Constellation Design for Improved Iterative LDPC Decoding

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Abstract— Recent advances in iteratively decoded channel codes such as low-density parity-check (LDPC) codes make it possible to operate close to channel capacity limits. The capacity of a coded modulation is therefore a very useful indicator when analyzing the performance of a near infinite-length code under maximum likelihood (ML) decoding. In this work, we analyze the behavior of LDPC codes of moderate lengths using belief propagation (BP) based decoding algorithms. We found that in many cases analyzing the capacity of the coded modulation is not enough to predict which constellation will yield the best error rate performance in a Gaussian noise channel. For the case of M-ary constellations with M=8 and 16 we show how a lower error rate can be achieved by making simple changes to constant envelope phase-shift keying (PSK) constellations. Modifying these constellations translates to an increase in the mean log-likelihood ratio of some of the received bits, resulting in an improvement in performance.

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1. INTRODUCTION

Shannon in [1] defined the channel capacity C as the maximum rate at which information can be transmitted over a noisy channel. Turbo-like codes and LDPC are known for their near-capacity performance. Low-density parity-check (LDPC) codes were proposed by Gallager in the early 1960s [2,3]. The structure of Gallager's codes (uniform column and row weight) led them to be called regular LDPC codes. Gallager provided simulation results for codes with block lengths on the order of hundreds of bits. However, these codes were too short to approach Shannon capacity (which is achievable in the limit of infinite block length). Furthermore, the computational resources to support longer random codes were decades away from being broadly accessible. In his original work, Gallager also introduced several decoding algorithms to decode LDPC codes, such as the sum-product or belief propagation (BP) algorithm that has since been used in factor graphs.

Following the ground breaking demonstration by Berrou *et al.* [4] of the impressive capacity-approaching capability of long random linear (turbo) codes, MacKay [5] re-established interest in LDPC codes during the mid to late 1990s. Luby *et al.* [6] formally showed that properly constructed irregular codes can approach capacity more closely than regular codes. As in the case of turbo codes, LDPC codes belong to the class of codes that can be efficiently decoded via *iterative* techniques. Chung *et al.* show in [7] how an LDPC code of length 10^7 can perform within 0.04 dB of the Shannon limit using a maximum of 2000 iterations.

A significant research effort is underway in the area of studying unequal error protection as a method of improving error rate performance. A common approach is to design an interleaver to compensate the unequal error protection of certain modulations with the irregular protection from different node degrees in an LDPC code. In [8] the authors propose a bit reliability (BR) mapping such that the most significant bit (MSB) of an M-ary constellation are interleaved to the higher order variable nodes and the least significant bit (LSB) are mapped to low degree nodes. This approach tends to protect the most reliable bit-planes by mapping them to higher level variable nodes that are less susceptible to transmission errors. Maddock in [9] introduces a search algorithm to determine an interleaver that will minimize the number of check nodes with more than one unreliable bit connected to them. At the same time their algorithm minimizes the number of check nodes with no unreliable bits connected to them. The method proposed in this work provides a similar effect as [8] by rearranging the constellation points in space instead of using an interleaver that is dependent on the code structure. The proposed constellations also outperformed M-ary phase-shift keying (M-PSK) when regular LDPC codes were used, and the method in [8] could not be applied.

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For codes of practical lengths using two dimensional constellations, we evaluated the error rate performance of iteratively decoded LDPC codes codes using a BP decoder using the model shown in Figure 1. We designed an M-ary constellation where half of the symbols had greater energy (and therefore greater error protection) than the other half. Constant envelope PSK constellations, with symbols uniformly distributed along a circle of radii $\rho = 1$, were converted to amplitude phase-shift keying (APSK) constellations with radii $\rho = \sqrt{1 \pm \delta}$.

