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Normal Modes for Electrostatic Ion Waves in an Inhomogeneous Plasma

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The propagation of low-frequency electrostatic oscillations perpendicular to a density gradient in a low-density, fully ionized plasma in a strong magnetic field B is studied from the point of view of the linearized macroscopic equations. Two sets of waves are found in the limits where the electrons move freely along B or not at all along B. Among these are waves which travel at the electron and ion diamagnetic drift velocities, and a physical description of this phenomenon is given in terms of microscopic motions. The transition between these two sets of modes occurs at angles of propagation very close to 90° to B; to treat this region of angles one must take into account either the electron inertia or the plasma resistivity. Dispersion curves are given for this transition region. In addition, it is found that in a small range of angles one of the cyclotron waves and the wave traveling near the electron drift velocity are unstable and can be excited by the pressure gradient, even if no longitudinal currents exist in equilibrium.

I. INTRODUCTION

LOW-FREQUENCY oscillations in an inhomogeneous plasma created by thermal ionization have already been observed by D'Angelo and Motley.[1] These oscillations seem to propagate across the magnetic field with the same velocity as the pressure-gradient drift; that this should follow from the macroscopic equations of a collisionless plasma has been noted by D'Angelo.[2] However, it seems at first paradoxical that a density perturbation should propagate with the macroscopic drift when, in the microscopic picture, the ions must gyrate around fixed lines of force in the absence of collisions. It is the purpose of this article to show that the various electrostatic modes arising from the macroscopic equations have simple physical interpretations; that it is important to consider finite values of the wavenumber $k_\parallel$ parallel to the magnetic field; and that, when finite resistivity is taken into account, it is the wave traveling with the *electron* pressure-gradient drift which is unstable, in agreement with well-known results on "universal" instabilities. To achieve this we propose to study a particularly simple system in which the physical character of the pressure-gradient drift modes can be made clear.

II. FUNDAMENTAL EQUATIONS

For a fully ionized plasma, the first two moments of the Boltzmann equation give, for each species, the following fluid equations (in esu):

$$mn(\partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v}) = qn(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - KT\nabla n + \mathbf{P}_{12}, \qquad (1)$$

$$\partial n/\partial t + \nabla\cdot(n\mathbf{v}) = 0. \qquad (2)$$

Here $\mathbf{P}_{12}$ is the force per unit volume exerted on fluid 1 by collisions with fluid 2. We have neglected the viscosity and assumed that the temperature of each species is constant and uniform; if the plasma is adiabatic instead of isothermal, a factor of $\frac{5}{3}$ should appear in front of the density-gradient term; but this would not greatly affect the results. These equations are valid[3] as long as collisions are frequent enough to keep the velocity distributions Maxwellian. In the cesium and potassium plasmas with which this paper is primarily concerned, this assumption is probably closer to the truth than the assumption of no collisions for the low frequencies of interest. Of course, the condition on the collision frequency is relaxed if we restrict our attention to certain types of phenomena; for example, to those in which the parallel wave velocity is much larger than the thermal velocity and the Larmor radii are smaller then the scale of macroscopic gradients. In particular, we shall avoid considering zero-order drifts along B, since this would lead to excitation of waves with a growth rate which can be computed accurately only by including the effects of Landau damping.

We take the density n to be the same for ions as for electrons, and the charge q to be $+e$ for ions and $-e$ for electrons. We shall further assume that the density is so low that $\nabla\times\mathbf{B} = 0$ and that the frequencies are so low that $\nabla\times\mathbf{E} = 0$; then $\mathbf{B}$ is constant and uniform, and $\mathbf{E}$ can be written $-\nabla\phi$. If only frequencies much less than the ion plasma frequency are considered, we may omit

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[1] N. D'Angelo and R. W. Motley, Phys. Fluids **6**, 422 (1963).
[2] N. D'Angelo, Phys. Fluids **6**, 592 (1963).
[3] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1962), 2nd ed., p. 24.

Maxwell's equations altogether and use only Eqs. (1) and (2). In neglecting gradients in B we miss waves associated with the ∇B drifts; however, these occur at extremely low frequencies if the vacuum field is uniform, and there has been no experimental evidence that such waves are important.

