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EXCITATION OF DRIFT INSTABILITIES IN THERMIONIC PLASMAS

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Abstract—Low-frequency oscillations travelling near the electron pressure gradient drift velocity in Cs or K plasmas have been observed to exist only in the presence of an ion sheath at the cathodes. It is suggested that these waves are self-exciting 'universal' instabilities in the presence of finite resistivity and that their excitation is connected with the insulating properties of an ion sheath, which allow the perturbation amplitude to be finite at the sheath edge. This leads to overstable oscillations with $\lambda_{\|}$ longer than the length of the machine, a strange concept which is explained physically. Although the solution of the problem involves sheath-type boundary conditions, for an isothermal plasma the criterion for onset of instability can be stated rather simply. The predicted critical sheath voltage agrees well with observations. It appears that alkali–metal plasmas with electron sheaths are quiescent only by virtue of stabilization by conducting end-plates; however, this feature makes such plasmas particularly valuable for the study of the onset of long-wavelength drift instabilities.

1. INTRODUCTION

RECENT experiments (D'ANGELO and MOTLEY, 1963; D'ANGELO *et al.*, 1963; LASHINSKY, 1964; BUCHEL'NIKOVA, 1964; WOLF and ROGERS, 1964; HARTMAN and MUNGER, 1964; D'ANGELO, 1964) in thermally ionized K and Cs plasmas have revealed oscillations travelling with the electron diamagnetic drift velocity $v_D = -(KT_e/eB)(n_0'/n_0)$, where n_0' is the radial density gradient. This phase velocity, as well as the fact that no driving forces other than the pressure gradient are present in the experiments, suggests that what has been observed is the well-known 'universal' instability (KADOMTSEV and TIMOFEEV, 1962) of an inhomogeneous plasma. However, the collision frequency in these plasmas is so high that the collisionless universal instability due to resonant particles is not likely to occur. This universal instability has a counterpart in the resistive-fluid limit; and we believe that what has been observed is, in a straight magnetic field **B**, the universal resistive overstability found by MOISEEV and SAGDEEV (1963; 1964) and by CHEN, (1964; 1965a) and in a curved **B**, the resistive gravitational instability (CHEN, 1965a; JUKES, 1964). These instabilities are characterized by very long wavelengths $\lambda_{\|}$ along **B**, and sometimes (D'ANGELO and MOTLEY, 1963) are observed to have $\frac{1}{2}\lambda_{\|}$ larger than the length $2L$ of the machine. It is our purpose to show how this is made possible by the sheaths at the cathodes and how one can explain the observed (D'ANGELO and MOTLEY) excitation of these waves in the presence of ion sheaths but not in the presence of electron sheaths.

Briefly, what happens physically is as follows. Consider a cylindrically-symmetric plasma column maintained between two identical hot tungsten cathodes spaced $2L$ apart, as shown in Fig. 1. For the present, let **B** be uniform, so that $R = \infty$. In front of each cathode will be a collisionless sheath a few Debye lengths in thickness, much less than either L or any collision mean free path λ. In the plasma, λ is assumed to be $\ll L$, so that the macroscopic fluid equations can be used; this will be justified in

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Section 2. A drift wave in the plasma is characterized by filament-like perturbations in density $\nu = n_1/n_0$ and potential ϕ which are almost in phase: (CHEN, 1965b) $n \approx n_0 \exp(e\phi/KT_e)$, $\nu \approx e\phi_1/KT_e$. When the wave is unstable, there is a phase shift between ν and ϕ, which means that the electrons are not exactly in thermal equilibrium along each line of force. A current $j_{\parallel}$ must then flow; and, because of the resistivity, an electric field $E_{\parallel}$ is needed to drive this current. In an infinitely long plasma, regions with an excess of electrons can get rid of them because $E_{\parallel}$ drives them into regions

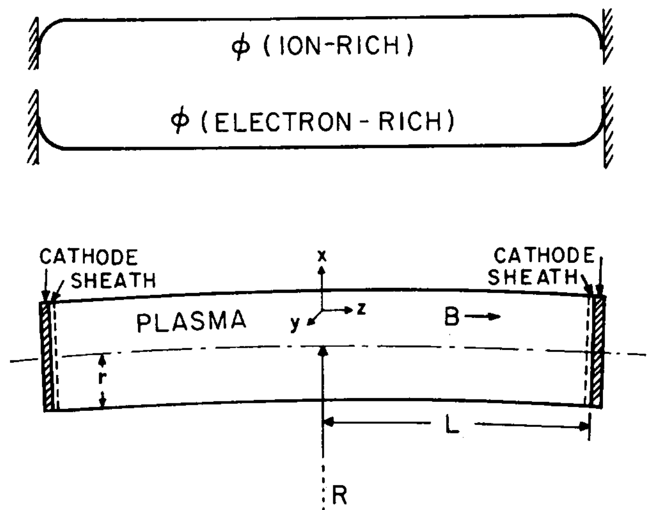

FIG. 1.—Schematic of geometry and potential distribution assumed. The magnetic field **B** is generally uniform but may have a slight curvature with radius R. The analysis will be done in a local Cartesian co-ordinate system located as shown with respect to the centre of curvature.

with a deficiency of electrons, $\lambda_{\parallel}$ being finite. Suppose, however, that we have $L \ll \lambda_{\parallel}$; now a filament of plasma with an excess of electrons no longer 'sees' a region with an electron deficiency. The emitting cathode, however, can take up the flow of electrons providing that the sheath drop, which determines whether electrons flow in or out of the cathode, has the proper magnitude. For $\lambda_{\parallel} \gg L$, the perturbations, ν and ϕ_1 are finite at the sheath edge and are taken up by a small change in sheath thickness and voltage drop at different parts of the cross section. Under certain conditions the perturbation in the sheath drop is just that necessary to give the proper $j_{\parallel}$, and therefore wavelengths $\lambda_{\parallel} \gg L$ can exist. It turns out that this occurs only for large ion sheaths, when the plasma is highly positive relative to the cathodes, so that few plasma electrodes can reach the ends. The sheaths are then good insulators, and one can say that the perturbation is not 'tied down' by the conducting end-plates. For electron sheaths the conductivity is good, and the usual boundary conditions are valid. Note that although the fluid is well tied to the lines of force in the plasma, there is a large slippage between fluid and field in the sheath because of the large value of $E_{\parallel}$ there.

In Section 3 the sheath conditions relating $j_{\parallel}$ to ϕ will be derived. The matching of the collisionless sheath to a collisional plasma is usually a difficult problem requiring some knowledge of what happens in the transitional 'plural-collision' region. It will be shown, however, that thermionic plasmas are particularly simple, and the matching can be done effortlessly. In Sections 4 and 5 the theory of resistive and gravitational overstabilities with current-matching boundary conditions at the sheath edge will be presented; in Section 6 damping mechanisms will be discussed; and in Section 7 the results will be compared with experiments. The concept of oscillations with $\lambda_{\parallel} \gg L$ will seem strange to many physicists, but there is nothing wrong with it once the role of the sheath is understood. We wish to stress that this understanding is important to the study of drift instabilities and 'anomalous diffusion' for the following reason. In plasmas with high conductivity the relevant wavelengths $\lambda_{\parallel}$ are extremely long. To study such phenomena in the laboratory, one must use either a closed geometry or an extremely long one with end-plates. In closed geometries the

