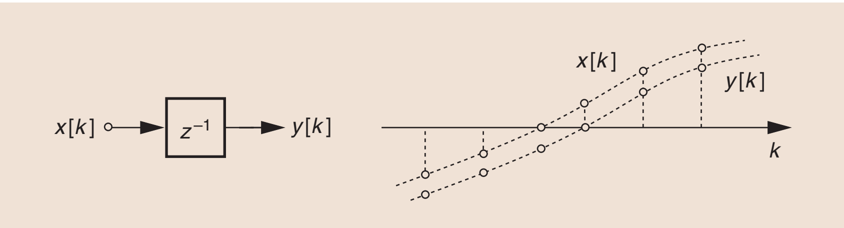

FIGURE 4: A slowly varying signal passing through unit delay.

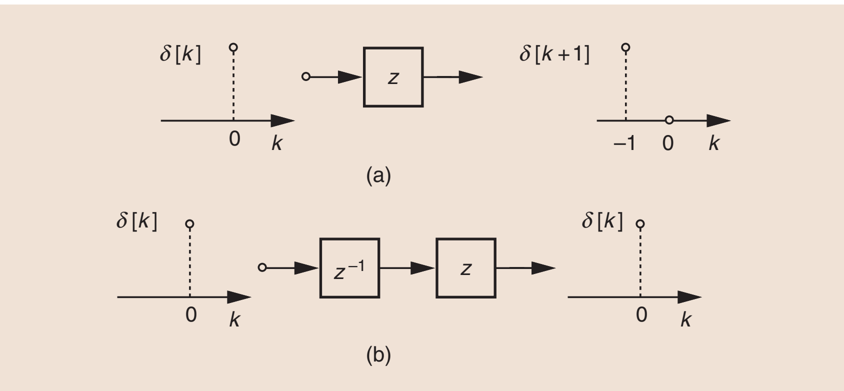

FIGURE 5: The impulse response of the (a) unit advance stage and (b) cascade of z^{-1} and z stages.

each other. The circuit then passes low frequencies and attenuates high frequencies.

Our results merit three remarks. First, the step response of the DT LPF, shown in Figure 6(d), does exhibit a somewhat gradual rising behavior, as expected of an LPF. Second, the steady-state value of the step response is equal to 2 and represents the magnitude of the transfer function at low frequencies (like the gain of an amplifier). That is $|1+z^{-1}| \to 2$ if the input DT signal varies slowly, which agrees with our previous result, namely, $z^{-1} \to 1$. Third, for a sinusoidal input with a period of T_{in}, the output in Figure 6(c) drops to zero if the delay is equal to $T_{\text{in}}/2$, i.e., if the input frequency is half of the clock frequency. Sketched in Figure 6(e), this behavior stands in contrast to that of first-order CT LPFs. To compute $|H(z)|$, we assume a sinusoidal input, i.e., $z = e^{T_{CK} j\omega}$, and write

$$|H(z)| = |1 + e^{-T_{CK} j\omega}| \tag{12}$$

$$= |1 + \cos\omega T_{CK} - j\sin\omega T_{CK}| \tag{13}$$

$$= \sqrt{2 + 2\cos\omega T_{CK}}. \tag{14}$$

Our development of the first-order DT LPF leads to two other useful observations. First, the broadening effect depicted in Figure 6(b) can employ a nonunity delayed impulse [Figure 7(a)], providing a transfer function given by $1 + \alpha z^{-1}$. We assume α is real, positive, and no greater than unity. Second, as shown in Figure 7(b), the step response displays a steady-state value of $1 + \alpha$, and so does $|H(z)|$ at low frequencies. Interestingly, however,

