F. Chen

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PLASMA PHYSICS LABORATORY

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PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

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AEC RESEARCH AND DEVELOPMENT REPORT

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INTRODUCTION

An instability of partially ionized gases subject to an electric field perpendicular to a strong magnetic field has recently been discovered by Simon[1] and by Hoh.[2] This mechanism seems to explain satisfactorily the low-frequency oscillations and anomalous transport of plasma across a magnetic field observed by the author and his co-workers in hot-cathode discharges. These results will be reported in another paper. Similar low-frequency fluctuations are found also in fully ionized gases suffering from anomalous diffusion, such as in a stellarator; and it is the purpose of this paper to investigate whether or not a similar mechanism of instability may be operative in the absence of neutral atoms.

The physical reason for the instability is most easily explained with reference to the case of partial ionization. Imagine a cylindrical, isothermal plasma column, as shown in Fig. 1, immersed in a constant and uniform magnetic field B and subject to a zero-order electric field E_o and density gradient ∇n_o, both directed toward the axis. This choice of sign is normal for ∇n_o, but it is normal for E_o only if an external potential is applied, as in a reflex discharge, or in a magnetic field strong enough that the ions diffuse across B faster than the electrons. In the absence of collisions, both ions and electrons would drift with a velocity v_o in the θ-direction because of E_o. In addition, the ions and electrons have a drift in opposite directions due to the pressure gradient. This produces an azimuthal current such that $\underset{\sim}{j} \times \underset{\sim}{B} = \underset{\sim}{\nabla} p$. As is well known, this differential drift cannot cause an instability because it is always perpendicular to the density gradient.

If now we admit collisions with neutral atoms, the drift due to E is modified in two ways. There is first a drift in the direction of E of magnitude $v_\mu = \mu_\perp E$, where $\mu_\perp$ is the perpendicular mobility coefficient and is of opposite sign for the two species. Secondly, there is a decrease in v_o because of the collisions; this decrease is usually larger for ions than for electrons, and therefore a relative drift occurs between the two species. Both effects tend to create a charge separation, and the first effect, v_μ, is approximately $\omega_c \tau$ times larger than the second, Δv_o, where ω_c is the ion gyrofrequency and τ the ion-neutral collision time.

We assume that the zero-order conditions are maintained by appropriate sources and inquire what will happen to a density perturbation. It is easy to convince oneself that an axially symmetric perturbation will cause a charge separation but no instability. If a perturbation in the θ-direction occurs, however, the differential drift Δv_o will cause a charge separation, as shown in Fig. 1. An azimuthal electric field E_1 then grows until the drift $\mu_\perp E_1$ is just sufficient to stop the separation of charge. The secondary effect of E_1, however, is to produce a radial drift v_1 in the same direction for both species. If the zero-order density gradient is in the direction shown, this radial drift brings a region of higher density into a region where the density perturbation is already positive, and vice versa; and hence the perturbation grows until the zero-order gradient is destroyed. The growth rate is of course damped by collisional diffusion both across B and along B, if the wavelength in the direction of B is finite. Near the threshold of instability, one might expect to observe discrete frequencies at harmonics of the rotation frequency. Well above the threshold, one would expect the plasma to break up into a distribution of eddies of different sizes, with lifetimes determined by the rates of classical diffusion. An enhanced transport of particles across B then results from the fluctuating azimuthal electric fields; or, to put it another way, the instability makes a large density gradient impossible in the interior of the plasma, and hence there must be a large gradient at the edge, so that classical diffusion can remove the particles at a fast rate.

This mechanism differs from the helical instability of Kadomtsev and Nedospasov[3] in two main respects. First, the value of $\omega_c \tau$ must be finite here but may be zero in the case of Ref. 3. Second, in the latter case, an azimuthal charge separation is caused by the longitudinal displacement of a helix; here, the charge separation is caused by a differential rotation, and hence no helix and no currents or gradients in the B direction are necessary.