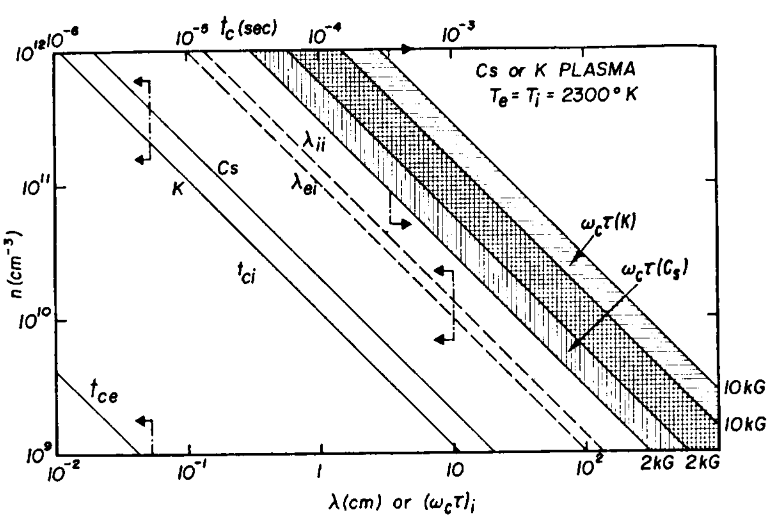

FIG. 2.—Self-collision times t_c, mean free paths λ, and ion $\omega_c\tau$ values for 2300°K Cs and K plasmas in fields of 2–10 kG. For each plotted quantity a set of arrows shows the range in which the fluid theory is applicable to the experiments.

curvature of the magnetic field usually prevents a quiescent equilibrium state from being achieved. End-plate stabilization with electron sheaths in a thermionic plasma makes a quiescent, fully ionized plasma possible, and the fortunate behaviour of the sheaths makes it possible to achieve large effective values of L with short machines and to observe the onset of drift instabilities as $\lambda_{\parallel}$ is changed or as the magnetic field is slowly perturbed from uniformity.

2. APPLICABILITY OF FLUID EQUATIONS

Figure 2 shows several collision parameters for a fully ionized caesium or potassium plasma at 2300°K as a function of plasma density n. The self-collision times t_{ci} and t_{ce} for ions and electrons, respectively, are according to SPITZER (1962). Since observed frequencies ω lie in the range below 30 kc, one has $\omega t_{ce} \ll 1$ for any density above about 10^9 cm^{-3}; and the fluid theory is valid for electrons. For the ions, however, $\omega t_{ci} < 1$ obtains only for $n > 5 \times 10^{11}$ cm^{-3}. Fortunately, the fluid theory is still applicable to the ions because their motion is primarily transverse to **B**, and in this case one requires (CHEW *et al.* 1956) only $\omega \ll \omega_c$. As shown by CHEN (1965a) ion motion along **B** is negligible whenever the wave travels at the electron drift velocity. The ion–ion mean free path λ_{ii} and the ion–electron mean free path λ_{ei} computed from the 90° deflexion time t_D (SPITZER) are independent of mass and are seen to be much shorter than the machine length (generally 20–100 cm) for densities greater than about 10^{10} cm^{-3}. Also shown in Fig. 2 are the values of $\omega_c\tau$ which enter in the computation of ion viscosity. As B is varied between 2 and 10 kG, the values for Cs and K overlap. The computations by CHEN (1965a) are valid for $(\omega_c\tau)^2 \gg 1$ but $\omega_c\tau < 10^2$; hence they are valid for densities roughly between 10^{10} and 3×10^{11} cm^{-3}. Since $n = 10^{10}$–10^{11} cm^{-3} is a typical range for the experiments, (D'ANGELO and MOTLEY; D'ANGELO *et al*; LASHINSKY; BUCHEL'NIKOVA; WOLF and ROGERS; HARTMAN and MUNGER; D'ANGELO) the fluid equations can be expected to provide a much better description of plasma phenomena than the Vlasov equation. In particular, phenomena connected with resonant particles or Landau damping would not be observable unless n were well below 10^9 cm^{-3}.

We thus omit from consideration the collisionless universal instability (KADOMTSEV and TIMOFEEV) as well as a whole body of literature (EICHENBAUM and HERNQVIST, 1961) dealing with sheath instabilities in caesium diodes. The latter works show that in a collisionless thermionic plasma various stable potential distributions are possible, including non-monotonic ones, and that the plasma may oscillate between these states. In the experiments under consideration the collisionless assumption is certainly not valid in the body of the plasma. We have not been able to prove that a non-monotonic potential distribution is impossible in this case, but for simplicity we shall *assume* that the potential varies monotonically from the cathode to the centre of the plasma. This is probably what happens in practice, since collisions tend to destroy local potential wells.

3. THE SHEATH CONDITIONS

(a) *Electron-rich plasma*

Let the potential ϕ be zero at the mid-plane of the plasma and rise monotonically to $\phi_c > 0$ at the cathode. Let the electron temperature T_e be equal to the cathode temperature T_c, and let the ion temperature be given by $T_i = \lambda T_e$. We assume the sheath to be so thin that plane-parallel geometry can be used. At some distance s from the cathode, to be clarified later, we define a sheath edge $x = s$, at which the potential ϕ_s is essentially zero and within which collisions may be neglected. Electrons of the plasma arriving at the sheath edge are accelerated toward the cathode, passing only once through the sheath. If the distribution at s is Maxwellian, the density at x, where the potential is $\phi(x)$, is well known to be

$$n_e^p = n_{es}^p \, e^{\chi} (1 - \operatorname{erf} \chi^{1/2}), \tag{1}$$

where

$$\chi \equiv e\phi/KT_e, \quad \operatorname{erf} y = (2/\pi^{1/2}) \int_0^y e^{-t^2} \, dt, \tag{2}$$

and the superscript p indicates particles coming from the plasma. The density of 'incoming' electrons n_{es}^p at $x = s$ is just $j_{er}/2v_{the}$, where j_{er} is the random electron current in the plasma, and

$$v_{the}^2 \equiv KT_e/2\pi m_e. \tag{3}$$

The electrons emitted by the cathode are assumed to be a half-Maxwellian distribution at a temperature $T_c = T_e$, and they encounter a potential barrier. If the superscript c indicates particles originating at the cathode, their density is given by

$$n_e^c = n_{ec}^c \, e^{\chi - \chi_c} (1 + \operatorname{erf} \chi^{1/2}), \tag{4}$$

where

$$n_{ec}^c = j_T/2v_{the}, \tag{5}$$

j_T being the Richardson current emitted at temperature T_c. This density is composed of electrons moving in both directions, and the error function term gives the deviation from an exponential distribution due to electrons which go over the potential hill into the plasma. This term is exactly equal to the erf term in equation (4) if no net current flows into the plasma. The incoming flux j_e^p is just j_{er}, while the outgoing flux j_e^c is $j_T\,e^{-\chi_c}$; hence the net flux into the plasma is

$$j_e = j_e^c - j_e^p = j_T\,e^{-\chi_c} - j_{er}. \tag{6}$$

If $j_e^p = j_e^c$, we have $n_{es}^p = n_{ec}^c\,e^{-\chi_c}$; in this case the total electron density in the sheath is

$$n_e = n_e^p + n_e^c = (j_{er}/v_{the})\,e^{\chi}, \tag{7}$$

and the cathode acts like a perfectly transmitting grid which merely sets the potential at $x = 0$. This occurs in equilibrium when diffusion and volume recombination are negligible compared to end losses, as we shall assume.