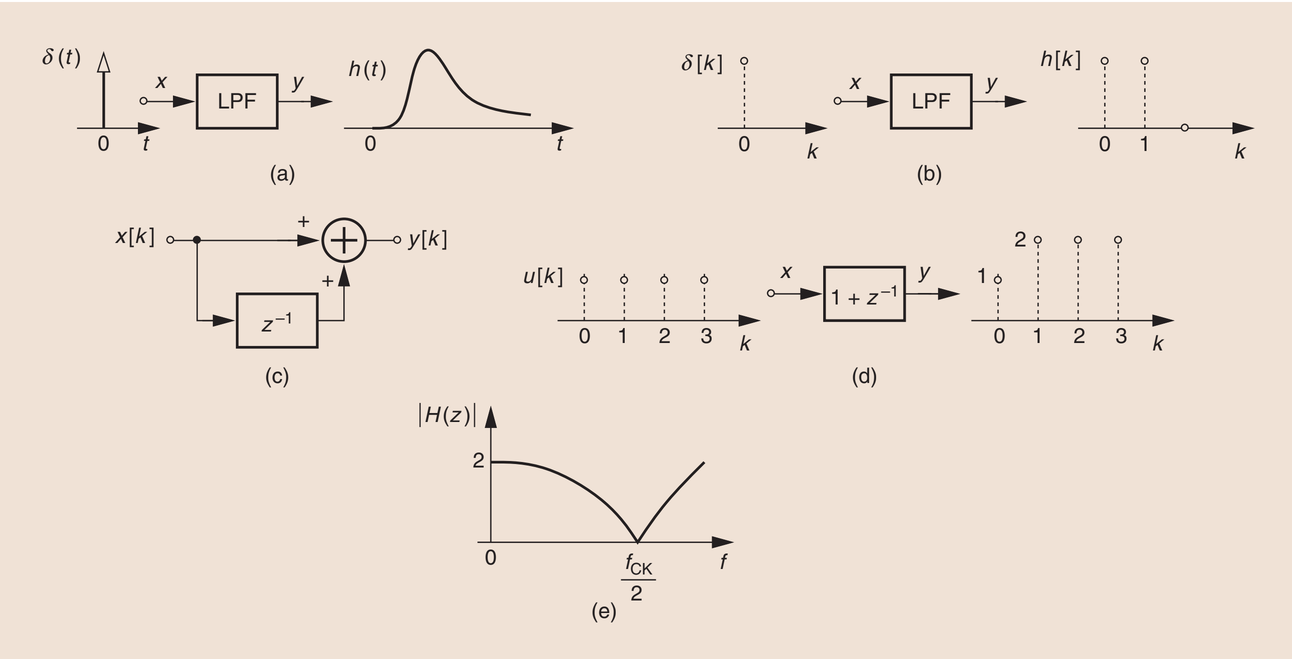

FIGURE 6: The (a) impulse responses of a CT system, (b) broadening of the impulse in a DT system, (c) realization of a first-order DT LPF, (d) step response of a DT LPF, and (e) frequency response of a DT LPF.

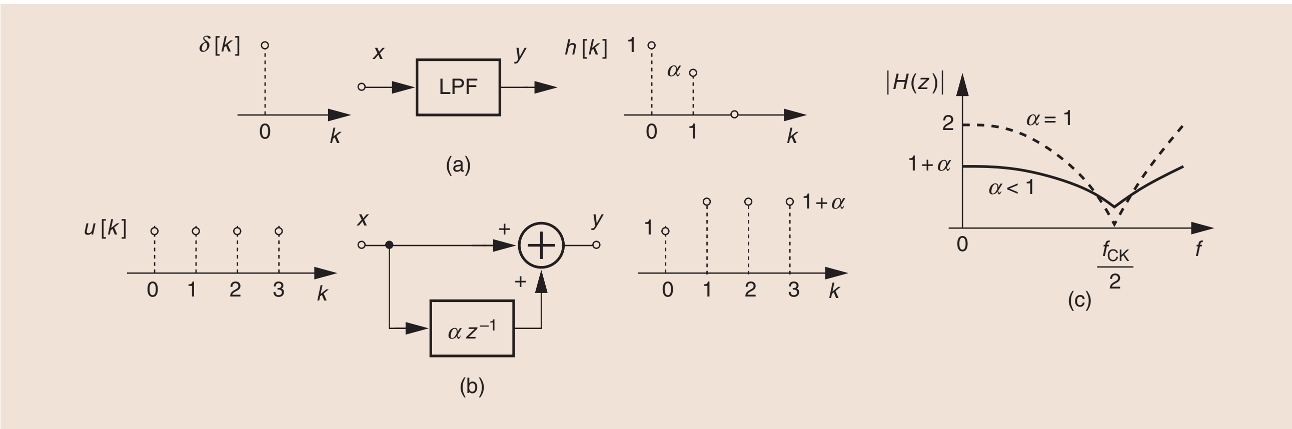

FIGURE 7: (a) An LPF with shorter tail, (b) the LPF step response, and (c) the LPF frequency response.