1. Equilibrium

In zero order we assume that the electric field and all time derivatives vanish, and that all quantities vary only in the x direction. With $\mathbf{P}_{12}$ neglected temporarily, Eq. (1) for ions becomes

$$m_i n_0 \mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(0)} = e n_0 \mathbf{v}^{(0)} \times \mathbf{B} - KT_i \nabla n_0. \tag{3}$$

For clarity we shall use Cartesian coordinates; the case of cylindrical symmetry is considered elsewhere.[4] With $\mathbf{B} = B\hat{\mathbf{z}}$ and a prime denoting $\partial/\partial x$, the x and y components of Eq. (3) become

$$v_x^{(0)} v_x^{(0)\prime} = (eB/m_i) v_y^{(0)} - (KT_i/m_i)(n_0'/n_0) \tag{4}$$

$$v_x^{(0)}[v_y^{(0)\prime} + (eB/m_i)] = 0, \tag{5}$$

and Eq. (2) becomes

$$n_0 v_x^{(0)\prime} + n_0' v_x^{(0)} = 0. \tag{6}$$

If $v_x^{(0)}$ vanishes, Eqs. (5) and (6) are automatically satisfied, and Eq. (4) gives the diamagnetic drift velocity:

$$v_y^{(0)} = \frac{KT_i}{eB} \frac{n_0'}{n_0} = \frac{v_{\mathrm{th}}^2}{\omega_c} \lambda(x) \equiv v_0, \tag{7}$$

where

$$\omega_c \equiv eB/m_i, \quad \lambda(x) \equiv n_0'/n_0, \quad v_{\mathrm{th}} \equiv (KT_i/m_i)^{\frac{1}{2}}. \tag{8}$$

If $v_x^{(0)}$ does not vanish, Eqs. (4)–(6) are three differential equations for the two quantities $v_x^{(0)}$ and $v_y^{(0)}$; and therefore a condition is imposed on the equilibrium density gradient $\lambda(x)$. This condition is not satisfied in general, and in particular is not satisfied for λ constant, the case with which we are especially concerned. Thus we henceforth assume that $v_x^{(0)}$ vanishes and $v_y^{(0)}$ is given by Eq. (7). A similar equation for electrons gives

$$v_{0e} = -(T_e/T_i) v_0. \tag{9}$$

The equilibrium configuration, then, contains a uniform magnetic field in the z direction, a density gradient in the x direction, and oppositely directed drifts of ions and electrons in the y direction. These drifts, of course, refer to a fluid element; the individual particles gyrate about fixed lines of force and do not drift.[5]

[4] F. F. Chen, Princeton Plasma Physics Laboratory Report MATT-227 (1963).
[5] Reference 3, p. 32.

2. Perturbation

We now wish to consider waves propagating perpendicular to the zero-order density gradient; that is, with a propagation vector $\mathbf{k}$ lying in the y–z plane such that the primary component $k_\perp$ is in the y direction, with a small component $k_\parallel$ in the z direction. Assuming that all perturbed quantities vary as $\exp(-i\omega t)$, we linearize Eq. (1) by writing $n = n_0 + n_1$, $\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}$, and $\mathbf{E} = 0 - \nabla\phi$, dropping higher order terms, and subtracting $(1 + n_1/n_0)$ times Eq. (3). Again specifying ions, we have

$$m_i n_0(-i\omega \mathbf{v}^{(1)} + \mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(1)} + \mathbf{v}^{(1)} \cdot \nabla \mathbf{v}^{(0)}) = -e n_0 \nabla\phi + e n_0 \mathbf{v}^{(1)} \times \mathbf{B} - KT_i(\nabla n_1 - n_1 n_0^{-1} \nabla n_0). \tag{10}$$

It will now be convenient to measure times in units of ω_c^{-1} and velocities in units of v_{th}, and thus lengths in units of v_{th}/ω_c. We thus introduce the following dimensionless quantities, denoted by Greek letters:

$$\begin{aligned}
&\boldsymbol{\upsilon} \equiv \mathbf{v}^{(1)}/v_{\mathrm{th}}, \quad \boldsymbol{\upsilon}_0 \equiv \mathbf{v}^{(0)}/v_{\mathrm{th}}, \quad \Omega \equiv \omega/\omega_c, \\
&\delta \equiv \lambda v_{\mathrm{th}}/\omega_c = \lambda r_L/\sqrt{2} \ll 1, \\
&\gamma \equiv k_\perp v_{\mathrm{th}}/\omega_c = k_\perp r_L/\sqrt{2} \ll 1, \\
&\gamma_\parallel \equiv k_\parallel v_{\mathrm{th}}/\omega_c = k_\parallel r_L/\sqrt{2} \ll 1, \\
&\chi \equiv e\phi/KT_i, \qquad \nu \equiv n_1/n_0, \\
&\psi \equiv \Omega - \gamma \upsilon_0, \qquad \psi_e \equiv \Omega + \beta\gamma\upsilon_0, \\
&\beta \equiv T_e/T_i, \qquad \mu \equiv m_e/m_i, \qquad \theta \equiv \gamma_\parallel/\gamma, \\
&\kappa \equiv k_\perp v_0/\omega_c = \gamma\upsilon_0 = \gamma\delta, \qquad \bar{\beta} \equiv 1 + \beta, \\
&A \equiv 1 + \bar{\beta}\gamma^2,
\end{aligned} \tag{11}$$

where $r_L = \sqrt{2} v_{\mathrm{th}}/\omega_c$ is the ion Larmor radius. Note that $\boldsymbol{\upsilon}$ is simply the velocity in units of v_{th}, while γ and δ are essentially the wavenumber and density gradient in terms of the Larmor radius. Also, χ and ν are the perturbations in potential and density, and κ is the Doppler shift of the zero-order ion drift, so that ψ and ψ_e are the frequency measured in the ion and electron drift frames, respectively. We are interested in waves propagating nearly perpendicular to $\mathbf{B}$; that is, in small values of θ, which is then essentially the angular deviation from perpendicularity.