As for the ions, let the probability of reionization upon striking the cathode be p, and let the flux emitted due to ionization of a neutral beam impinging on the cathode be j_0. Incoming ions are retarded by the potential. If the ions have a Maxwellian distribution at s, their density in the sheath is given by

$$n_i^p = n_{is}^p\,e^{-\chi/\lambda}\{1 + \mathrm{erf}\,[(\chi_c - \chi)/\lambda]^{1/2}\}, \tag{8}$$

where $n_{is}^p = j_{ir}/2v_{thi}$ is the density of incoming ions at s, and

$$v_{thi}^2 \equiv KT_i/2\pi m_i. \tag{9}$$

Ions originating at the cathode are accelerated; and if they are emitted in a half-Maxwellian at temperature T_c, their density is given by

$$n_i^c = n_{ic}^c\,e^{\chi_c - \chi}\,[1 - \mathrm{erf}\,(\chi_c - \chi)^{1/2}]. \tag{10}$$

Here n_{ic}^c is the density of outgoing ions at the cathode, created by ionization of the neutral beam and reionization of incident ions which have penetrated the potential barrier:

$$n_{ic}^c = \lambda^{1/2}(j_0 + pj_i^p)/2v_{thi}, \tag{11}$$

where $j_i^p = j_{ir}\,e^{-\chi_c}$. The outgoing flux is $j_i^c = 2n_{ic}^c v_{thi}/\lambda^{1/2}$, and hence the net ion flux into the plasma is

$$j_i = j_i^c - j_i^p = j_0 - (1 - p)\,j_{ir}\,e^{-\chi_c}. \tag{12}$$

If, as shown by VON GOELER (1964), end-plate losses are dominant, we may set $j_i = 0$ in equilibrium. Then if $\lambda = 1$, the error function terms in equations (8) and (10) are again equal and cancel each other in the expression for the total density:

$$n_i = (j_{ir}/v_{thi})\,e^{-\chi}. \tag{13}$$

Thus even for the ions the cathode acts as a transparent grid; the distribution function is Maxwellian everywhere, and the density follows the Boltzmann law. This is true only if $T_i = T_e$ and if j_0 is just sufficient to compensate for end-plate recombination so that the net ion current is zero.

(b) *Ion-rich plasma*

When $\chi_c < 0$, electrons from the plasma are repelled and contribute a density

$$n_e{}^p = n_{es}{}^p\, e^{\chi}\, [1 + \operatorname{erf}\,(\chi - \chi_c)^{1/2}], \tag{14}$$

where $n_{es}{}^p = j_{er}/2v_{the}$. Electrons emitted from the cathode are accelerated and contribute a density

$$n_e{}^c = n_{ec}{}^c\, e^{\chi-\chi_c}\, [1 - \operatorname{erf}\,(\chi - \chi_c)^{1/2}], \tag{15}$$

where $n_{ec}{}^c = j_T/2v_{the}$. Again the distribution is Maxwellian everywhere if $j_e = 0$. In any case, j_e is given by

$$j_e = j_T - j_{er}\, e^{\chi_c}. \tag{16}$$

Ions coming from the plasma are accelerated and contribute a density

$$n_i{}^p = n_{is}{}^p\, e^{-\chi/\lambda}\, [1 - \operatorname{erf}\,(-\chi/\lambda)^{1/2}], \tag{17}$$

where $n_{is}{}^p = j_{ir}/2v_{thi}$. Ions emitted from the cathode are retarded by the potential, and those with insufficient energy to reach the plasma are reflected and re-ionized at the cathode with a probability p. If $j_{ic}{}^c\, e^{\chi_c}$ is the fraction escaping into the plasma, the emitted flux $j_{ic}{}^c$ is given by

$$j_{ic}{}^c = j_0 + pj_{ic}{}^c(1 - e^{\chi_c}) + pj_i{}^p, \tag{18}$$

where the incoming flux $j_i{}^p$ is equal to j_{ir} in this case. The ion density is then

$$n_i{}^c = n_{ic}{}^c\, e^{\chi_c-\chi}\, [1 + \operatorname{erf}\,(-\chi)^{1/2}], \tag{19}$$

with

$$2n_{ic}{}^c v_{thi}/\lambda^{1/2} = j_{ic}{}^c = (j_0 + pj_i{}^p)/[1 - p(1 - e^{\chi_c})]. \tag{20}$$

The net ion current into the plasma is given by equation (20) and

$$j_i = j_i{}^c - j_i{}^p = j_{ic}{}^c\, e^{\chi_c} - j_i{}^p. \tag{21}$$

If $j_i = 0$ and $\lambda = 1$, the total ion density $n_i = n_i{}^c + n_i{}^p$ is again given by equation (13), and the ion distribution is Maxwellian everywhere. The behaviour of the partial densities with χ for this case is shown in Fig. 3. Note that the acceleration of emitted electrons by the sheath voltage drop should not increase the electron temperature, as is often erroneously supposed (D'ANGELO and MOTLEY); these high-energy electrons simply replace those reaching the cathode by penetrating the Coulomb barrier. However, thermoelectric currents arising from differences in temperature of the two end-plates could heat the electrons.

(c) *The sheath edge*

For convenience we consider only the case of ion sheaths ($\chi_c < 0$); the opposite case follows analogously. The distribution of potential in the sheath is found by integrating Poisson's equation,

$$d^2\phi/dx^2 = 4\pi e\,(n_e^p + n_e^c - n_i^p - n_i^c), \quad (22)$$

with the densities given by equations (14), (15), (17) and (19). To ensure a smooth transition to the plasma solution $n_i = n_e$, this equation must be solved with the boundary conditions $\phi = \phi' = 0$ at $x = s$, together with $\phi'' = 0$, which determines the position of the sheath edge s. It has been pointed out by BOHM (1949) that a

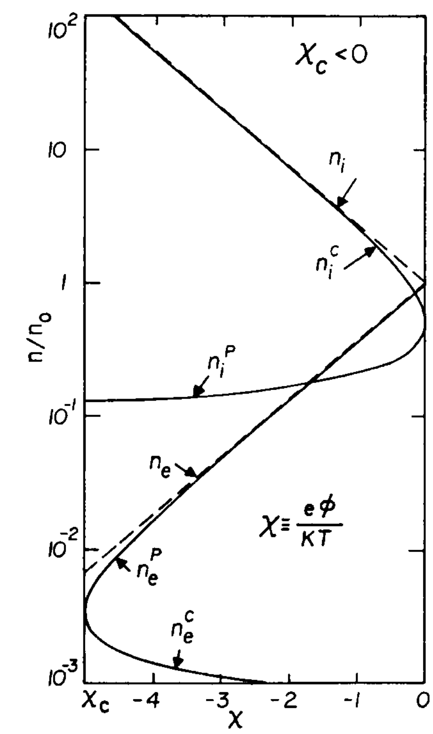

FIG. 3.—Behaviour of ln n_i and ln n_e and their components in an ion sheath as a function of potential χ. The superscripts p and c stand for particles entering the sheath region from the plasma and the cathode, respectively.

monotonic solution with such boundary conditions does not always exist. We are not concerned with the actual solution, which will only tell us the sheath thickness is some multiple of the Debye length and hence is thin in comparison to macroscopic gradients, but only with whether or not the assumed velocity distributions are consistent with a monotonic potential distribution. For instance, suppose the plasma ions were cold, so that $\lambda \ll 1$. Then in order for the incoming and outgoing fluxes to be approximately equal, the density n_i^p must be larger than n_i^c at $x = s$. Because of the large slope of n_i^c near $\chi = 0$ (see Fig. 3), the ion density will fall below n_e just inside the sheath, and the curvature given by equation (22) will be of the wrong sign to give a monotonic solution. This will also happen if ion emission is insufficient, so that the n_i^c curve lies below that shown in Fig. 3. In the extreme case of a cold cathode and cold ions, it is well known (BOHM) that the ions must stream into the sheath with the acoustic velocity in order to give a monotonic solution. This requirement can be derived from the sheath criterion (CHEN, 1965c) $dn_i/d\chi \le dn_e/d\chi$ at $\chi = 0$. For ion-emitting cathodes one might think that this criterion could be used to find the streaming velocity that gives the proper boundary condition for equation (22). However, we note on Fig. 3 that the slopes of n_i^c and n_i^p are infinite at $\chi = 0$, the singularity coming from the derivative of the erf terms in equations (17) and (19), whereas the slopes of n_e^c and n_e^p are finite. The singularity in $dn_i^p/d\chi$ can be removed by assuming a directed drift velocity for ions entering the sheath, but the singularity for the emitted ions remains. To remove it requires treating the ion motion more carefully in the region of small $|\chi|$; that is, in the region where collisions set in.