We wish to point out several interesting features of the mechanism described qualitatively above. First, since a finite wavelength along B adds to the damping, those perturbations with very long parallel wavelengths will grow fastest. This is in agreement with measurements both in a reflex discharge[4] and in a stellarator.[5] Furthermore, if the diffusion process can be described in terms of random walks,[6] the selection of long parallel wavelengths automatically ensures that electrons and ions diffuse out at the same rate, since the streaming of electrons along the field cannot, in this case, change their rate of random walking. Second, since the frequencies depend on the rate of diffusion and the speed of rotation, they would be expected to lie in a low-frequency band not related to ω_{pi} or ω_{ci}. Since high frequencies are associated with small eddies, which are more easily damped by diffusion, the frequency spectrum would be expected to decrease with frequency. This is in qualitative agreement with the observations.[4,5] These measurements of parallel wavelength and frequency would seem to eliminate ion waves propagating along B as a possible cause of the electrostatic fluctuations. Third, this instability is found from a parabolic differential equation, not a wave equation, and hence does not involve the synchronism between particles and waves. As a consequence, a nonlinear limit to the amplitude of the fluctu ations, of the type considered by Drummond and Pines,[7] does not apply. Finally, perhaps the most interesting aspect of this mechanism is the correlation between the density fluctuations and the velocity v_1 (Fig. 1). This correlation is such that v_1 is always outward where the density is above average and inward where the density is below average. Anomalous transport can therefore result from coherent, as well as random, fluctuations. In the analysis of Spitzer,[6] no account was taken of the possible relation between the fluctuating electric field and the positions of the particles creating this field; with this mechanism, it is possible to take this correlation into account. This type of turbulence can be illustrated by the example of a block of ice suspended above a hot radiator in such a way that the average velocity of the air between them is zero. Yet there is a net transport of heat upwards because of the correlation between the heat content of an eddy and its direction of motion.

FUNDAMENTAL EQUATIONS

We now inquire whether a similar mechanism of instability can occur in a fully ionized gas. It is clear from the physical picture that what is needed is a difference of drift velocity between ions and electrons rotating under a radial electric field. In the absence of collisions with neutrals, there are two effects which may cause such a differential rotation: the viscosity and the centrifugal force. The latter has the added attraction that it is always in the same direction regardless of the direction of the electric field, and this direction is the proper one to produce the instability. In addition, a differential drift can occur because of gradients in B. This has been studied extensively by other authors and will not be considered here.

We shall employ the macroscopic equation of motion for the (singly charged) ion and electron fluids and the equations of continuity, as follows:

$$ne(\underset{\sim}{E} + \underset{\sim}{v}_i \times \underset{\sim}{B}) - \underset{\sim}{\nabla} p_i = Mn\underset{\sim}{v}_i \cdot \underset{\sim}{\nabla}\underset{\sim}{v}_i + \underset{\sim}{\nabla}\cdot\underset{\sim}{\pi}_i \tag{1}$$

$$-ne(\underset{\sim}{E} + \underset{\sim}{v}_e \times \underset{\sim}{B}) - \underset{\sim}{\nabla} p_e = 0 \tag{2}$$

$$\frac{\partial n}{\partial t} + \underset{\sim}{\nabla}\cdot(n\underset{\sim}{v}_i) = 0 \tag{3}$$

$$\frac{\partial n}{\partial t} + \underset{\sim}{\nabla}\cdot(n\underset{\sim}{v}_e) = 0 \tag{4}$$

Here $\underset{\sim}{\pi}_i$ is the ion viscosity tensor given, for instance, by Bernstein and Trehan.[8] The two terms on the right of Eq. 1 represent the effects of centrifugal force and viscosity, which are normally neglected. We have neglected the corresponding effects for the electrons, because of their small mass, and have assumed the density to be so small that B is constant and uniform and that the frequency is so small that a) quasi-neutrality obtains, b) Poisson's equation need not be used, and c) the inertia terms can be neglected. We now further assume that the fluctuations are electrostatic and the temperature uniform, so that $\underset{\sim}{E} = -\underset{\sim}{\nabla}\phi$ and $\underset{\sim}{\nabla} p_{i,e} = KT_{i,e}\underset{\sim}{\nabla} n$.