To take resistivity into account we now retain the term $\mathbf{P}_{12}$ in Eq. (1) and relate it to the resistivity η by the well-known expression

$$\mathbf{P}_{ei} = -\mathbf{P}_{ie} = en\eta\mathbf{j} = e^2 n^2 \eta(\mathbf{v}_i - \mathbf{v}_e). \tag{12}$$

If the parameter ϵ defined by

$$\epsilon \equiv e n_0 \eta / B \tag{13}$$

is much smaller than unity, as we shall assume, it can be shown that the effect of resistivity on the perpendicular components of $\mathbf{v}$ is negligible. We therefore make the basic approximation that the term $\mathbf{P}_{12}$ need be included only in the z component of Eq. (1). We also neglect the small velocity component $v_x^{(0)}$ introduced into the equilibrium solution by the collisions.

Since the zero-order quantities are independent of y and z, we can perform a further Fourier analysis and assume that ν, χ, and v vary as $\exp i(k_\perp y + k_\parallel z)$. The algebra becomes appreciably simpler if we now make the assumption that v_0, and hence ψ and ψ_e, is constant; that is, that λ is constant and the zero-order density gradient is exponential:

$$n_0 \sim e^{\lambda x}. \tag{14}$$

This is a physical assumption made for convenience; it also happens to be approximately fulfilled in certain experiments.[1]

III. RESULTS

Following standard procedure, we can with this approximation solve Eq. (10) and its electron counterpart for the components of $\mathbf{v}_i$ and $\mathbf{v}_e$. These are then inserted into the linearized form of the equations of continuity (2) to obtain a pair of coupled differential equations for $\nu(x)$ and $\chi(x)$. It can be shown[4] that when these equations are solved for the lowest mode, the effect of the x dependence is negligible on the drift modes and gives a frequency shift of order $\frac{1}{4}(\delta/\gamma)^2$ on the cyclotron modes. We shall therefore neglect the x dependence and assume that ν and χ are constant. The components of the velocity perturbations then become:

$$\begin{aligned} v_{xi} &= \frac{i\gamma}{\psi^2 - 1}(\chi + \nu), \quad v_{xe} = \frac{-i\gamma}{\mu^2\psi_e^2 - 1}(\beta\nu - \chi), \\ v_{yi} &= \frac{\psi\gamma}{\psi^2 - 1}(\chi + \nu), \quad v_{ye} = \frac{\mu\psi_e\gamma}{\mu^2\psi_e^2 - 1}(\beta\nu - \chi), \\ v_{zi} &= D^{-1}\gamma_\parallel[\mu\psi_e(\chi + \nu) + i\bar{\beta}\epsilon\nu], \\ v_{ze} &= D^{-1}\gamma_\parallel[\psi(\beta\nu - \chi) + i\bar{\beta}\epsilon\nu], \end{aligned} \tag{15}$$

where

$$D \equiv \mu\psi\psi_e + i\epsilon(\psi + \mu\psi_e). \tag{16}$$

When these expressions are inserted into the equations of continuity, one obtains a pair of algebraic equations for χ and ν. The condition that the determinant vanish then gives the following dispersion relation:

$$-\theta^2 = \frac{[\mu\psi\psi_e + i\epsilon(\psi + \mu\psi_e)]\psi^2(\delta\gamma^{-1}\psi + 1)}{\psi^2(\psi^2 - A) - \bar{\beta}\kappa\psi + (\psi^2 - 1)(\mu\psi\psi_e - \bar{\beta}\gamma^2\theta^2) - i\bar{\beta}\epsilon\gamma^2\psi(\delta\gamma^{-1}\psi + 1)}. \tag{17}$$

In obtaining this result we have assumed $\mu \ll 1$.