$|H(z)|$ does not fall to zero for any sinusoidal input because a delayed and scaled copy of the input cannot yield complete cancellation unless $\alpha = 1$. For small values of α, the LPF exhibits little selectivity [Figure 7(c)].

Let us further extend our impulse broadening notion and add more delayed replicas of the input. Figure 8(a) illustrates an example, for which we have

$$H(z) = 1 + z^{-1} + z^{-2}. \qquad (15)$$

We expect that the greater broadening of the input impulse leads to a narrower bandwidth for the filter. Indeed, from the step response shown in Figure 8(b), we observe a more gradual transition. Equation (15) exemplifies a simple finite impulse response filter. A more general form is

$$H(z) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots. \qquad (16)$$

High-Pass Filters

The developments in the previous section can be repeated for high-pass filters (HPFs) as well. In this case, it is simpler to begin with the step response of a CT HPF, sketched in Figure 9(a). Note that $y(t)$ must decay to zero because the circuit does not pass dc. Taking the derivative, we obtain the impulse response, Figure 9(b), observing that the output contains a positive impulse and a negative broadened tail. This points to the DT counterpart in Figure 9(c) and the realization, in Figure 9(d), where $H(z) = Y/X = 1 - z^{-1}$. Sketched in Figure 9(e), the filter's frequency response begins from zero at $f = 0$, where $z^{-1} \to 1$, and reaches 2 at $f = f_{CK}/2$, where $z^{-1} \to -1$.

Figure 9(d) implies that subtracting a delayed replica of a signal from itself performs high-pass filtering. This agrees with our previous results: a slowly varying signal and its delayed copy nearly cancel each other upon subtraction.

Our first-order HPF merits one more remark. We plot the step response as shown in Figure 10, recognizing that the two signals cancel each other except at $k = 0$. Thus, the step response is simply an impulse, revealing that $1 - z^{-1}$ represents a

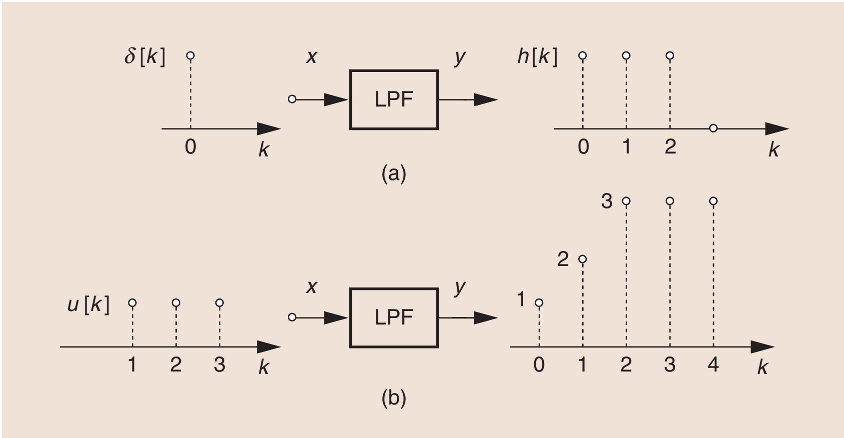

FIGURE 8: An LPF with greater impulse broadening: the (a) impulse response and (b) step response.