We then assume an essentially azimuthal perturbation of the form

$$n_1 = \nu n_o(r) \exp i(m\theta - \omega t) \tag{5}$$

$$\phi_1 = \bar{\phi} \exp i(m\theta - \omega t) \quad , \tag{6}$$

where ν is a real constant and $\bar{\phi}$ a complex one. We have thus assumed that n_1 has the same radial dependence as n_o (which cannot be true near the axis) and that the radial dependence of $\bar{\phi}$ can be neglected; these approximations should not be serious. Eqs. 1-4 are then linearized about an axially symmetric equilibrium and combined in the usual fashion so that Eqs. 3 and 4 form two equations for ν and $\bar{\phi}$ whose determinant gives the value of ω when set equal to 0.

VISCOSITY ALONE

We first consider the case in which the $\underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}$ term may be neglected. For simplicity we confine the discussion to a plane geometry in which all zero-order gradients are in the x direction, with B in the z direction. If $\alpha = \omega_c \tau$ is much larger than 1, the zero-order viscosity term is approximately[8]

$$(\underset{\sim}{\nabla}\cdot\underset{\sim}{\pi})^{(o)} = -\mu\left(\frac{1}{3} v_x^{(o)'} + \frac{1}{2\alpha} v_y^{(o)'}\right)' \hat{x} - \mu\left(\frac{1}{\alpha^2} v_y^{(o)'} - \frac{2}{\alpha} v_x^{(o)'}\right)' \hat{y} \tag{7}$$

where $\mu = \frac{1}{3} nKT_i \tau$ is the classical viscosity coefficient, τ is the ion-ion collision time, and a prime denotes differentiation in x. Following Simon,[9] we perform a simultaneous expansion in the two small parameters α^{-1} and ϵ, the ratio of the ion Larmor radius r_{Li} to the length of the macroscopic gradients; Eq. 1 then yields in zero-order

$$v_y^{(o)} = \frac{1}{B} \left(\frac{KT_i}{e} \frac{n_o'}{n_o} - E_o \right) \tag{8}$$

$$v_x^{(o)} = \frac{\mu}{n_o eB} \left(\frac{1}{\alpha^2} v_y^{(o)'} \right)' \tag{9}$$

$$\Delta v_y^{(o)} = \frac{-\mu}{2n_o eB} \left(\frac{1}{\alpha} v_y^{(o)'} \right)' \tag{10}$$

Here $v_y^{(o)}$ is the ordinary ion drift across B and $\Delta v_y^{(o)}$ the change in this drift due to the viscosity. Since both μ and α contain a factor of τ, Eq. (10) shows that $\Delta v_y^{(o)}$ does not depend on collisions. It is instead an effect of the finite Larmor radius of the ions, which brings them into regions of different $v_y^{(o)}$ during a gyration.

Since the electrons are assumed to make no collisions, they do not diffuse across B; hence the ion diffusion velocity $v_x^{(o)}$ must also vanish. Eqs. (8) and (9) then give a second order differential equation for E_o if n_o' is prescribed. If E_o is such that $v_y^{(o)}$ vanishes, the plasma is static; and none of the effects considered in this paper will occur. If E_o is such that $v_y^{(o)'}$ vanishes (no shear), both $v_x^{(o)}$ and $\Delta v_y^{(o)}$ vanish, and the instability under consideration cannot occur because the ions and electrons drift at the same speed. However, E_o can be such that $v_x^{(o)}$ vanishes while $\Delta v_y^{(o)}$ remains finite, due to the fact that α^{-1} is proportional to n for a fully ionized gas. In this case the instability will occur if $\Delta v_y^{(o)}$ is of the proper sign; moreover, the smallness of μ will not limit the amplitude, since μ also provides the only damping mechanism considered. We have not computed the growth rate because the value of $\Delta v_y^{(o)}$ is quite arbitrary, even if the equation $v_x^{(o)} = 0$ must be satisfied with a prescribed potential drop in the plasma.

It is dubious that a large shear velocity can be maintained in the plasma and we shall henceforth assume that $v_y^{(o)\prime}$ vanishes and hence that $(\underset{\sim}{\nabla}\cdot\underset{\sim}{\pi})^{(o)}$ is zero.