These problems connected with anisotropic distributions can be avoided, at least in equilibrium, by considering only completely isothermal systems. We therefore henceforth assume that $T_i = T_e = T_c \equiv T$, and that losses occur only at the end-plates, so that no current flows in equilibrium. Then the erf terms cancel each other, $dn_i/d\chi$ is finite, and the distributions are isotropic and in fact Maxwellian everywhere. The densities n_e and n_i are given by equations (17) and (13), and equation (22) has a well-defined solution. Since equations (7) and (13) are valid regardless of the number of collisions, the exact position of the sheath edge is irrelevant.

(d) *The boundary conditions*

The ion and electron fluxes into the plasma are given by equations (12) and (6) for $\chi_c > 0$ and by equations (21) and (16) for $\chi_c < 0$. For given T and j_T, the equilibrium condition $j_e = 0$ gives a relation between the sheath drop χ_c and the density n_0 occurring in $j_{er} = n_0 v_{the}$. The condition $j_i = 0$ then gives n_0 in terms of the neutral flux, occurring in j_0. Since no currents flow, n_0 and $\chi_0(= 0)$ are constant in the plasma. During an oscillation, n and χ may take values $n = n_0 + n_1$, $\chi = \chi_0 + \chi_1$ at the sheath edge $x = s$. If the oscillation frequency is much less than the ion plasma frequency, so that the sheath has time to come to equilibrium, the perturbations in n and χ will cause currents to flow in or out of the cathode and the sheath. These currents are given by the above-mentioned equations for j_i and j_e, with $\chi_c - \chi$ replacing χ_c and $(1 + \nu)\, j_r$ replacing j_r, where $\nu \equiv n_1/n_0$. These currents must then be matched to those existing in the plasma and found from the linearized drift wave analysis of the fluid equations. Defining

$$\iota_T \equiv j_T/n_0 v_{the}, \qquad \iota_e \equiv j_e/n_0 v_{the}, \tag{23}$$

we can write equations (6) and (16) as follows:

$$\begin{aligned} \chi_c > 0&: \ \iota_e = \iota_T\, e^{\chi-\chi_c} - (1+\nu) \\ \chi_c < 0&: \ \iota_e = \iota_T - (1+\nu)\, e^{\chi_c-\chi}. \end{aligned} \tag{24}$$

A similar but more complicated set for the ions can be found from equations (12) and (21); however, we shall not need this since ion motion along the magnetic field will be neglected. To linearize equation (24), we expand the exponential for $\chi \ll 1$ and subtract the steady-state equation $\iota_T = e^{\chi_c}$; we then obtain the required boundary conditions:

$$\begin{aligned} \chi_c > 0&: \ \iota_e = \chi - \nu \\ \chi_c < 0&: \ \iota_e = \iota_T(\chi - \nu). \end{aligned} \tag{25}$$

These currents flow because the ions do not have time to move a distance L to readjust n to its equilibrium value. Hence the validity of equation (25) requires $\omega_{pi} \gg \omega \gg 2\pi v_s/L$, where $v_s^2 = KT/m_i$. Since we shall be concerned with $\lambda_{\parallel} \gtrsim L$ this condition is satisfied by drift waves, for which $\omega/k_{\parallel} \gg v_s$.

(e) *Deviations from the idealized case*

Diffusion and recombination in the plasma will cause small fluxes of both ions and electrons into the plasma in equilibrium. The effect will be to raise the n_i^c and n_e^c curves from the positions shown in Fig. 3 for the idealized case. It is clear that the condition $n_i \geqslant n_e$ is still satisfied everywhere in this case, and only a small change in sheath thickness will result. The electron-rich case is analogous. On the other hand, perturbations in plasma potential can cause particle fluxes either into or out of the plasma, corresponding respectively to raising or lowering the n_i^c and n_e^c curves in Fig. 3. In the latter case the space charge just inside the sheath edge would be negative instead of positive, unless one postulates an ion drift at the sheath edge, caused by a leakage of the sheath electric field into the quasi-neutral region. The same occurs for electrons when $\chi_c > 0$. The sheath edge must then be taken a few mean free paths into the plasma in order for equations (6) and (16) to be valid. We assume that $\lambda_{\parallel}$ is still much larger than s, so that s can be neglected in the evaluation of the length of the plasma column.

In most experiments the neutral beam is applied to only one of the cathodes. Recombination at the other hot cathode then causes a small ion current to flow, driven by a pressure gradient along the column. Since electrons are in approximate thermal equilibrium, there must be an associated potential gradient in the plasma, and the sheath drops at the two cathodes will be slightly different. We assume that both this difference and the Doppler shift due to the mass motion are negligible.

When the machine is operated 'single-ended', that is, with one cold cathode, a large loss of both species will occur at the cold end. In an ion-rich plasma the effects of mass flow between the plates will be like those in the preceding paragraph, but larger. In an electron-rich plasma, distinction is made between a floating cold cathode and one grounded to the hot cathode. It is not possible to have an electron sheath next to the cold cathode because of the mobility of the electrons. If the electron emission is increased indefinitely and the cathodes are connected, a double sheath will form in front of the hot cathode. If the cold cathode floats, it will charge up negatively, and an ion sheath will form there while an electron sheath forms at the hot cathode. In 'single-ended' operation an exact analysis is not possible because of the difficulty of matching the sheath to the plasma in the presence of one-sided distribution functions. However, the Doppler shift is still probably negligible, and the results of the next section can be applied qualitatively: the sheath with the greater conductivity will control the damping of the drift instabilities, but it is effectively twice as far from the middle of the plasma as in the case of symmetrical operation.

Finally, we note that the equilibrium sheath condition $j_T = n_0 v_{the}\, e^{\chi_c}$ implies that a radial electric field E_r must exist if the density has a radial profile. If n_0 falls exponentially with radius r, then E_r is constant, and the results of the next section are valid in a rotating system in which E_r vanishes. Experimentally it is often found (Grieger, private communication) that E_r is smaller than the above relation would predict. This is due to a decrease of T, and hence j_T, at large radii.

4. RESISTIVE OVERSTABILITY

We now wish to perform a perturbation analysis of a plasma of finite length using the macroscopic equations and simplifying assumptions examined by Chen, (1965a) together with the boundary conditions derived in Section 3. With the assumptions of quasi-neutrality, electrostatic waves, low β, and constant and uniform temperature, the appropriate equations are

$$m_i n(\partial \mathbf{v}_i/\partial t + \mathbf{v}_i.\nabla \mathbf{v}_i) = en(-\nabla\phi + \mathbf{v}_i \times \mathbf{B})$$
$$- KT\nabla n + (n/R)(KT + m_i v_{zi}^2)\,\hat{\mathbf{x}} + n^2e^2\eta(\mathbf{v}_i - \mathbf{v}_e) - \nabla.\pi_i \quad (26)$$

$$\partial n/\partial t + \nabla.(n\mathbf{v}_i) = 0, \quad (27)$$

and a similar pair, equations (26′) and (27′), for the electrons, but with m_e set equal to zero. Here **B** is constant and essentially uniform but may have a small curvature of radius R, and the resistivity η is a constant scalar. In this simplified treatment only the collision-independent part of the viscosity tensor π will be included. We use a local Cartesian co-ordinate system (see Fig. 1) with **B** in the z-direction, $-\nabla n_0$ and the effective gravitational force due to the curvature of **B** in the x-direction, and the propagation vector **k** in the y–z plane. The mid-plane of the machine is at $z = 0$, and the sheath edges at $z = \pm L$.