The *capacity* of a channel is the tightest upper-bound on the amount of information that can be reliably transmitted over that channel. In [1], Shannon showed that the channel capacity of a given channel is the limiting information rate that can be achieved with arbitrarily small error probability. For an additive white gaussian channel (AWGN), the capacity limit can be achieved by an input signal with a gaussian distribution (not a practical modulation). The constrained modulation capacity (CMC) [10] indicates the information rate that can be achieved with a particular modulation. Higher values of CMC indicate that more information can be transmitted through a particular channel. The CMC can therefore be used as an indicator of the relative performance between two constellations, where modulations with larger CMC are likely to produce lower error rates. In [11], the authors develop a technique where capacity is used as a metric to careful place constellation points to design efficient constellations. The authors show how this method improves the bit error rate performance for PAM modulations using LDPC codes. In this work, we show that for some values of δ, PSK constellations with lower CMC had better error performance than constant envelope modulations with larger CMC. For M-ary constellations with M=8 and 16, we experienced a gain of between 0.2-0.5 dB by simply modifying the constellation points by choosing proper values of δ. Transmitting some symbols with additional energy results in larger log-likelihood ratios (LLRs) for the corresponding demodulated bits at the decoder. We make the conjecture that the overall code performance under iterative decoding can be improved when these larger LLRs are propagated through the graph. We will show how the increased bit reliabilities due to the additional energy, quickly compensates the least reliable information from bits corresponding to lower energy symbols.

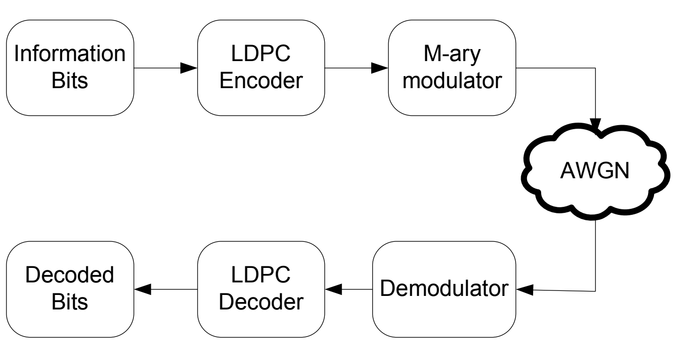


Figure 1. Block diagram of communication system

The rest of the paper is organized as follows. Section 2 analyzes the constrained capacity of different M-ary constellations. Section 3 describes the technique used to find the proposed constellations. Simulation results are presented in Section 4. The effects of using the proposed constellations in practical software-defined radios are considered in Section 5. Finally, Section 6 concludes the paper.

2. CONSTRAINED CAPACITY OF M-ARY CONSTELLATIONS

The capacity of a channel can be written as the mutual information between the channel's input signal X and output Y, maximized over all possible input distributions. The channel capacity for an AWGN channel with a one dimensional input can be expressed as:

$$C = max_{p(x)} \left[I(X;Y) \right] = \frac{1}{2} log_2 \left(1 + 2 \frac{r.E_b}{N_0} \right) \quad (1)$$

where E_b is the energy per information bit, r is the information rate and N_0 is the noise power spectral density [12]. For an M-ary modulation, the channel's input distribution is constrained, so no maximization in (1) is required. The CMC is simply the mutual information between the channel input and output. The mutual information can be measured by means of the following expectation:

$$\begin{aligned} C_m &= I(X;Y) = E_{X_k,n} \left[log(M) + log(p(x_k|y)) \right] \; [nats] \\ &= log_2(M) + \frac{E_{X_k,n}[\Lambda_k]}{log(2)} \; [bits] \end{aligned} \quad (2)$$

where Λ_k is the log-likelihood of symbol x_k [10]. Under ideal conditions, using a maximum likelihood (ML) decoder, a constellation with higher capacity should outperform one of lower capacity. Therefore, as we mentioned in Section 1, the constrained capacity of a particular modulation is therefore a good indicator of the relative performance between two constellations [10, 12]. The expectation in (2) was computed for different M-ary constellations through Monte Carlo simulation using [13]. Figure 2 shows the capacity (in bits per symbol) as a function of signal-to-noise ratio (SNR). Consider for M=16, a classic 4-12-APSK constellation (sometimes denoted as 12-4-APSK), with symbols uniformly distributed in two concentric rings with $\rho_1/\rho_2 = 3.15$. This constellation has an outer ring of radii ρ_2 with 12 symmetrically-distributed symbols, and a smaller one with 4 symmetric symbols. We can see in Figure 2 how this constellation has the largest capacity for all SNRs. On the other hand, the relative CMC of 16-PSK and 8-8-APSK varies as a function of SNR. For M=8 the capacity of 8-PSK is always larger than the 4-4-APSK case.