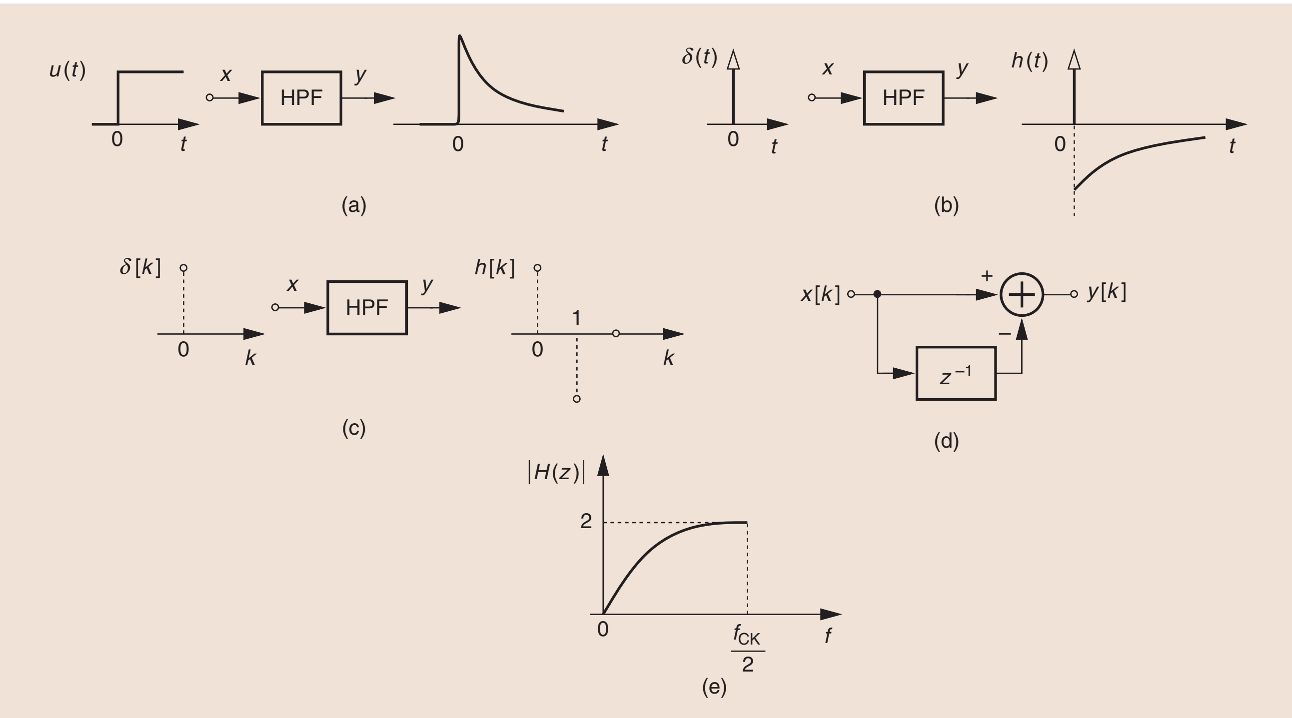

FIGURE 9: The (a) step response of a CT HPF, (b) resulting impulse response, (c) DT HPF impulse response, (d) DT HPF realization, and (e) DT HPF frequency response.

differentiator. As explained in the next section, this point paves the way for constructing integrators and noise-shaping loops.

Selecting a nonunity coefficient for z^{-1} alters the HPF response. For $1-\alpha z^{-1}$ and $0<\alpha<1$, $|H(z)|$ begins from $1-\alpha$ at $f=0$ and reaches $1+\alpha$ at $f=f_{CK}/2$, yielding less selectivity (Figure 11).

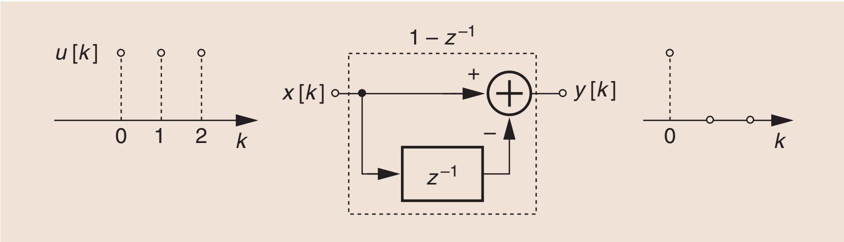

FIGURE 10: The step response of a first-order DT HPF.

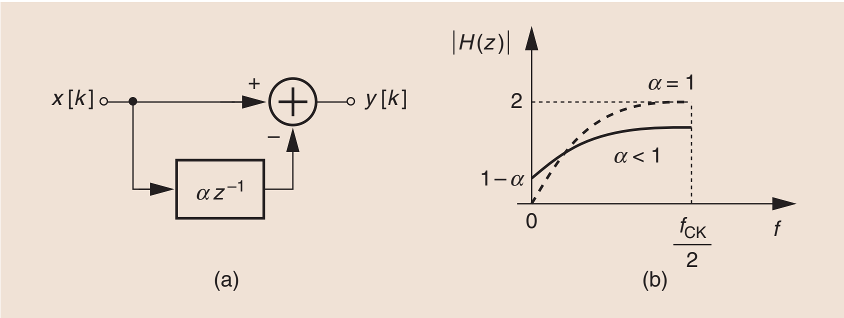

FIGURE 11: The DT HPF with (a) a shorter tail. (b) Its frequency response.