CENTRIFUGAL FORCE ALONE

If $\underset{\sim}{\nabla}\cdot\underset{\sim}{\pi}$ may be neglected in Eq. (1), we find that in zero-order, the term $Mn_o\underset{\sim}{v}^{(o)}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}^{(o)}$ is just the centrifugal force $-Mn_o v_\theta^{(o)2}/r$ and that $v_r^{(o)} = 0$ and

$$v_\theta^{(o)}\left(1 + \frac{Mv_\theta^{(o)}}{reB}\right) = \frac{1}{B}\left(\frac{KT_i}{e}\frac{n_o'}{n_o} - E_o\right) . \tag{11}$$

The ion drift velocity can thus be faster or slower than the electron drift velocity, depending on the sign of E_o; however, the charge separation is always in the proper direction to produce instability, according to the simple picture of Fig. 1. In first order, we must evaluate the terms $n_o\underset{\sim}{v}^{(o)}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}^{(1)}$ and $n_o\underset{\sim}{v}^{(1)}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}^{(o)}$. If $rv_\theta^{(o)}$ is assumed constant, and $v_\theta^{(1)}$ is neglected relative to $v_r^{(1)}$, as a consequence of the essentially one-dimensional perturbation (Eqs. 5 and 6), these terms are approximately

$$\underset{\sim}{v}^{(o)} \cdot \underset{\sim}{\nabla} \underset{\sim}{v}^{(1)} \approx v_{\theta}^{(o)} \frac{1}{r} \frac{\partial v_r^{(1)}}{\partial \theta} \hat{r} \tag{12}$$

$$\underset{\sim}{v}^{(1)} \cdot \underset{\sim}{\nabla} \underset{\sim}{v}^{(o)} \approx 2\, v_{\theta}^{(o)} \frac{v_r^{(1)}}{r} \hat{\theta} \tag{13}$$

Solving the system (Eqs. 1-4) with this approximation, we find that ω may be complex if $\underset{\sim}{\nabla} \times \underset{\sim}{E}_o \neq 0$ but that if $\underset{\sim}{\nabla} \times \underset{\sim}{E}_o = 0$, ω is purely real. This somewhat surprising result is a consequence of the fact that we have included only one effect, the term of Eq. (12), which can offset the centrifugal force. This has the effect of shifting the phase of $v_r^{(1)}$ until is is exactly 90° out of phase with the density perturbation n_1, and therefore the perturbation does not grow.

CENTRIFUGAL FORCE AND VISCOSITY

We now include both the terms on the right of Eq. (1) under the assumption that $rv_{\theta}^{(o)}$ is constant and hence that $(\nabla \cdot \pi)^{(o)}$ vanishes. The first order viscosity term, $(\nabla \cdot \pi)^{(1)}$, is in general quite complicated, and we shall be content with retaining only the dominant term in each direction. If $rv_{\theta}^{(o)}$ is constant, $v_{\theta}^{(1)}$ is much less than $v_r^{(1)}$, m is not too small, and all r-derivatives of first-order quantities and θ-derivatives of zero-order quantities are neglected, these dominant terms are, very approximately,

$$\nabla \cdot \pi = \frac{m^2}{r^2} \frac{\mu v_r^{(1)}}{\alpha} \left(\frac{\hat{r}}{\alpha} - \frac{\hat{\theta}}{2} \right). \tag{14}$$

Solution of the dispersion equation then yields

$$\mathrm{Re}\,(\omega) \approx m\omega_o \equiv mv_{\theta}^{(o)}/r \tag{15}$$

$$\mathrm{Im}(\omega) \approx - \frac{m}{r} \frac{\mu}{M n_o} \frac{1}{\alpha^2} \frac{n_o'}{n_o} . \tag{16}$$

In obtaining this result we have made the approximations $m\Omega_o \ll 1$, $|\gamma a| \gg |$, $\delta \ll 1$, $|\gamma a| m\Omega_o \gg \delta$, and $\alpha m\Omega_o \gg \delta$, where

$$\Omega_o \equiv v_\theta^{(o)}/r\omega_c , \quad \delta \equiv \gamma^2 \mu/(\alpha n_o eB) , \quad \gamma \equiv m/r , \quad \text{and } a \equiv n_o/n_o' , \tag{17}$$

the prime denoting differentiation with respect to r. If n_o' is negative, as is normal, Eq. (16) shows that there is an instability. This is because the viscosity prevents the term $\gamma v_\theta^{(o)} v_r^{(1)}$ of Eq. (12) from completely canceling the effect of the centrifugal force. However, the growth time is of the order of 10^{-2} seconds for stellarator conditions, and this is too slow to be of interest.