In Figure 3, different error rate performance curves are shown for the capacity scenarios shown in Figure 2. For the case of M=8 although the CMC of 8-PSK is higher, the bit error rate (BER) curves of Figure 3(a) show a 0.2 dB gain at BER=10^{-5} when 4-4-APSK is used. Lower error rates were explored by means of a field-programmable gate array (FPGA) LDPC decoder using DVB-S2 codes [14]. For this case, shown in

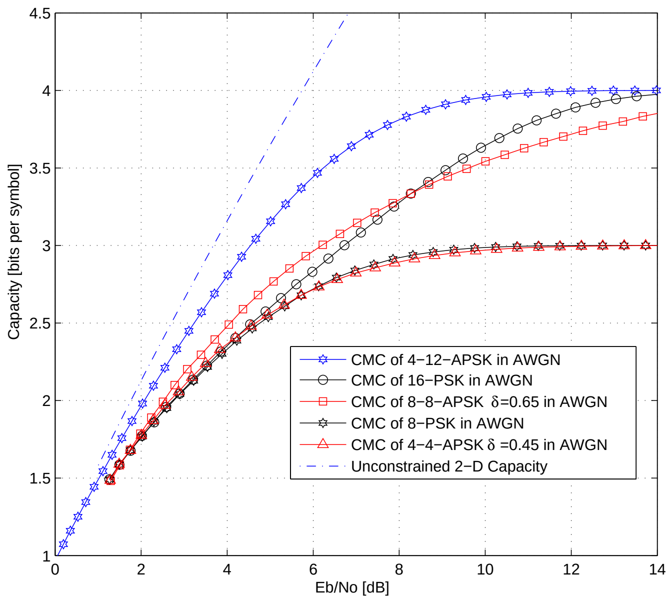


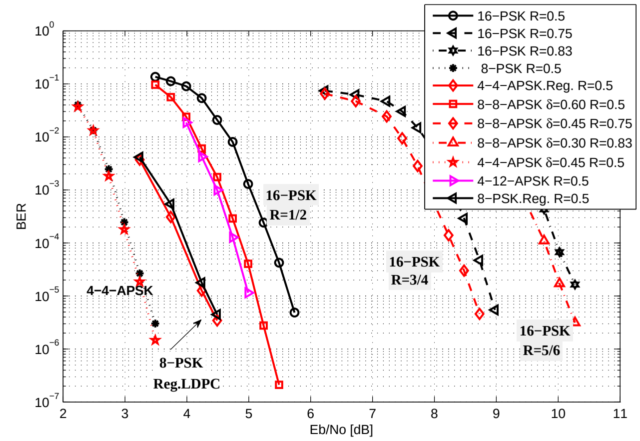
Figure 2. Constrained Modulation capacity as a function of E_b/N_0

Figure 3(b), a performance gain of 0.4 dB was achieved at BER=10^{-7} using 15 iterations and a length n =1600 code. For M=16, the CMC of 16-APSK with $\delta = 0.65$ is higher than that of 16-PSK at low SNRs, and for higher SNRs, the opposite effect occurs. However, the BER plots show that the modified constellations always outperform M-PSK for all SNRs. In both Figures 2 and 3(a), we explicitly included the case of a 4-12-APSK constellation whose CMC is much larger than the other 16-ary cases. We can see that the large gap in capacity cannot be compensated by manipulating the constellation. For this reason, the error rate performance of 4-12-APSK outperforms the other two cases.