If the resistivity ϵ vanishes, one obtains the disperson relation for finite $k_\parallel$ when the electron velocity along $\mathbf{B}$ is limited only by the electron inertia. We shall omit the discussion of this case, which is given elsewhere,[4] and proceed to the physically interesting case $\mu \ll \epsilon \ll 1$. A simpler expression which still retains all the essential features of Eq. (17) can be obtained if we assume, in addition, $\theta^2 \ll 1$ and $\epsilon \ll \delta\gamma^{-1}$, as is reasonable in most physically interesting situations; Eq. (17) then becomes

$$i\epsilon^{-1}\theta^2 = \frac{\psi^3(\delta\gamma^{-1}\psi + 1)}{\psi^2(\psi^2 - A) - \bar{\beta}\kappa\psi + \bar{\beta}\gamma^2\theta^2}. \tag{18}$$

Numerical results for the relation (17) are shown in Fig. 1 for the case $\bar{\beta} = 2$, $\mu = 1.4 \times 10^{-5}$, $\epsilon = 8 \times 10^{-5}$, which corresponds to a 2000°K potassium plasma. In Fig. 1 the real and imaginary parts of Ω/κ, which is the wave velocity in units of the ion pressure-gradient drift velocity, are plotted against $\epsilon^{-1}\theta^2$, which is a measure of $k_\parallel$, for the cases $\delta < \gamma$ and $\delta > \gamma$. Since δ and γ, and hence κ, have been chosen positive, positive values of Im (Ω/κ) correspond to growth and negative values to damping.

IV. THE NORMAL MODES

1. Perpendicular Propagation

If $k_\parallel$, and hence θ, is strictly zero, the numerator of Eq. (17) or (18) must vanish. This gives the five roots on the Ω/κ axis of Fig. 1; four of these roots appear in the approximate formula (18). The real part of ω is given by:

$$\psi^3 = 0 \quad \text{or} \quad \omega = k_\perp v_0 \qquad (\mathrm{DI}_\perp), \tag{19}$$

$$\psi = -\gamma/\delta \quad \text{or} \quad \omega = k_\perp v_0 - (k_\perp/\lambda)\omega_c \quad (\mathrm{CE}_\perp), \tag{20}$$

$$\psi_e = 0 \quad \text{or} \quad \omega = k_\perp v_{0e} \qquad (\mathrm{DE}_\perp). \tag{21}$$

(*a*) *The Ion Drift Mode.* The triple root (19) is the one discussed previously by D'Angelo.[2] It is a wave traveling at the zero-order ion pressure-gradient drift velocity. This mode is designated by $\mathrm{DI}_\perp$, the subscript indicating the limit of per-

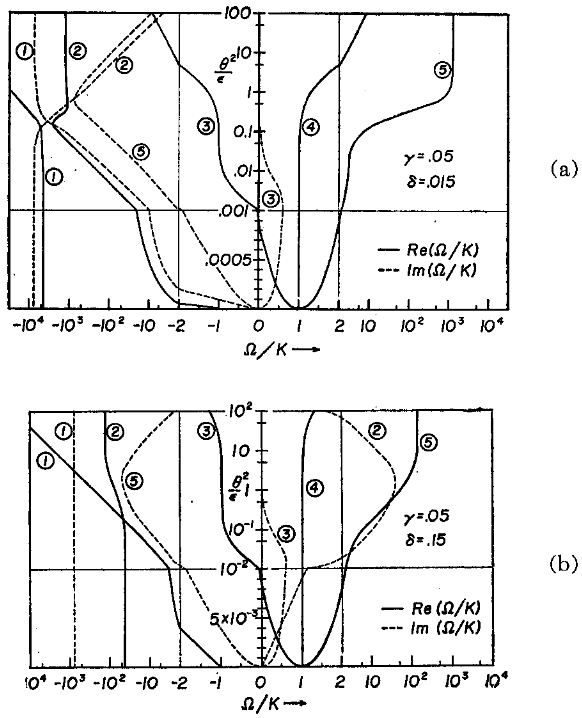


FIG. 1. Variation of normalized wave velocity with angle of propagation θ for a given wavelength γ and two values of the density gradient δ, in the case of finite (normalized) resistivity ϵ. The value $\gamma = 0.05$ corresponds, for example, to the $m = 1$ mode of a thermally ionized potassium plasma 1.5 cm in radius, in a field of 4 kG, and the values of ϵ and μ used correspond to a 2000°K plasma of density 3×10^{10} cm^{-3}. Note that the scales are linear near the axes and logarithmic elsewhere, causing the abrupt changes in slope.

pendicular propagation. We shall try to understand this mode physically from both the fluid and the single-particle points of view.