In the steady state we assume $\mathbf{E} = \partial/\partial t = \partial/\partial y = \partial/\partial z = v_{ze} = v_{zi} = 0$. Solution of equations (26) and (26′) then gives the diamagnetic drifts

$$v_{yi}^{(0)} = -v_{ye}^{(0)} = \frac{KT}{eB_0}\left(\frac{n_0'}{n_0} - \frac{1}{R}\right), \quad (28)$$

where the prime indicates $\partial/\partial x$ and B_0 is the magnitude of **B** at $x = 0$. The variation of $|\mathbf{B}|$ with x due to the curvature will be neglected. We also neglect the small velocity $v_x^{(0)}$ due to finite resistivity. For simplicity we consider density profiles such that $v_y^{(0)}$ is a constant; if R is large, as we shall assume, the density falls off nearly exponentially with radius. It is convenient to use dimensionless variables with $v_s \equiv (KT/m_i)^{1/2}$ as the unit of velocity and $\omega_c^{-1} \equiv m_i/eB_0$ as the unit of time, and hence $a \equiv v_s/\omega_c = r_L/\sqrt{2}$ as the unit of length. Equation (28) then reads simply

$$v_{yi}^{(0)} = -v_{ye}^{(0)} = \delta - \rho \equiv v_0, \quad (29)$$

where $v \equiv v/v_s$, $\delta \equiv an_0'/n_0$, and $\rho \equiv a/R$.

We now consider perturbations of the form

$$\begin{aligned} n_1 &= \nu(z)n_0(x) \exp i(ky - \omega t) \\ \chi_1 &= \chi(z) \exp i(ky - \omega t) = e\phi_1/KT \\ \mathbf{v}^{(1)} &= \mathbf{v}(z) \exp i(ky - \omega t). \end{aligned} \quad (30)$$

Dividing by $m_i n\omega_c v_s$, non-dimensionalizing and linearizing equations (26) and (26′) as in CHEN (1965a), we obtain the first-order equations of motion:

$$-i\psi\mathbf{\upsilon} = -\nabla(\chi + \nu) + \mathbf{\upsilon} \times \hat{\mathbf{z}} + \epsilon(\mathbf{\upsilon}_e - \mathbf{\upsilon}) - 2\epsilon v_0\nu\hat{\mathbf{y}}$$
$$+ \tfrac{1}{2}i\delta\kappa(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}) + \tfrac{1}{2}\kappa^2(v_x\hat{\mathbf{y}} - v_y\hat{\mathbf{x}}) \quad (31)$$

$$0 = \nabla(\chi - \nu) - \mathbf{\upsilon}_e \times \hat{\mathbf{z}} + \epsilon(\mathbf{\upsilon} - \mathbf{\upsilon}_e) + 2\epsilon v_0\nu\hat{\mathbf{y}}, \quad (32)$$

where $\psi \equiv \Omega - \kappa v_0$, $\Omega \equiv \omega/\omega_c$, $\epsilon \equiv n_0 e\eta/B$, $\kappa \equiv ka$, and $\mathbf{\nabla}$ is in ξ-space, where $(\xi,\eta,\zeta) = (x,y,z)/a$. We have suppressed the subscript i on ion quantities. Electron viscosity has been neglected, and the ion viscosity evaluated as by CHEN (1965a). The curvature ρ does not appear explicitly but is contained in v_0 and ψ. The centrifugal term in $v_{zi}{}^2$ in equation (26) is higher order. ψ is a constant by assumption. If the normalized resistivity ϵ is small, as we shall assume, it can be neglected in the x and y components of equations (31) and (32). These equations then yield the following velocity components:

$$Cv_x = -i\kappa(1 - \tfrac{1}{2}\kappa^2)(\chi + \nu) \tag{33}$$

$$Cv_y = -\kappa(\psi + \tfrac{1}{2}\delta\kappa)(\chi + \nu) \tag{34}$$

$$v_z \approx 0, \quad v_{ye} \approx 0 \tag{35}$$

$$v_{xe} = -i\kappa(\chi - \nu) \tag{36}$$

$$\epsilon v_{ze} = (\chi - \nu)^*, \tag{37}$$

where

$$C = (1 - \tfrac{1}{2}\kappa^2)^2 - (\psi + \tfrac{1}{2}\delta\kappa)^2, \quad * = \partial/\partial\zeta.$$

Equation (37) is essentially the generalized Ohm's law giving the current along **B**. Despite its smallness, ϵ cannot be neglected because we wish to consider small values of $k_{\parallel}$ such that the ζ derivative is very small. Equation (36) can be recognized as the sum of the $\mathbf{E} \times \mathbf{B}$ and $\mathbf{\nabla}p \times \mathbf{B}$ drifts; equation (33) gives the same for ions but with a finite Larmor radius correction. Equation (34) gives a component v_y which arises because of ion inertia; v_{ye} vanishes with the neglect of E_x, $\eta_{\perp}$, and electron Larmor radius. The approximation $v_{zi} = 0$ was examined by CHEN (1964; 1965a) and found to be valid for $k_{\parallel}/k \ll \delta$, which is always true for waves travelling at the diamagnetic drift velocity.

We next non-dimensionalize and linearize the equations of continuity (27) and (27′). This gives

$$\psi\nu + i\delta v_x - \kappa v_y = 0 \tag{38}$$

$$\psi_e\nu + i\delta v_{xe} + iv_{ze}{}^* = 0, \tag{39}$$

where $\psi_e \equiv \Omega - \kappa v_{0e} = \Omega + \kappa v_0$. Insertion of equations (33) and (34) into (38) gives a relation between χ and ν. If we postulate low frequencies $\omega \ll \omega_c$, so that ψ^2 can be neglected in C, and assume $\delta \ll 1$, $\kappa \ll 1$, so that higher powers of these quantities can be neglected, this relation can be expressed as follows:

$$\nu - \chi = \sigma\nu, \quad \sigma = [2\delta\kappa + (1 + \kappa^2)\psi]/(\delta\kappa + \kappa^2\psi). \tag{40}$$

Substituting equations (36), (37) and (40) into the electron equation of continuity (39), we obtain a differential equation for ν:

$$(\sigma/i\epsilon)\partial^2\nu/\partial\zeta^2 + (\psi_e - \delta\kappa\sigma)\nu = 0, \tag{41}$$

in which the coefficients do not depend on ζ. The symmetric solution is

$$\nu = \nu_0 \cos \kappa_{\parallel}\zeta, \tag{42}$$

where

$$\kappa_{\parallel}{}^2 = i\epsilon\sigma^{-1}(\psi_e - \delta\kappa\sigma). \tag{43}$$

For physical reasons we require $\kappa_{\|}$ to be real and look for complex values of Ω which make it so.

To find the eigenvalues of $\kappa_{\|}$, we apply the boundary conditions given by equation (25), which can be written

$$(2\pi\mu)^{1/2}v_{ze} = \pm(\nu - \chi)\begin{Bmatrix}1\\ \iota_T\end{Bmatrix} \text{ at } \zeta = \pm\, L/a \text{ for } \begin{Bmatrix}\chi_c > 0\\ \chi_c < 0\end{Bmatrix}, \tag{44}$$

where $\mu = m_e/m_i$. The value of v_{ze} existing in the plasma is given by equations (37), (40) and (42):

$$v_{ze} = \epsilon^{-1}(\chi - \nu)^* = -\epsilon^{-1}\sigma\nu^* = \epsilon^{-1}\sigma\nu_0\kappa_{\|}\sin\kappa_{\|}\zeta. \tag{45}$$

Evaluating this at $\zeta = \pm L/a$ and substituting into equation (44), we obtain

$$k_{\|}L\tan k_{\|}L = \frac{L}{a}\frac{\epsilon}{(2\pi\mu)^{1/2}}\begin{Bmatrix}1\\ \iota_T\end{Bmatrix} \text{ for } \chi_c \gtrless 0\,. \tag{46}$$

The right-hand side is determined by the operating parameters: T, n_0, B, L and m_i. Equation (46) then gives the permissible values of $\kappa_{\|}$. To see whether these values correspond to unstable Ω's, we substitute equation (40) into equation (43) and write $\psi_e = \Omega + 2\kappa(\delta - \rho)$ to obtain

$$\psi^2 + (\delta\kappa - 2\kappa\rho + iY)\,\psi + 2\delta(i\kappa Y - \rho) = 0, \tag{47}$$

where $Y \equiv \kappa_{\|}^2/\epsilon\kappa^2$ and $\psi \equiv \Omega - \kappa(\delta - \rho)$. This is just the dispersion relation, equation (88) of CHEN (1965a) for an infinitely long plasma with $T_i = T_e$. This relation is shown in Fig. 4, in which we have plotted Re $\Omega/\delta\kappa$ and Im $\Omega/\delta\kappa$ against $(Y/\delta\kappa)^{1/2}$ for $\rho = 0$ and one non-vanishing value of ρ/κ. It is seen that Im Ω is positive for any finite value of Y because no damping process has been taken into account. To include damping one has merely to substitute for equation (47) one of the more complete dispersion relations given by CHEN (1965a).