FINITE RESISTIVITY

To take collisions between electrons and ions into account, it is more convenient to use the single-fluid equations of Spitzer:

$$\underset{\sim}{j} \times \underset{\sim}{B} - \underset{\sim}{\nabla} p = M n \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \underset{\sim}{v} + \underset{\sim}{\nabla} \cdot \underset{\sim}{\pi} \tag{18}$$

$$\underset{\sim}{j} \times \underset{\sim}{B} - \underset{\sim}{\nabla} p_e = ne (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} - \eta \underset{\sim}{j}) \tag{19}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{j} = 0 \tag{20}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underset{\sim}{v}) = Q . \tag{21}$$

Here p is the total pressure, η the resistivity, and $\underset{\sim}{j}$ and $\underset{\sim}{v}$ the macroscopic current and velocity. Terms of order m_e/M have been neglected. Since there is a small but finite value of $v_r^{(o)}$, we have introduced a source term Q to achieve a steady state; but we assume that Q is so small that it can be neglected in first order. The calculation proceeds in an analogous manner to the preceding case, and we find

$$\mathrm{Im}(\omega) \approx - \frac{m}{r} \frac{\mu}{Mn_o} \frac{1}{\alpha^2} \frac{n_o'}{n_o} - \frac{m^2}{r^2} \eta \frac{n_o KT}{B^2} + \frac{2\eta M n_o \omega_o^2}{B^2} - \frac{\omega_o}{2} \frac{m^2}{r^2} \frac{\eta}{B^2} \frac{\mu}{\alpha} . \tag{22}$$

The first term is the same as in Eq. (16). The second is always negative and is obviously the dissipative effect of cross-field diffusion. The third is always positive and represents the combined effect of resistivity and centrifugal force. The last term may be of either sign but is very small because it involves both η and μ. None of the terms in (22) is large enough to be of interest.

ION INERTIA

We now take into account the inertia of the ions by adding a term $Mn\, \partial \underset{\sim}{v}/\partial t$ to the right hand side of Eq. (1) or Eq. (18). In the case of zero resistivity, we find the dispersion equation

$$S\gamma a \Omega^2 - [m\gamma a \Omega_o + C - S\xi]\, \Omega + m\Omega_o (C - \xi) = 0 , \tag{23}$$

where $\Omega = \omega/\omega_c$, $S = 1 - m\Omega_o$,

$$C = 1 + 2\Omega_o - \frac{1}{2}\delta$$
$$\xi = 1 - \frac{a\delta}{rC} - \gamma a m \Omega_o + \frac{i\gamma a \delta}{\alpha} \quad , \tag{24}$$

and the other symbols are defined in Eq. (17).

If the viscosity can be neglected, this equation is simple enough to be solved with the approximations $|\gamma a| >> 1$ and $m\Omega_o << 1$. We then find instability if $\gamma a < 0$ and

$$m^2\Omega_o^2 |\gamma a| < 4 , \tag{25}$$

with a maximum growth rate

$$\mathrm{Im}(\omega)_{max} \approx |v_\theta^{(o)}| \sqrt{-\frac{m}{r}\frac{n_o'}{n_o}} = m|\omega_o| \; (-\gamma a)^{-\frac{1}{2}} . \tag{26}$$

These expressions are independent of the direction of E_o, and Eq. (26) gives a growth rate $(-\gamma a)^{-\frac{1}{2}}$ times the observed frequency. If $|\gamma a| = 25$ and the observed frequency is 50 kc, the growth time is of the order of 100 μs, which is sufficiently fast to be of interest. Although we have not found simple expressions for the effect of resistivity or viscosity, these effects are probably small in view of the fast growth rate.