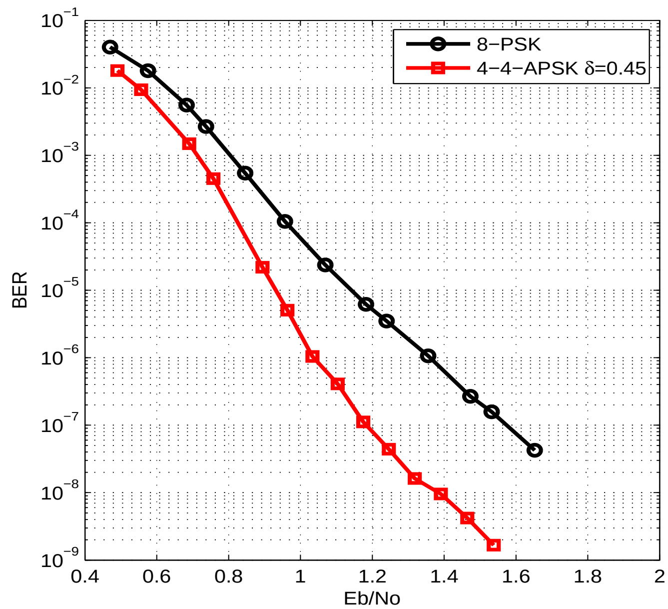
As we explained in Section 1, we believe that the increase in magnitude of LLRs corresponding to symbols with higher energy is responsible for the gain in error rate performance. Highly reliable beliefs propagate through the graph correcting unreliable information. In Figure 4, we show how the values of LLRs in the graph increase for the modified constellations. The increase in reliabilities occurs for all variable node degrees in the IEEE 802.11n $(n, k) = (1944, 972)$ code. Not only are the final values of the LLRs larger for the case of $\delta = 0.6$, but also the convergence rate is significantly faster for the modified constellations. This effect can be also appreciated in Figure 6.

3. Constellation Modification Method

For the case of M-ary modulations, the different transmitted bits $(c_{m-1}, \ldots, c_1, c_0)$, $m = log_2(M)$ have different levels of error protection. For BPSK, every bit c_0 is mapped to two possible symbols $S = \{-\rho, \rho\}$ on the real axis. Since both bits are equally protected, enhancing the energy of one of these two symbols implies equally reducing the energy of



(a)802.11n LDPC codes. Iterations=50, code length $n = 1944$.



(b)DVB-S2 LDPC codes. Iterations=15, code length $n = 16200$.

Figure 3. Bit error rate performance for different constellations, codes, and information rates.

the other symbol. The same analysis applies to QPSK where the two bits that form every symbol lie on the same bit-planes. However, for higher order constellations with $m > 2$, c_{m-1} and c_{m-2} have equal protection but the remaining bits do not.

Traditional M-PSK constellations have symbols uniformly distributed along a circle of radii $\rho = 1$. Reducing the energy of $M/2$ symbols by moving them to a ring with $\rho = \sqrt{1-\delta}$ and enhancing the energy of the remaining $M/2$ symbols (keeping the total energy constant) by translating them to a larger ring with $\rho = \sqrt{1+\delta}$ results in a $(M/2) - (M/2)$-APSK constellation. The two concentric rings in this new APSK constellation, unlike classic APSK where all points are uniformly distributed in space, have symbols that are not uniformly distributed within the two rings.