In the fluid picture, we note that the first-order velocity for the ion fluid vanishes. This follows from the fact that for $\gamma_{\parallel} = 0$ and $\mu \ll 1$, the electron continuity equation is simply

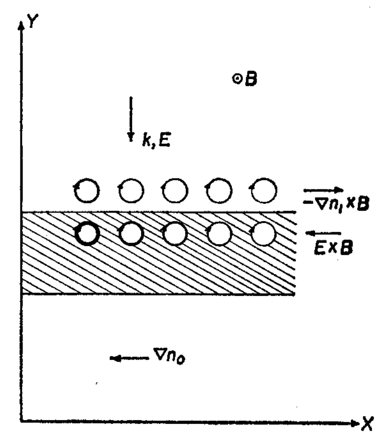
$$\gamma v_0 \chi = -\Omega \nu. \qquad (22)$$

For this mode, we have $\psi = \Omega - \gamma v_0 = 0$, and hence $\nu + \chi$ vanishes; and, by Eq. (15), v_i is identically zero. It appears that a density perturbation in the ion fluid will be carried along with the zero-order drift in the y direction.

In the microscopic picture, each particle gyrates about a fixed line of force since the effect of collisions on perpendicular motions has been neglected, and density perturbations cannot be "carried along" with the zero-order drift. However, density perturbations can still propagate because the guiding centers can drift along the zero-order density gradient. To see this more clearly, we must first recognize that the χ terms in the expressions (15) for the velocities represent the $\mathbf{E} \times \mathbf{B}$ drift due to the first-order electric field, while the ν terms represent the $\nabla p \times \mathbf{B}$ drift due to the density perturbation. This is illustrated in Fig. 2. According to the physical picture given by Spitzer,[5] the density perturbation in the y direction gives rise to a macroscopic velocity v_x which does not exist in the microscopic picture. The perturbed electric field E_y (E_x being zero by assumption) gives rise to a drift of the guiding centers in the x direction. For the ions these two drifts exactly cancel so that the macroscopic velocity perturbation is zero, while for the electrons the two drifts add. A density perturbation propagates in the y direction by virtue of the $\mathbf{E} \times \mathbf{B}$ drift of both the ion and the electron guiding centers in the x direction, bringing in alternately regions of higher and lower density in the zero-order distribution. It is clear from this picture that if the ion inertia had been neglected along with the electron inertia, the ions and electrons would respond exactly the same way to the electric field, and there would be no charge separation to perpetuate the electric field; it is the lag of the ions in setting up their $\mathbf{E} \times \mathbf{B}$ drift that makes this wave possible.

From Eq. (15) we see that in the limit $\mu \to 0$ the only nonvanishing velocity component is v_{xe}; the electrons drift back and forth in the x direction with a negligibly small Larmor radius. This wave is therefore longitudinal only in the sense that $\mathbf{E}$ is parallel to $\mathbf{k}$; it is transverse in the sense that the fluid velocity is perpendicular to $\mathbf{k}$. In the microscopic picture it is easy to see intuitively why the wave velocity should be proportional to the ion drift velocity. The frequency should be proportional to the rate at which the density can change; that is, proportional to the drift speed E/B times the density gradient λ. Now, E is $-ik_{\perp}\phi$, and $e\phi$ must be proportional to KT_i, since it is the ion inertia which creates the potential; thus we have that

FIG. 2. Pictorial representation of the density-gradient wave $\mathrm{DI}_{\perp}$ traveling with the ion drift velocity. The shaded region represents a density perturbation propagating in the negative y direction, and the thickness of the ion Larmor orbits represents the relative local density. In this wave $\mathbf{E} \times \mathbf{B}$ drifts in the x direction exactly cancel the macroscopic velocity due to the density perturbation.



$\omega/k_\perp \propto KT_i\lambda/eB = v_0$. Why the constant of proportionality should be exactly 1 seems to be fortuitous; indeed, one finds that $\omega/k_\perp$ differs slightly from v_0 if v_0 is not constant. Although we have not considered propagation in the x–y plane, it is clear that the wave velocity will be fastest in the y direction, since then the $\mathbf{E}\times\mathbf{B}$ drifts are parallel to the zero-order density gradient.

(*b*) *Modified Ion Cyclotron Wave.* We now turn our attention to the root (20) of the dispersion relation (18); this mode is labeled (2) in Fig. 1. This is a modified ion cyclotron wave in which electrons do not stream along $\mathbf{B}$ in order to preserve charge neutrality, as in the usual case, but rather drift in the x direction along ∇n_0. This mechanism of charge neutralization does not exist in a uniform plasma. We note that the first term in Eq. (20) is usually small compared to the second, so that the wave almost always travels in the direction of the zero-order electron drift; it is therefore denoted by $CE_\perp$, the "E" indicating the direction of the wave velocity. The frequency can be larger or smaller than ω_c depending on the relative size of the wavelength and the density gradient. In wave $CE_\perp$ both the ion and the electron velocity perturbations are finite.