Our principal result is given by equations (46) and (47) and the latter's asymptotic form for $Y \gg |\delta\kappa|$, $|\delta| \gg \rho$:

$$\psi \approx -2\delta\kappa + 2iY^{-1}(\delta^2\kappa^2 - \delta\rho). \tag{48}$$

From equation (46) we see that if the right-hand side is large, $k_{\|}L$ must be nearly $\pi/2$, or an odd multiple thereof, and the perturbation is effectively tied at the ends. In thermionic plasmas this is usually the case for electron sheaths, for which the '1' in the curly brackets is to be used. For ion-rich plasma, the 'ι_T' is to be used, and this factor, indicating the ratio of emitted current to random electron current in the plasma, may be very small if the sheath drop is large. Then the left-hand side can be approximated by $k_{\|}^2L^2$, and the resulting value of $k_{\|}$ may be substituted into equation (48) to obtain the growth rate. If the growth time is larger than the duration of the plasma or than a characteristic damping time such as that for classical diffusion, then the overstability will not occur. Thus unless ρ is large or the machine is very long, oscillations will generally be observed only with sufficiently strong ion sheaths; otherwise the conductivity of the sheaths is so high that $\chi = 0$ at the sheath edge is maintained, and

$k_{\parallel}$ is forced to be so large that the growth rate given by equation (48) or Fig. 4 is extremely small. On the other hand, with strong ion sheaths the potential barrier for plasma electrons escaping to the cathodes is so large that only a small fraction reaches the cathode; the sheath conductivity is then effectively so low that χ can be finite at the sheath edge without violating the current-matching conditions there. Then it is possible to have $\lambda_{\parallel} \gg L$.

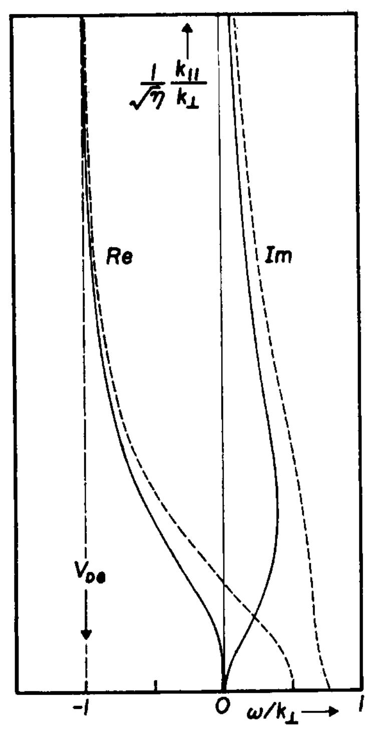

FIG. 4.—The dispersion relation for resistive overstabilities with (dashed line) and without (solid line) curvature in the magnetic field. The real and imaginary parts of $\omega/k_{\perp}$ (in units of the electron diamagnetic drift v_{De}) are plotted against $k_{\parallel}$ (arbitrary scale) for the case $T_i = T_e$.

The average physicist is so unaccustomed to this concept that we feel it is necessary to give a further physical explanation. Note that if one plots the perturbation χ on any line of force for an unstable case in which $2\pi/k_{\parallel} \gg 4L$, he would find that χ is indeed zero at the two end-plates, rises abruptly to a finite value at the sheath edge and varies slowly in the main plasma. The Fourier transform of this function would indeed have no component with $k_{\parallel} < \pi/2L$. However, this does not prevent the perturbation from behaving as if it had $k_{\parallel} \gg \pi/2L$ *in the region between the sheaths*, as long as the proper conditions at the sheath edge are satisfied. The sheath varies in thickness on different lines of force in order to accommodate the finite perturbation in χ and ν at the sheath edge. Equation (42) shows that the oscillation is a standing wave in the z-direction, while it is a travelling wave in the y- or azimuthal direction. The perturbations are therefore perfectly straight in the z-direction, whereas they would be helical in an infinite cylinder. However, the density perturbation varies in amplitude along **B**, and this allows a parallel current to cancel charge accumulations caused by ion motions perpendicular to **B**. A physical picture of the instabilty mechanism has been given by CHEN (1965b). At the ends of the machine the cathodes act as reservoirs of electrons which are able to absorb or emit just that current which would have flowed had the perturbation been able to continue on past the sheath edge.

The generalization of this theory to include finite v_{zi} would be straightforward but quite tedious. To the ion equation (38) we must add a term $iv_{zi}{}^*$, which in general will be quite complicated because v_{zi} is coupled to v_{ze} by resistivity and to the perpendicular components by viscosity. In the limit $\epsilon \gg \mu$ which we are considering, v_{zi} is caused mainly by the drag due to collisions with electrons. From the z-components of equations (26) and (26′), one then obtains approximately $v_{zi} \approx -2i\nu^*/\psi$. Equation (38) then becomes a second-order differential equation, and the elimination of χ between equations (38) and (39) gives a fourth-order differential equation for ν. The solution of this requires the use of the ion sheath conditions (12) and (21), which are somewhat more complicated than the electron conditions. Fortunately, the neglect of v_{zi} is usually a good approximation.

5. THE RAYLEIGH–TAYLOR INSTABILITY

In the collisionless limit no instability occurs in the fluid theory unless ρ is finite. Then the well-known gravitational instability with finite Larmor radius correction occurs for $k_{\|} = 0$. The dispersion relation is given by equation (47) with $Y = 0$. The damping of this instability by the end-plates must be treated a little differently since ϵ and Y now vanish. The ion equations (38) and (40) are unaffected; but in the electron equation (39) we set $v_{ze}{}^* = 0$, since $\partial/\partial\zeta$ is now zero, and add a source term representing the influx of electrons from the cathodes. If the transit time of electrons along the length of the machine can be neglected, so that this influx fills the entire tube of force uniformly, we must add a term j_e/L to the right-hand side of equation (27′) or a term $iv_{ze}a/L$ to the right-hand side of equation (39). With v_{ze} given by the sheath condition (44), the electron continuity equation now reads

$$\psi_e \nu + \delta\kappa(\chi - \nu) = ig(\chi - \nu), \tag{49}$$

where

$$g \equiv a/(2\pi\mu L^2)^{1/2}\{1, \iota_T\} \geqslant 0 \text{ for } \chi_c \gtrless 0. \tag{50}$$

Substituting equation (40) for $\chi - \nu$, we obtain

$$\psi^2 + (\delta\kappa - 2\kappa\rho + ig\kappa^{-2})\psi - 2\delta\rho + 2i\delta\kappa^{-1}g. \tag{51}$$

The damping term in g is always 90° out of phase with the exciting term in ρ; hence the end-plates *per se* cannot eliminate this instability. The effect of the g terms is always to diminish the growth rate, as can be verified by tracing the roots of equation (51) in the complex plane as g is increased from zero.