The rearrangement of the constellation was done with the

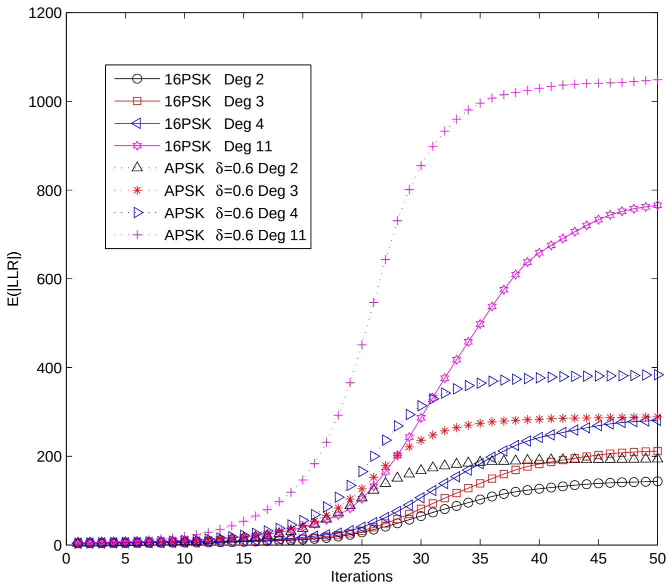
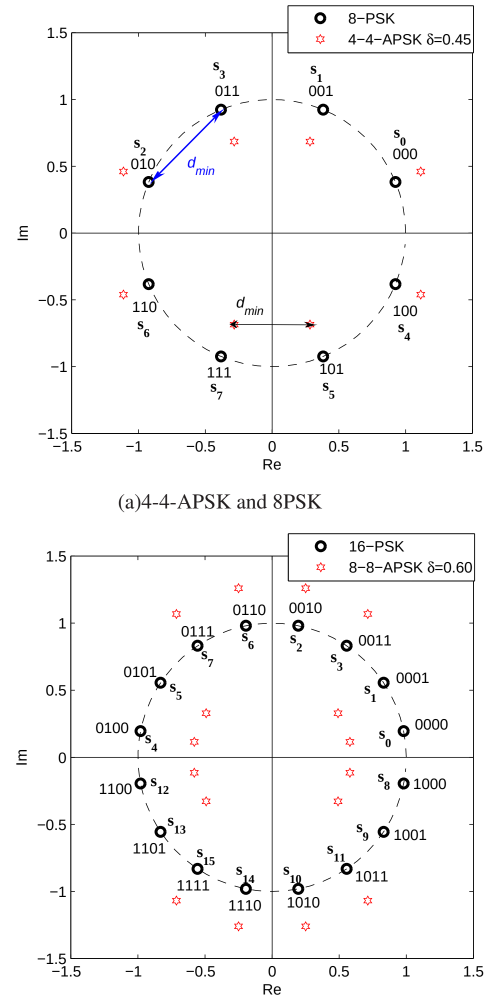


Figure 4. Mean log-likelihood ratios (LLRs) for a (1944,972) LDPC code using 16-PSK and 8-8-APSK for E_b/N_0=7.5 dB. All four variable-node degrees present in the LDPC code are shown. In all cases the LLR values of APSK modulated nodes increases faster and converges to a larger value than its 16-PSK counterpart.



(a)4-4-APSK and 8PSK

(b)8-8-APSK and 16PSK

Figure 5. Proposed APSK constellations compared with traditional M-PSK .

goal of providing extra protection to the MSB of the constellation. Figure 5(a) shows that for an 8-PSK constellation with symbols $s(i) = [c_2(i), c_1(i), c_0(i)]$, the energy for c_2 was augmented by increasing the power of the real component of symbols $s(0), s(2), s(4)$ and $s(6)$. The same BER performance is obtained by increasing the reliability of c_1 since, for 8-PSK, both c_1 and c_2 are equally protected. For the 16-PSK case, a similar criterion was used. Note that in Figure 5(b) the symbols with smallest minimum distance (d_{min}) differ in the LSB: c_0.

Rearranging an M-PSK constellation by moving some symbols together while spreading the rest apart will decrease the d_{min} of the constellation. For the M=8 constellations in Figure 5(b), d_{min}=0.7653 for 8-PSK and d_{min}=0.5676 for 4-4-APSK with δ=0.45. For M=16, d_{min}=0.39018 for 16-PSK and d_{min}=0.23083 for 8-8-APSK with δ=0.65. The reduction d_{min} causes a significant error rate degradation when these APSK constellations are used in uncoded scenarios. For uncoded scenarios Foschini *et al.* [15] develops a numerical method to design unequally spaced constellations. However when an *iterative* error-correcting code is used, the decrease in minimum distance is compensated by the BP decoder which propagates messages with stronger LLRs (corresponding to the symbols with increased power). These reliable LLRs, quickly propagate throughout the graph and correct weaker beliefs from symbols whose energy was reduced by the constellation modification. The overall effect is a larger reliability of the received bits, as shown in Figure 4.

The M-ary constellations that we use in this work are by no means the best possible option among all possible two-dimensional constellations (from a BER sense). For large values of M the complexity of analyzing all possible combinations of M points in space becomes intractable. Our goal is to show how some simple modifications to a given constellation can produce an error rate improvement at no extra cost.

The value of δ that offers the best error rate performance for constellations created with our proposed method was found by a search algorithm. The value of δ depends on the constellation size M and on the rate of the code ($r = k/n$). In general, we found that for a given modulation and information rate, the same value of works well under all SNRs. Some experimental values of δ are shown in Table 1 for the constellations in Figure 5 and different information rates. The constellation with M=8 used a similar value of δ for all code rates whereas for the larger constellation, δ was reduced as the code rate increased approaching an uncoded scenario.