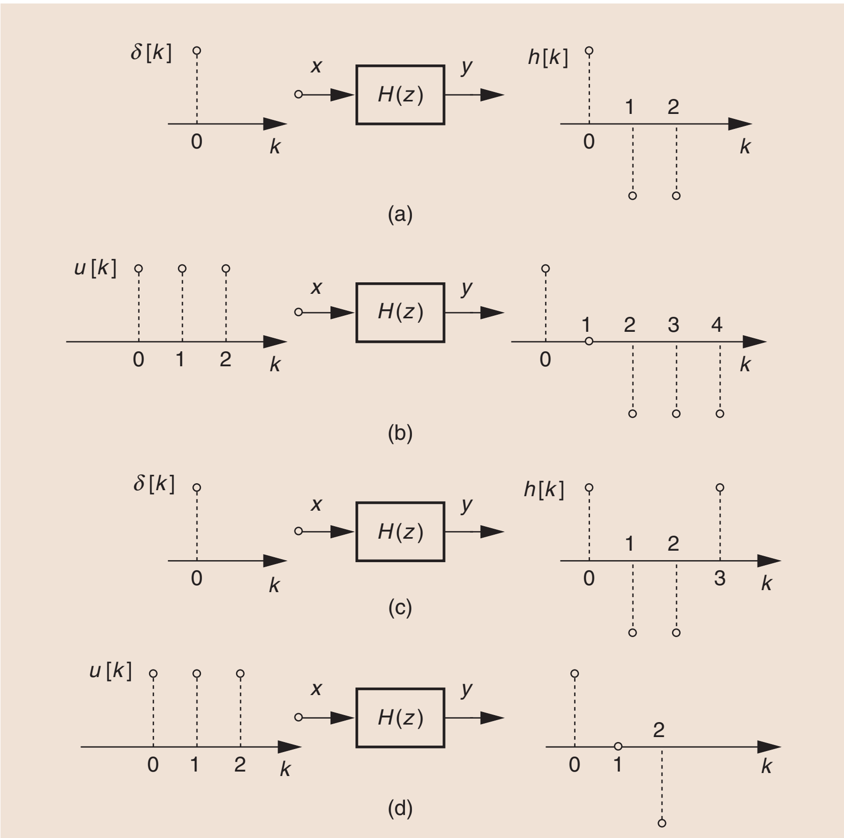

FIGURE 12: (a) A system exhibiting two negative tail impulses, (b) the step response of the system in (a), (c) a circuit with an impulse response consisting of two positive impulses and two negative impulses, and (d) the step response of the system in (c).

Let us now study the effect of broadening the impulse response in Figure 9(c). We add one more impulse at the output, Figure 12(a), recognizing that the step response

Most of our intuition will draw upon the z-transform of the unit-delay system.

[Figure 12(b)] exhibits a nonzero steady-state value and concluding that $1-z^{-1}-z^{-2}$ is not an HPF. This situation arises because $u[k], -u[k-1]$, and $-u[k-2]$ do not cancel in the steady state. We must therefore add one more positive impulse to the tail [Figure 12(c)] to obtain $H(z)=1-z^{-1}-z^{-2}+z^{-3}$ and hence an equal number of positive and negative steps. As shown in Figure 12(d), the steps cancel each other beyond $k=2$.

Interestingly, $H(z)=1-z^{-1}-z^{-2}+z^{-3}$ does not represent a simple HPF. In fact, we have $H(z)=1-z^{-1}-z^{-2}(1-z^{-1})=(1-z^{-1})^2(1+z^{-1})$, i.e., a cascade of two differentiators and one LPF. The circuit thus acts as a bandpass filter.

Integrators and Noise Shaping

As observed previously, a $1-z^{-1}$ system acts as a differentiator. Noting that the cascade of a differentiator and an integrator displays a unity transfer function [Figure 13(a)], we conclude that $(1-z^{-1})^{-1}$ represents integration. How do we realize this system using a unit delay? Since $1-z^{-1}$ appears in the denominator, we surmise that a feedback loop is necessary, arriving at the topology in Figure 13(b). We can view z^{-1} as a register that adds the present value of x to the previous value of y. From Figure 11, we can readily plot the integrator response, as shown in Figure 13(c) for the general case of $(1-\alpha z^{-1})^{-1}$. Such a system is called a *leaky* integrator if $\alpha<1$.