6. DAMPING MECHANISMS

The sheath condition (46) fixes the minimum value of $\kappa_{\|}$, which for any given mode determines a horizontal line on Fig. 4. To see whether or not a wave can grow, one must compare the growth rate given by this line with a damping rate. Five damping mechanisms neglected in the derivation of equation (47) are discussed below.

(a) *Resistive damping*

In CHEN (1965a), we have calculated the effect of perpendicular diffusion. For $\lambda = 1$, $\rho = 0$, $Y \gg |\delta\kappa|$, the growth rate is approximately

$$\operatorname{Im} \Omega = 2\kappa^2(\delta^2/Y - \epsilon). \tag{52}$$

Thus the overstability occurs if $Y < \delta^2/\epsilon$, or $(\kappa_{\|}/\kappa^2) < \delta^2$.

(b) *Viscous damping*

In CHEN (1965a) we have also calculated the effect of ion–ion collisions for $1 \ll (\omega_c \tau_{ii})^2 < 10^4$. For $\lambda = 1$, $\rho = 0$, $Y \gg |\delta\kappa|$, the result reduces to

$$\text{Im}\,\Omega = 2\kappa^2(\delta^2/Y - \kappa^2/4\omega_c\tau_{ii}). \tag{53}$$

The viscous damping is due to the diffusion of ions by like-particle collisions. As is well known, the diffusion velocity is proportional to $\partial^3 n/\partial y^3$, and the divergence of this gives rise to the κ^4 dependence of the damping term. Overstability therefore occurs for $\kappa_{\|}{}^2 < 4\epsilon\delta^2\omega_c\tau_{ii}$.

(c) *Ion lifetime*

It is clear that appreciable growth of an oscillation can occur only if $\tau\,\text{Im}\,\omega \gg 1$, where τ is the duration of the plasma. In a steady-state thermionic plasma, τ is effectively the lifetime of an ion. If ions hitting the cathode are reionized in a time short compared to ω^{-1}, a density perturbation is preserved in spite of the flux of ions to the ends. However, ions recombining at the cathodes are replenished by ionization of a neutral beam, which does not 'see' the perturbation; and the latter is gradually 'forgotten'. To estimate τ, we set

$$\tau = N[(1 - \text{p})\partial N/\partial t]^{-1}, \quad \partial N/\partial t = 2n_0 v_{thi}\{\text{e}^{-\chi_c}, 1\} \text{ for } \chi_c \gtrless 0, \tag{54}$$

where $N = 2Ln_0$ and $1 - p$ is the probability of recombining at the cathode. For $\chi_c < 0$, a random current of ions is lost at both cathodes. For $\chi_c > 0$, the escape flux is reduced and τ increased by the Coulomb barrier. For single-ended operation with one hot cathode, the other cathode being a cold, floating plate, the ion loss occurs mainly at the cold plate, and τ is simply $2L/v_{thi}$. Thus we have:

$$\text{ion-rich:} \quad \tau = (L/[(1 - p)v_{thi}])[1 - p(1 - e^{\chi_c})] \tag{55a}$$

$$\text{electron-rich:} \quad \tau = L\,\text{e}^{\chi_c}/[(1 - p)v_{thi}] \tag{55b}$$

$$\text{single-ended:} \quad \tau = 2L/v_{thi}. \tag{55c}$$

This is usually the limiting factor for instability.

(d) *End-plate diffusion*

GRIEGER (private communication) has pointed out that even if reionization of ions at the cathodes were perfectly efficient, a diffusion results from the collisions with the cathodes because the phase of cyclotron gyration is lost in the process. Thus an ion random walks across **B** with a step length $\approx r_L$ and a frequency given by the time of flight to the cathode. The effective diffusion coefficient is approximately

$$D_\perp \approx 2a^2/\tfrac{1}{2}Lv_{thi}^{-1} = 4a^2 v_{thi}/L. \tag{56}$$

The diffusion velocity is then $kD_\perp$ and its divergence $k^2D_\perp$. Putting this into our dimensionless units, and adding the usual growth rate, we find

$$\text{Im}\,\Omega = 2\kappa^2(\delta^2/Y - a/L\sqrt{\pi/2}), \tag{57}$$

so that overstability requires $Y < (\pi/2)^{1/2}L\delta^2/a$.

(e) *Landau damping*

As found in CHEN (1965a), Landau damping can be neglected if $\omega/k_{\|} \ll v_{thi}$. This requires $(\kappa_{\|}/\kappa)^2 \ll \delta^2$, which is the same condition as for the neglect of resistive damping.

7. COMPARISON WITH EXPERIMENT

(a) *Straight* **B**

The original experiment of D'ANGELO and MOTLEY was performed in potassium with approximately the following parameters: $L = 30$ cm, $r = 1{\cdot}5$ cm $T = 2300°$K, $B = 4$ kG, $n_0 = 5 \times 10^{10}$ cm^{-3}. These are well within the range of applicability of the fluid theory, as seen from Fig. 2. Unfortunately, only one hot cathode was used, and our results must be modified in the manner discussed at the end of Section 3. Since $\epsilon = 10^{-4}$, $\mu = 1{\cdot}4 \times 10^{-5}$, $L/a = 400$ in this case, equation (46) for the electron sheaths gives $k_{\|}L \tan k_{\|}L = 4$ or $k_{\|}L = 1{\cdot}3$, $\lambda_{\|} \approx 5L$. The perturbation is almost tied at the end-plates. For this value of $k_{\|}$, we obtain $Y = 40/m^2$, where $m = kr$ is the azimuthal mode number. From equation (48) we obtain, for the measured value $\delta \approx 0{\cdot}2$, Im $\omega \approx 5\, m^4$ sec^{-1}. Since an electron sheath exists at only one end, L and Im ω should be approximately doubled. The ion lifetime is given by equation (55c): $\tau \approx 2 \times 10^{-3}$ sec. Hence τ Im $\omega \approx 2 \times 10^{-2} m^4$, and the $m = 1, 2$ and 3 modes cannot grow, although higher modes should have been observable; perhaps it was an experimental factor such as cable capacitance which kept them from being seen. For large ion sheaths, we have $(k_{\|}L)^2 = 4\iota_T$, with $\iota_T \ll 1$. Then $Y = 10^2\, \iota_T/m^2$ and Im $\omega = 0{\cdot}8m^4/\iota_T$ sec^{-1}, while τ is the same as for the electron-rich case because of single-ended operation. If one assumes τ Im $\omega \approx 10$ for an oscillation to reach detectable amplitude, one obtains $\iota_T = 1{\cdot}8 \times 10^{-4} m^4$ and therefore $-\phi_c = (KT/e) \ln \iota_T = 1{\cdot}7$, $1{\cdot}2$, $0{\cdot}8$ V for $m = 1, 2, 3$, respectively. The observed change in probe floating potential between electron-rich conditions and the condition for onset of instability was 0·8 V agreeing with that expected for the highest observable mode. No phase change was detected along **B**, in agreement with theory, but the variation of amplitude with z was not checked. The theory predicts $\lambda_{\|}/L \approx 25$ at onset.

In the D'ANGELO–MOTLEY experiment the other damping mechanisms of Section 5 were negligible. Since $\kappa_{\|}^2 \leqslant 2{\cdot}5 \times 10^{-5}$ and $\kappa^2 \leqslant 2{\cdot}5 \times 10^{-3}$, $(\kappa_{\|}/\kappa)^2 < 10^{-2} < \delta^2 = 4 \times 10^{-2}$ for all conditions; hence resistive damping is negligible. End-plate diffusion can be neglected if $(\kappa_{\|}/\kappa)^2 < (\pi/2)^{1/2} \times L\delta^2\epsilon/a = 2 \times 10^{-3}$. This is not always satisfied, but it is satisfied whenever oscillations were observed; that is, under ion-rich conditions when $\kappa_{\|}^2$ is reduced by a factor ι_T. Viscous damping is negligible if $\kappa_{\|}^2 < 4\epsilon\delta^2\omega_c\tau_{ii} = 3{\cdot}7 \times 10^{-4}$, with $\omega_c\tau_{ii} = 23$ in this case. This is satisfied for any value of m.