(*c*) *Electron Drift Wave.* The mode labeled (1) in Fig. 1 and described by Eq. (21) is a wave $DE_\perp$ traveling with the electron pressure-gradient drift. This wave does not appear in the approximate dispersion relation (18), and it is an entirely different phenomenon from the wave $DI_\perp$. Its nature can be seen from the expression for v_{ze} given by Eqs. (15) and (16) if we neglect ϵ for the moment. If $\gamma_\parallel$ had been set equal to zero at the outset, this wave would not have appeared. But if we allow $\gamma_\parallel$ to approach zero from finite values, v_{ze} may become infinite if $D = \mu\psi\psi_e$ approaches zero faster than $\gamma_\parallel$. The root $\psi_e = 0$ therefore corresponds to infinite parallel velocities of the electrons. When collisions with ions are taken into account, one would expect this wave to be damped; and indeed it is clear from Fig. 1 that this mode always has a large negative imaginary part. This mode is therefore physically unimportant.

2. "Large" Angle Propagation

We now consider the opposite limit of propagation in the y–z plane with a component $k_\parallel$ of the propagation constant along $\mathbf{B}$ sufficiently large that electrons can move freely in the z direction. The quantity $\epsilon^{-1}\theta^2$ then becomes large and the denominator of Eq. (17) or (18) must vanish. We then obtain the four waves near the top of Figs. 1(a) and 1(b).

(*a*) *Ion Cyclotron Waves.* When $\kappa = \delta\gamma$ is finite, two of the roots of the denominator of Eq. (18) occur around $\psi \sim A \sim 1$. For $\gamma_\parallel^2 \ll \gamma^2 \ll 1$, we may neglect the last term in this denominator and obtain, approximately,

$$\psi^2 = 1 + \bar{\beta}\gamma(\gamma + \delta), \qquad (CI_\theta, CE_\theta)$$

or

$$(\omega - k_\perp v_0)^2 = \omega_c^2 + (1 + T_e/T_i)k_\perp^2(v_{th}^2 + v_0^2). \qquad (23)$$

When v_0 vanishes, we recover the Motley–D'Angelo[6] oscillations near the ion cyclotron frequency. When v_0 is finite, the frequency is modified by the density gradient perpendicular to the direction of propagation. The correction occurs because electrons can cancel charge imbalance by drifting in the x direction as well as by streaming in the z direction. These modes will be labeled CI_θ and CE_θ, the subscript denoting the limit of "large" angle propagation.

(*b*) *Electron Drift Wave.* The drift waves, for which $\psi^2 \ll 1$, are found by neglecting ψ^2 relative to A and to 1. We then have

$$A\psi^2 + \bar{\beta}\kappa\psi - \bar{\beta}\gamma_\parallel^2 = 0. \qquad (24)$$

Furthermore, if we neglect the term $\bar{\beta}\gamma_\parallel^2$, we find a wave traveling near the electron diamagnetic drift velocity $-\beta v_0$:

$$\Omega = \frac{\kappa(A - \bar{\beta})}{A} = -\kappa\frac{\beta + \bar{\beta}\gamma^2}{1 + \bar{\beta}\gamma^2} \approx -\beta\kappa,$$

or

$$\omega \cong k_\perp v_{0e} \qquad (DE_\theta). \qquad (25)$$

For this wave to exist, we obviously require $\bar{\beta}\gamma_\parallel^2 \ll \bar{\beta}\,|\kappa\psi| \approx \bar{\beta}^2\kappa^2$, or $\theta^2 \ll \bar{\beta}\delta^2$. On the other hand, the large-angle approximation required $\theta^2 \gg \epsilon\delta\gamma^{-1}$. Thus this wave exists for a range of angles θ provided that $\epsilon \ll \bar{\beta}\kappa$, a condition which is easy to satisfy in practice.

For larger values of θ or for small density gradients such that $\gamma_\parallel^2 \gg |\kappa\psi|$, we may neglect the middle term in the denominator of Eq. (18) and obtain

$$\psi^2 = \bar{\beta}A^{-1}\gamma_\parallel^2 \text{ or } (\omega - k_\perp v_0)^2 \approx (1 + T_e/T_i)k_\parallel^2 v_{th}^2. \quad (26)$$

Thus for sufficiently large θ both the wave DE_θ and the wave DI_θ discussed below turn into acoustic waves with a parallel phase velocity nearly equal to the sound velocity. Using the result (26), we see that the drift waves turn into sound waves for $\theta \gg \bar{\beta}^{\frac{1}{2}}A^{-\frac{1}{2}}\delta$. This is seen in Fig. 1, where the drift waves, labeled (3) and (4), are shown to deviate at large angles from the velocities $\Omega/\kappa = \pm 1$.

[6] R. W. Motley and N. D'Angelo, Phys. Fluids **6**, 296 (1963).