From another view point, a DT integrator adds its present input to the sum of its past inputs. That is,

$$y[k] = x[0] + x[1] + \cdots + x[k] \quad (17)$$
$$= y[k-1] + x[k]. \quad (18)$$

It follows that

$$Y(z) = z^{-1}Y(z) + X(z) \quad (19)$$

and hence

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1}}. \quad (20)$$

The circuit characterized by $(1 - z^{-1})^{-1}$ is called a *nondelaying* integrator. If the delay stage is placed in the forward path (Figure 14), we obtain $Y/X = z^{-1}/(1 - z^{-1})$ and call the structure a *delaying* integrator. Except for the one-cycle delay, this topology has the same properties as the nondelaying counterpart.

The very high gain provided by the integrator near $f = 0$ results from the fact that $z^{-1} \to 1$ and proves useful in many applications. For example, suppose a circuit experiences additive noise at its output [Figure 15(a)]. This can occur in the digital domain if the output of a register is truncated (quantized). We precede this stage by a delaying integrator and embed the cascade in a negative-feedback loop, as in Figure 15(b). The noise now experiences a transfer function given by

$$\frac{Y(z)}{N(z)} = \frac{1}{1 + \dfrac{z^{-1}}{1 - z^{-1}}} \quad (21)$$
$$= 1 - z^{-1}. \quad (22)$$

The differentiation action suppresses the low-frequency noise components. This occurs because the integrator in Figure 15(b) provides a high loop gain near $f = 0$, strongly opposing the injection by n. We say the noise spectrum is "shaped." Note also that, except for a one-cycle delay, the input–output transfer function is equal to unity.

Other Insights

Analyzing z-transforms in terms of the unit delay, z^{-1}, offers further

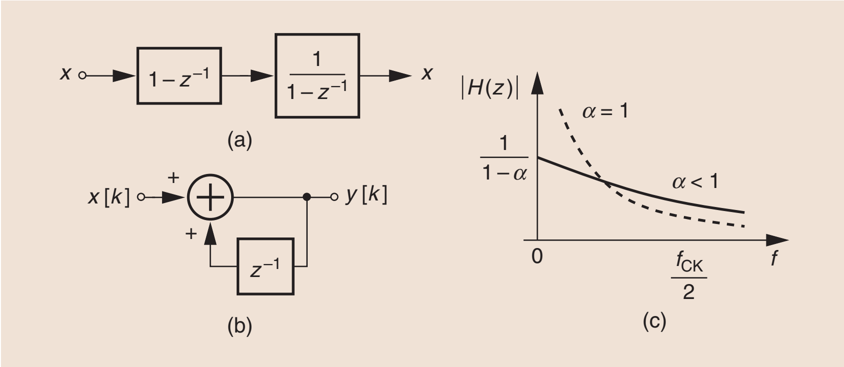

FIGURE 13: (a) The cascade of a differentiator and an integrator, (b) an integrator realization, and (c) the frequency response of the integrator.

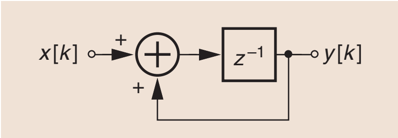

FIGURE 14: A delaying integrator.

insights. Let us return to LPFs and consider the two transfer functions $H_1(z) = 1 + z^{-1}$ and $H_2(z) = 1 + z^{-2}$. Recall from Figure 4 that a slowly varying signal and its delayed replica tend to add constructively. For $H_2(z)$, however, the signal is delayed by two clock cycles and must therefore vary even more slowly for constructive addition. This means that $H_2(z)$ exhibits a narrower bandwidth than $H_1(z)$ does [Figure 16(a)]. The slower step response in Figure 16(b) confirms this point as well.

Next, we study a $1 - z^{-2}$ system. Owing to the two-cycle delay, the input must vary even more slowly than in a $1 - z^{-1}$ system to cancel its replica. That is, the high-pass corner frequency is reduced. Moreover, writing $1 - z^{-2} = (1 - z^{-1})(1 + z^{-1})$, we recognize that this system comprises a cascade of an HPF and an LPF (Figure 17).