Figure 6 shows the frame error rate (FER) behavior of a length n=1944, irregular LDPC code from [16], as a function of δ for two different code rates: $R = 1/2$ and $R = 3/4$. As δ increases ($\delta = 0$ corresponds to classic 16-PSK and $\delta \neq 0$ corresponds to 8-8-APSK) , the d_{min} of the constellation decreases. The FER initially decreases with larger δ's up to a point where the d_{min} of the constellation is so small that the error events corresponding to symbols at d_{min} dominate the overall performance. Note how in Figure 6(a) for $\delta > 0.7$ the FER rapidly increases. Similarly for the rate 3/4 code in Figure 6(b), $\delta > 0.5$ has negative effects on FER. The results in Figure 6 were obtained using the length n=1944 codes from the IEEE 802.11n standard [16]. We also tested these constellations for the different code lengths in the standard observing similar results. The performance of length n=16200 DBV-S2 codes [14] was evaluated using an FPGA LDPC decoder. For this case, a gain of 0.4 dB was observed for 8-ary constellations at a BER=10^{-7} using 15 iterations.

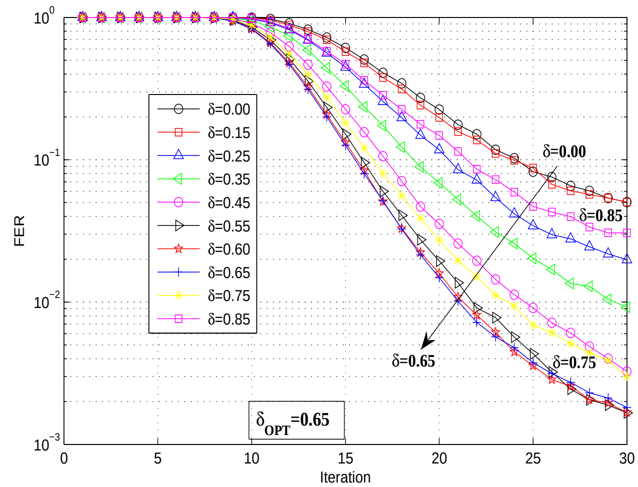
δ	M=$\{2, 4\}$	M=8	M=16
k/n=5/6	0	0.45	0.30
k/n=3/4	0	0.45	0.45
k/n=1/2	0	0.45	0.60

Table 1. Experimental values of δ for different information rates and constellation size.

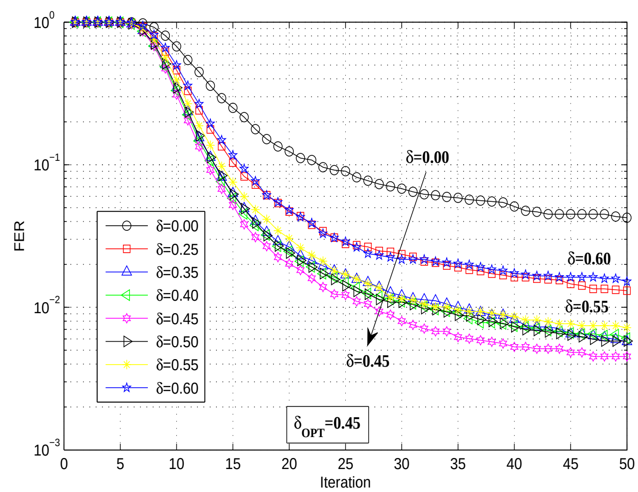
4. SIMULATION RESULTS

A block diagram of the system that was simulated appears in Figure 1. An M-ary modulator takes $m = log_2(M)$ code bits and maps them to a complex symbol using a given labeling scheme. Gray-labeled systems have been shown to perform better than natural labeled systems and are therefore used throughout this work [8, 17]. Following the notation from [8], we use a consecutive-bit (CB) bit mapping strategy. In CB mapping, m consecutive bits from a binary codeword are mapped (without interleaving) to a constellation symbol. The BR technique proposed in [8] and introduced in Section 1, was also tested in conjunction with our technique, but offered no performance improvement for the scenarios shown in Figure 3. For our system model, AWGN noise with zero mean and variance $\sigma^2 = N_0/2$ per dimension was assumed. A maximum $a - posteriori$ (MAP) demodulator generates bit metrics based on the received channel symbols. These metrics are finally passed on to an iterative decoder.