When $\epsilon^{-1}\theta^2$ is large, the electron continuity equation becomes simply

$$\chi = \beta\nu, \tag{27}$$

in contrast to Eq. (22). A physical interpretation of the electron drift wave may be obtained by computing the fluid velocities from Eq. (15). For $\chi = \beta\nu$, we see that v_{xe} and v_{ye} vanish, and v_{xi} is the main component of ion velocity, the components v_{yi} and v_{zi} being $\bar{\beta}\kappa$ and $\theta/\bar{\beta}\kappa$ times smaller, respectively. DE_θ is therefore the counterpart of mode $\text{DI}_\perp$; it is now the electrons whose $\mathbf{E} \times \mathbf{B}$ and $\nabla p \times \mathbf{B}$ drifts cancel to give a zero macroscopic velocity, while the ion drifts add to give a large ion drift velocity in the x direction. Because of the ion inertia, there is also a small but finite component of v_i in the y direction which gives rise to a charge separation which is cancelled by electron streaming along the magnetic field. It is intuitively reasonable that the mode DE_θ should propagate in the direction opposite to the zero-order ion drift, since the roles of the ions and electrons are interchanged as compared with mode $\text{DI}_\perp$.

(*c*) *Ion Drift Wave*. Finally, there is a solution of Eq. (24) for $\psi \approx 0$. In this case we may neglect the first term of Eq. (24) and write

$$\psi = \gamma_\parallel^2/\kappa \text{ or } \omega = k_\perp v_0 + (k_\parallel^2 v_{\text{th}}^2/k_\perp v_0) \quad (\text{DI}_\theta). \tag{28}$$

The condition for the existence of this wave is that $\bar{\beta}\gamma_\parallel^2 \gg A\psi^2$ in Eq. (24). Using the result (28), we find that we require $\theta^2 \ll \bar{\beta}A^{-1}\delta^2$, which, for $A \sim 1$, is essentially the same condition as found above for the electron drift wave. Although the mode DI_θ travels at very nearly the ion drift velocity, it is physically an entirely different phenomenon from the mode $\text{DI}_\perp$. Referring again to the velocities in Eq. (15), we see that for the wave DI_θ the electron velocities v_{xe} and v_{ye} still vanish by virtue of Eq. (27), and the ion velocities v_{xi} and v_{yi} are essentially the same as in the previous case (DE_θ), but that v_{zi} is now very large: $v_{zi} = \delta\theta^{-1}\bar{\beta}\nu$. This diverges as θ goes to zero, and it appears that charge cancellation via *ion* streaming along $\mathbf{B}$ is a characteristic of this mode. Indeed, if parallel ion motions had been neglected originally, this mode would not have appeared. DI_θ is therefore the large-angle counterpart of $\text{DE}_\perp$, in which *electrons* preserved quasineutrality by streaming along $\mathbf{B}$. From Fig. 1 it is seen that this mode, labeled (4), maintains its identity even for small values of $\epsilon^{-1}\theta^2$.

When the resistivity ϵ is finite, one would expect this mode to be damped. Surprisingly, the imaginary part of Ω/κ for the mode DI_θ vanishes almost everywhere. The reason for this can be seen from the expressions (15) for the parallel velocities. The second term in the brackets of v_{zi} and v_{ze} represents the effect on v_z of the drag due to the other species. For $\mu \ll \epsilon$ and $\psi \ll \epsilon$, this term is dominant in both equations and we find $v_{zi} \approx v_{ze}$. Apparently the electrons are entrained by the ions because of the collisions, and both species stream along z with approximately the same velocity. The damping, which is proportional to $v_{zi} - v_{ze}$, appears only for *extremely* small values of θ^2, of order $\epsilon\mu$.

Although v_{zi} can be finite as $\theta \to 0$ if ν vanishes, it is not possible to interpret this mode as an initial perturbation in v_{zi}, with no perturbation in n, which is propagated along with the ion drift. This is because the method by which we derived the dispersion relation depends on the fact that ν is finite. Since the large values of v_{zi} which are necessary for the existence of this mode at small values of θ cannot be reconciled with the microscopic picture, we conclude that DI_θ is not adequately described by this simple theory. At large values of θ, as we have already noted, this wave becomes an acoustic wave and no longer possesses the phase velocity v_0.

3. Intermediate Angles

From Fig. 1 we see that as $\epsilon^{-1}\theta^2$ is increased from zero, the mode $\text{DI}_\perp$ splits into three waves which become the modes DE_θ, DI_θ, and CI_θ at large values of $\epsilon^{-1}\theta^2$; this change in the character of mode $\text{DI}_\perp$ occurs at extremely small angles θ. The mode $\text{CE}_\perp$, however, maintains its identity for a larger range of θ. Turning our attention to the imaginary part of Ω/κ, we note that the cyclotron waves (2) and (5) tend to become real as $\epsilon^{-1}\theta^2$ approaches 0 or ∞; but for finite values of $\epsilon^{-1}\theta^2$ the "forward" cyclotron wave CI is always damped, while the "backward" cyclotron wave CE is damped if $\delta < \gamma$ and unstable if $\delta > \gamma$. The maximum growth rate is of the order of Re (ω), at least in the framework of a theory which neglects cyclotron damping. The author knows of no experimental evidence for this instability; it is not the same as that observed by Motley and D'Angelo,[6] because $\mathbf{k}$ is perpendicular to ∇n_0 in this theory.