As another example, consider a $1 + z$ system. Since we have $1 + z = z(1 + z^{-1})$, the system advances (shifts to the right) the input signal by one clock cycle and then subjects it to an LPF. Similarly, $1 - z$ is equal to $z(1 - z^{-1})$ and hence equivalent to a one-cycle advancement followed by an HPF.

Stability

For CT systems, the transfer function's poles must remain in the left half s plane to ensure stability. Poles on the $j\omega$ axis lead to constant-amplitude sinusoidal oscillation, and

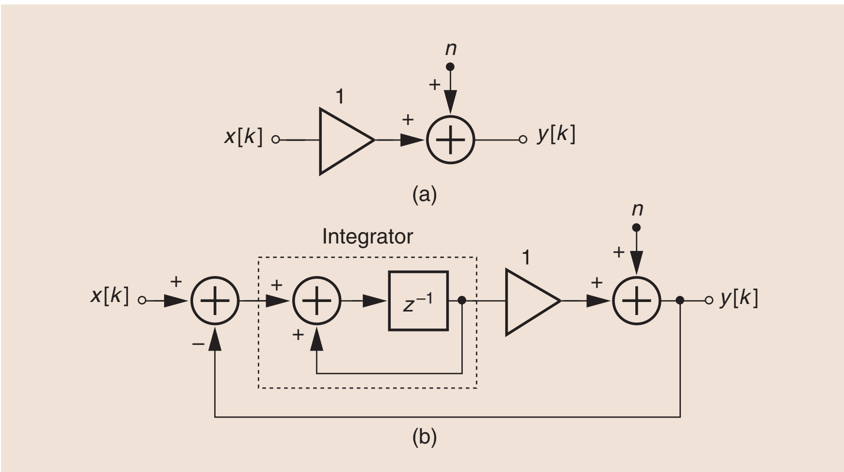

FIGURE 15: (a) A stage experiencing additive noise at its output and (b) the shaping of the noise spectrum through the use of negative feedback.

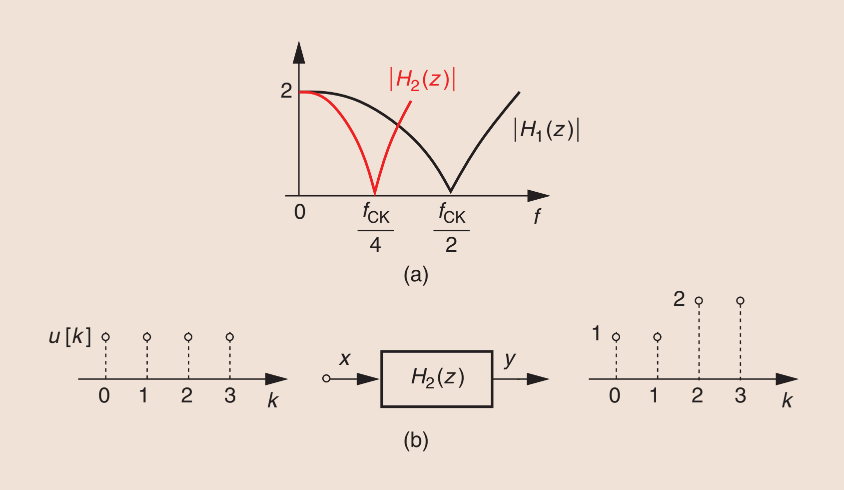

FIGURE 16: (a) The frequency responses of H_1 and H_2 and (b) the step response of H_2.

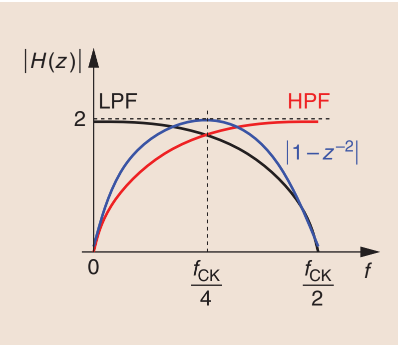

FIGURE 17: The frequency response of $1 - z^{-2}$.

those in the right half plane cause a growing sinusoidal response. We must now see how these phenomena manifest themselves in the z domain.