If the centrifugal force is neglected in addition to viscosity and resistivity, we obtain purely real values of ω for propagation perpendicular to B. If, however, small but finite values of the parallel wave number are admitted, and the oscillation of electrons along the field is taken into account, we find an instability driven only by the pressure gradient. This effect will be reported in a separate article.

We note that the introduction of the inertia term makes these oscillations into waves, and some of the considerations mentioned in the introduction no longer apply. In particular, n_1 and $v_r^{(1)}$ are no longer correlated in such a way that enhanced transport can occur even with coherent oscillations.

DISCUSSION

Various theories[10, 11] have previously been proposed to explain the fast loss of plasma ("pumpout") observed in stellarator discharges. These have been based on the existence of longitudinal gradients in potential or temperature. Inasmuch as a radial electric field is always observed[12] in a stellarator and has recently been shown[13] to affect the diffusion rate, we have investigated whether or not this agent alone can cause low-frequency oscillations to arise, without any nonuniformity along the field lines. It has been found that the centrifugal force on a rotating plasma can cause such instabilities. A radial electric field can be expected to arise in a toroidal discharge[14] in which $KT_e >> KT_i$ and the plasma is defined by an aperture limiter. Such fields are normally opposite in sign from those needed for the reflex-discharge instability. Local shear fields can also arise in the interior of the plasma because of charge separations caused by magnetic field inhomogeneities; in such a case the ion viscosity can conceivably lead to an instability.

The effects considered in this paper can be eliminated by stopping the rotation of the plasma or by imposing a periodicity in the z-direction.

The inclusion of the ion inertia term in the equation of motion is tantamount to considering the finite Larmor radius of the ions. If it should turn out that it is this term which is responsible for electrostatic instabilities, it would be ironic that the same effect is needed to stabilize a plasma against hydromagnetic instabilities.

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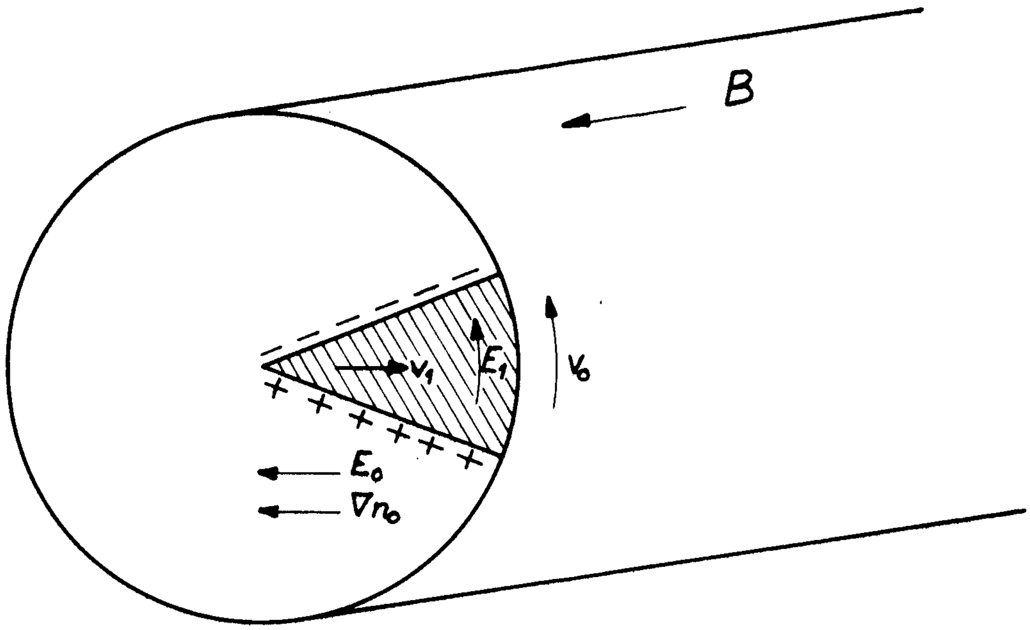

Figure: 1