The mode labeled (3) in Fig. 1 changes from the mode $\text{DI}_\perp$ to the DE_θ and finally to an acoustic wave as θ is increased from 0. In an intermediate range of θ, this wave is unstable, with growth rate of the same order as Re (ω). We can offer the following partial explanation of the instability of the $\text{DI}_\perp$–DE_θ wave. When θ equals zero, we saw in Eq. (22) that for the ion drift wave the density and potential perturbations are 180° out of phase: $\nu = -\chi$. This means that the electric field E_y and hence the $\mathbf{E} \times \mathbf{B}$ drift of guiding centers in the x direction are 90° out of phase with n_1. Similarly, we saw in Eq. (27) that for the wave DE_θ, ν and χ are exactly in phase. Again, this means that the $\mathbf{E} \times \mathbf{B}$ drift is 90° out of phase with n_1. In an intermediate range of θ, the streaming of electrons along $\mathbf{B}$ is neither nonexistent nor infinitely efficient, and this mechanism can change the phase relation between ν and χ. It turns out that the phase is shifted in such a direction as to make the $\mathbf{E} \times \mathbf{B}$ drift of the guiding centers (of both species) be in the direction of decreasing n_0 when n_1 is positive, and vice versa. Thus when the perturbed density is already greater than n_0, the $\mathbf{E} \times \mathbf{B}$ drift brings in guiding centers from a region of larger n_0 to increase n further; the perturbation therefore grows at the expense of the zero-order density gradient. This would appear to be a sort of "universal" instability which is driven only by the density gradient. This instability disappears when the conductivity is infinite.

When this problem is solved in cylindrical geometry, it is found[4] that the centrifugal force causes the mode $\mathrm{DI}_\perp$ to be unstable even for $\theta = 0$. This is related to the destabilizing effects of centrifugal force previously noted by the author.[7]

V. DISCUSSION

We have considered the propagation of low-frequency electrostatic waves in an inhomogeneous plasma in directions nearly perpendicular to the magnetic field. Particular attention was given to the physical interpretation of waves traveling with nearly the ion and electron diamagnetic drift velocities. This treatment is limited by the accuracy of the linearized fluid equations with an isotropic pressure term and by the electrostatic and low-β approximations. Since we have neglected Landau and cyclotron damping as well as zero-order drifts along $\mathbf{B}$ and anisotropies of the pressure tensor, this theory cannot give accurate results for the excitation of these waves. However, we note that even in the absence of all sources of energy other than the zero-order density gradient the wave DE_θ is found to be unstable in this theory. This may be one of the fundamental mechanisms which cause the low-frequency, long-parallel-wavelength fluctuations observed in plasmas subject to anomalous diffusion.

Dispersion curves for a thermally ionized plasma are given in Fig. 1. One concludes that the mode $\mathrm{DI}_\perp$ discussed by D'Angelo[2] is not likely to be the mode observed by D'Angelo and Motley,[1] since the former can exist only for values of θ^2 smaller than about 10^{-7}, corresponding to parallel wave velocities greater than 3×10^8 cm/sec. To excite this mode by means of a relative drift between ions and electrons, one would need a longitudinal current far in excess of what could have been present in the experiment.

On the other hand, the mode DE_θ, propagating in the opposite direction to $\mathrm{DI}_\perp$, not only maintains a velocity near the diamagnetic drift velocity over a large range of θ in the neighborhood of 10^{-2}, but also can be excited without longitudinal currents if the resistivity is finite. We therefore suggest that it was probably the mode DE_θ which was observed, and that its apparition at low electron emission from the end plates is connected with the insulating properties of an ion sheath, which can permit long-parallel-wavelength disturbances to exist without being short-circuited at the end plates.

Finally, the observed wave may have been the mode DI_θ, which has the proper velocity for all angles θ smaller than about 10^{-2}. However, this wave is not excited in the absence of longitudinal currents, and no such currents were deliberately imposed in the experiment. The modes DE_θ and DI_θ can easily be distinguished by measuring the direction of propagation.

Since the completion of the original manuscript, two publications have appeared which have relevance to the present work. A finite-resistivity instability reported by Moiseev and Sagdeev[8] appears to be the same as that found here. A measurement by Lashinsky[9] confirms our conjectures both as to the direction of propagation and as to the insulating effect of ion sheaths. Whether the instability is the usual "universal" instability of a collisionless plasma or the finite-resistivity effect discussed here is yet to be determined.

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