Recall that $z = e^{T_{CK}s}$, where $s = \sigma + j\omega$. If $\sigma = 0$, s appears on the $j\omega$ axis while $z = e^{T_{CK}j\omega}$ has a unity magnitude and is located on the unit circle [Figure 18(a)]. Thus, if the z-transform contains poles on the unit circle, the system exhibits constant-amplitude oscillation.

Next, we select $\sigma < 0$ to ensure stability in CT systems. Consequently, $z = e^{T_{CK}\sigma} e^{T_{CK}j\omega}$ has a magnitude less than unity [Figure 18(b)].

In our analysis of DT systems, we will often draw intuition from their CT counterparts.

This indicates that DT systems must be designed such that their z-transform displays poles only inside the unit circle. For example, the unit-delay stage is characterized by z^{-1} and hence a pole at the origin.

The third case occurs if $\sigma > 0$ and z falls outside the unit circle, Figure 18(c), causing oscillation with a growing amplitude. In other words, if the poles of $H(z)$ are too far from the origin, the system becomes unstable. Figure 18(d) depicts an example where a pole of $H(z)$ is varied from $z = 0$ along the real axis, and its counterpart in the s domain is shown. As the former crosses point A, the latter goes from the left half plane to the right half plane.

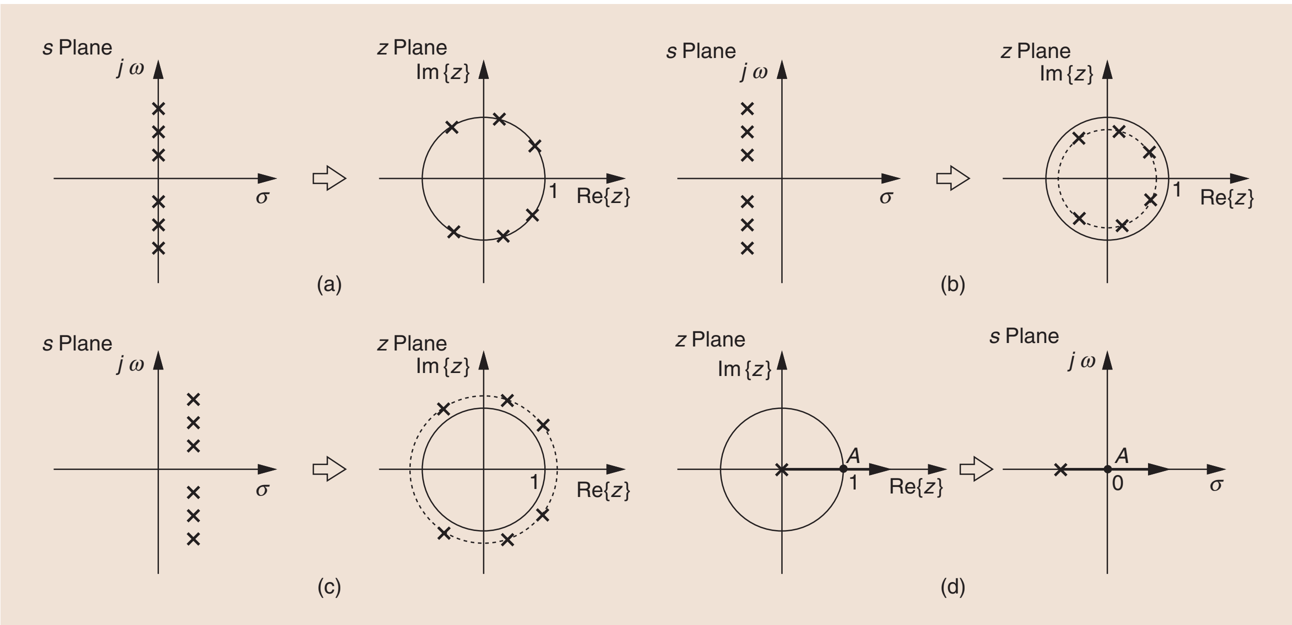

FIGURE 18: The (a) poles on the $j\omega$ axis mapping to the unit circle, (b) poles in the left half plane mapping to a region inside the unit circle, (c) poles in the right half plane mapping to a region outside the unit circle, and (d) correspondence between the real poles in the z and s planes.