Using the proposed constellation design method, Figure 6 shows the FER as a function of decoder iterations using an LDPC code. We can see that for a target FER, the best choice of δ uses a much smaller number of iterations. For a receiver operating at a E_b/N_0 of 5.00 dB, the FER rate decreases by more than an order of magnitude when the best possible δ is chosen. When BER is plotted as a function of E_b/N_0 in Figure 3, the SNR improvement is $\approx$ 0.5 dB for the rate 1/2 code and 0.3 dB for the rate 3/4 code using M=16, for a length n=1944 code. For M=8 a gain of 0.2 dB was observed at BER=10^{-5} for a length n=1944 code and 0.4 dB



(a)16-PSK($\delta = 0$) and 8-8-APSK ($\delta \neq 0$) using a (n,k)=(1944,972) LDPC code at E_b/N_0=5.00 dB.



(b)16-PSK($\delta = 0$) and 8-8-APSK ($\delta \neq 0$) using a (n,k)=(1944,1458) LDPC code at E_b/N_0=3.23 dB.

Figure 6. Frame error rate (FER) vs. Iterations for different code rates and $M = 16$.

for the DVB-S2 codes of length 16200.

5. HARDWARE IMPLEMENTATION

Belief propagation based iterative decoders operate by sequentially interchanging LLRs of message reliabilities. These LLRs are computed based on the energy of received symbols. An LLR indicates both the value of a bit (*i.e.*, '0' or '1') and the confidence in that metric. The closer the received symbol is to one of the constellation points, the higher the confidence. An LLR can be defined as,

$$LLR_k(r) = \ln \left(\frac{\sum_{i=0, i_k=\mathbf{1}}^{M} \exp(-\frac{E_b}{N_0} |C_i - r|^2)}{\sum_{i=0, i_k=\mathbf{0}}^{M} \exp(-\frac{E_b}{N_0} |C_i - r|^2)} \right) \quad (3)$$

where i_k is the k^{th} bit of symbol C_i. The ratio in (3) represents the probability that the k^{th} bit of the received symbol r is a '1' to the probability it is a '0'.

Almost all systems in use today use some sort of approximation to compute LLRs in real time. The regular and symmetric structure of most standard constellations makes this computation straightforward. The computation becomes less trivial if the constellation is not known or may change during the system operation. For the case of software defined ratios (SDRs), the communication system must be able to operate across large frequency spectrums, different modulations and codes rates. This is done by means of software-controlled programmable hardware. As we show in Figure 7, computing LLRs for these different scenarios requires the use of lookup tables. Using the proposed constellations with SDRs requires no additional complexity at the receiver.

An additional advantage of the proposed APSK constellations appears when carrier tracking is considered. Let ψ be the rotational invariance angle of the constellation. For M-PSK constellations we have that $\psi_{M-PSK} = 2\pi/M$. Let γ_c be the maximum carrier-phase offset that the system can handle without severely degrading its BER performance. A carrier recovery loop can usually track angles $\gamma_c < \frac{\psi}{2}$ without the use of outside information to correct phase ambiguity issues. The proposed modification to traditional M-PSK constellations cause ψ to increase from $\psi_{8PSK} = \pi/4$ and $\psi_{16PSK} = \pi/8$ to $\psi_{APSK} = \pi$ which is simpler to track [18].

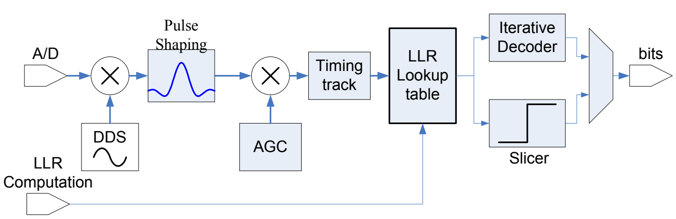


Figure 7. Schematic diagram of hardware implementation

6. Conclusion

A study of the performance of LDPC and turbo-coded modulation systems with M-PSK and APSK modulations was performed. The proposed constellations not only enhance the error rate performance under ideal AWGN scenarios where no carrier tracking is required, but also could provide enhanced carrier tracking by increasing the rotational invariance of the constellation. The proposed constellations were designed using the conjecture that the performance of iterative decoders can be enhanced by increasing (to some extent) the energy of some symbols, even if this implies a reduction in the minimum distance of the constellation. For low code-rates, large values of δ offer the best error rate performance. As the code rates increases and the effect of the error correcting code decreases, the values of δ must be reduced. The results shown in this work imply that when iterative decoding is used with M-ary constellations, scenarios of smaller CMC can sometimes outperform those with larger CMC. The results shown in this work are based on the modification of the radii of constant envelope constellations. Future work involves the study of this effect for non-constant envelope constellations such as QAM, which has not been thoroughly investigated yet.

Acknowledgment

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References

[1] C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, vol. 27, pp. 379–423 and 623–656, July and Oct. 1948.

[2] R. G. Gallager, "Low-density parity-check codes," *IRE Trans. Inform. Theory*, vol. IT-8, pp. 21–28, Jan. 1962.

[3] ——, *Low-density parity-check codes*. Cambridge, MA: MIT Press, 1963.

[4] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo codes," *Proc. IEEE Int. Conf. on Comm. (ICC)*, pp. 1064–1070, May 1993.

[5] D. J. C. MacKay, "Good error-correcting codes based on very sparse matrices," *IEEE Trans. on Inform. Theory*, vol. 45, pp. 399–431, Mar. 1999.

[6] M. Luby, M. Mitzenmacher, A. Shokrollahi, and D. Spielman, "Improved low-density parity-check codes using irregular graphs," *IEEE Trans. on Inform. Theory*, vol. 47, pp. 585–598, Feb. 2001.

[7] S. Chung, T. Richardson, and R.Urbanke, "On the design of low-density parity-check codes within 0.0045 db of the shannon limit," *IEEE Comm. Letters*, vol. 5, no. 2, pp. 58–60, Feb. 2001.

[8] Y. Li and W. E. Ryan, "Bit-reliability mapping in LDPC-coded systems," *IEEE Comm. Letters*, vol. 9, no. 1, pp. 1–3, Jan. 2005.

[9] R. Maddock and A. Banihashemi, "Reliability-based coded modulation with LDPC codes," *IEEE Trans. on Comm.*, vol. 54, no. 3, pp. 403–406, Mar. 2006.

[10] G. Caire, G. Taricco, and E. Biglieri, "Bit-interleaved coded modulation," *IEEE Trans. on Inform. Theory*, vol. 44, no. 3, pp. 927–946, May 1998.

[11] M. Barsoum, C. Jones, and M. Fitz, "Constellation design via capacity maximization," *Proc. IEEE Int. Symp. on Inf. Theory*, 24-29, Jun. 2007.

[12] T. Cover and J. Thomas, *Elements of Information Theory*. Wiley and Sons, New York, 2^{nd} Ed. 2006.

[13] M. Valenti, "Iterative solutions coded modulation library (ISCML)." [Online]. Available: http://www.iterativesolutions.com

[14] ETSI, "Digital video broadcasting (DBV) second generation framing structure for broadband satellite applications," EN 302-307 V1.1.1. [Online]. Available: www.dvb.org.

[15] G. Foschini, R. Gitlin, and S. Weinstein, "Optimization for two-dimensional signal constellations in the presence of gaussian noise," *IEEE Trans. on Comm.*, vol. 22, pp. 28–38, Jan 1974.

[16] *IEEE P802.11n/D1.05 Wireless LAN Medium Access Control(MAC) and Physical Layer (PHY) specifications - Enhancements for Higher Throughput*. (Draft), October 2006.

[17] S. ten Brink, "Designing iterative decoding schemes with the extrinsic information transfer chart," *AEO Int. J. Electron. Commun.*, vol. 54/6, pp. 389–398, 2000.

[18] E. L. Vallés, "Timing recovery using soft information feedback and efficiency of array codes," Ph.D. dissertation, University of California, Los Angeles, Mar. 2007.

BIOGRAPHY

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