In comparison with the above, the experiment of BUCHEL'NIKOVA differed mainly in the low value of B(600–1600 G), the use of two hot cathodes, and the density range ($n_0 = 10^8 - 5 \times 10^{11}$). The other parameters ($L = 18$ cm, $r = 2$ cm, $T = 2000°$K) were similar. The low value of B and correspondingly large Larmor radii ($r_L \approx 0{\cdot}2\, r$) meant that a radial electric field was necessary to confine the ions, and this could have caused instabilities not discussed here. The use of two hot cathodes greatly increases the ion lifetime. Both these effects make the situation more unstable, and indeed oscillations were observed regardless of the sheath conditions. We note, however, that the detector was more sensitive than in the D'ANGELO–MOTLEY experiment. Taking $n_0 = 10^{10}$ cm^{-3} and $B = 1$ kG, we find $\epsilon = 10^{-4}$ and $k_{\|}L$ tan $k_{\|}L = 0{\cdot}72$, $\kappa_{\|} = 10^{-2}$ and $\lambda_{\|}/L = 8$ for electron sheaths. This wavelength is about twice as long as that observed, but the agreement is adequate. However, our theory predicts a much longer wavelength with ion sheaths, and there would be a discrepancy if $\lambda_{\|}$ did not change with the sheath conditions, as BUCHEL'NIKOVA seems to imply. Taking $\delta = 0{\cdot}13$, $\kappa = 0{\cdot}13\, m$, we find Im $\omega \approx 2\, m^4$ sec^{-1}. The ion lifetime, given by equation (55b), is $\tau = 7 \times 10^{-3} \exp \chi_c$ for $p = 0{\cdot}9$. For $\chi_c = 0$, we have τ Im $\omega \approx 0{\cdot}014\, m^4$, so that only the higher modes should be excited. However, in addition to the usual instability with ion sheaths, we have now a possibility of an instability with electron sheaths if $\exp \chi_c \gg 1$ because the ion lifetime is increased. For τ Im $\omega = 10$, $m = 3$, we would require only $\chi_c = 2{\cdot}2$ or $\phi_c = 0{\cdot}38$ V. Unfortunately, the variation of amplitude with ϕ_c was not given. With large electron sheaths the growth would be limited only by resistive diffusion, since end-plate diffusion also decreases with e^{χ_c}. It is interesting to speculate on what happens near the edge of a plasma column, where the density becomes very low. Unless the cathode temperature falls, an electron sheath must exist there. Then $k_{\|}$ is effectively fixed by the distance between the plates, and Im $\omega \propto \epsilon \propto n_0$. However, the lifetime τ varies as $e^{\chi_c} = \iota_T \propto n_0^{-1}$. Hence the oscillations are able to reach the same limiting amplitude at the edge of the column.

BUCHEL'NIKOVA also finds these oscillations in the low-density regime where collisions may be neglected. The only known instability which can arise under such circumstances is the well-known 'universal' instability of a collisionless plasma. The growth rate for this is comparable to or less than that computed above for finite resistivity. We cannot understand how this instability can arise in a short system, since the excitation mechanism is associated with resonant electrons with $v_{\|} = \omega/k_{\|}$. In the time (Im $\omega)^{-1}$ such electrons make many collisions with the cathodes, and the resonance is lost.

Similarly, the effects observed by LASHINSKY are attributed to Landau damping and the 'universal' instability. A glance at Fig. 2 shows that at his densities of $n_0 \approx 7 \times 10^{10}$ cm^{-3} collisions should be dominant, and the theory presented here should be more applicable. In a double-ended caesium experiment, LASHINSKY found many harmonics, with amplitudes decreasing with increasing m. Although such a spectrum would be expected for the non-linear regime (CHEN, 1965b), this is in disagreement with what the theory predicts for the small amplitudes which were measured. The growth rate, and hence the amplitude, should increase with m, since none of the damping mechanisms has an m-dependence as strong as that of Im ω. In a single-ended experiment, LASHINSKY also measured the effect of changing B and L, and hence $k_{\|}$ and Im ω. At threshold a small change in B or L was found to cause a large change in amplitude. This is in accord with our theory if $\nu \propto \exp(\tau$ Im $\omega)$ and τ Im $\omega \gg 1$. At threshold, there must be an ion sheath large enough to give $\lambda_{\|} \gg L$; no measurement of $\lambda_{\|}$ was given. The observed variation of Re ω with L is not given by this theory; one would have to include the effect of finite v_{zi}. In all the experiments the magnitude of Re ω is in accord with the predicted value; moreover, LASHINSKY also found the direction of propagation to be correct. The radial variation of amplitude shows a peak where δ is large, as expected.

(b) *Curved* **B**

Experiments by D'ANGELO *et al.*, and D'ANGELO, in caesium plasmas with a small curvature $\rho = a/R$ in the lines of force seem to confirm the existence of the gravitational overstability with finite $k_{\|}$. The interpretation of the experiments, however, is not straightforward, since the equilibrium of the plasma is disturbed by the curvature. This has been discussed by ECKHARTT *et al.* (1964) and by COLGATE and YOSHIKAWA (1964). We ignore this problem in what follows, and our remarks are subject to the proviso that difficulties with the equilibrium are not important. In general, waves of a similar nature to those with $\rho = 0$ are observed, with amplitude increasing with ρ. This is in accord with equation (48), describing the upper region of Fig. 4, which predicts an additional growth rate $\text{Im}\,\omega = -2\omega_c\delta\rho/Y$ with $\text{Re}\,\omega$ essentially unchanged. This becomes larger than the $\rho = 0$ growth rate when $\rho > |\delta\kappa^2|$ or $R \lesssim 200$ cm for $m = 1$. This is in order of magnitude agreement with experiment, but an exact comparison is difficult because the oscillation amplitude depends sensitively on $\text{Im}\,\omega$ near threshold, as previously discussed. However, there are some difficulties with this simple interpretation. Large m values are found to occur only for large ρ. This is predicted only for small values of Y, when the instability is essentially the Rayleigh–Taylor instability, which is stabilized by finite-r_L effects at large m. But for double-ended operation with electron sheaths such small values of $k_{\|}$ are not likely, and in any case Fig. 4 shows that $\text{Re}\,\omega$ should differ from kv_D in the small-Y region. As in the $\rho = 0$ case, there seems to be a damping mechanism more strongly dependent on m than any we have considered.

As for the end-effects, no phase change was detected along B, as expected for standing waves; and oscillations were excited with electron sheaths, as expected when $\text{Im}\,\omega$ is made large by the curvature. If the decrease in maximum density can be taken as a measure of the degree of instability, the experiments show that with $L = 80$ cm the plasma is more unstable than with $L = 40$ cm, the density falling at a larger value of R in the longer machine. This is in qualitative agreement with theory, since the lifetime τ increases with L, and also the minimum permissible value of $k_{\|}$ decreases, and hence $\text{Im}\,\omega$ increases, with L. For a given L, the plasma seems more unstable at small densities than at large densities. This can be understood for electron sheaths, since then $k_{\|}L \approx \pi/2$ independent of ϵ (or n_0), while $\text{Im}\,\omega$ is proportional to $\epsilon/k_{\|}^2$. This would not be true for ion sheaths, where $k_{\|}^2 \propto \epsilon$. Obviously, many more data are necessary to verify the theory quantitatively.

The observation $|\chi| \ll |\nu|$ cannot be understood theoretically even when short-circuiting of electric fields through highly conducting electron sheaths is taken into account. The relation between χ and ν is specified by the ion equation (40) quite independently of the sheath conditions. From equation (40) one sees that since $\psi \approx -2\delta\kappa$ and $\kappa^2 \ll 1$, $\nu - \chi$ is much less than ν. If χ is kept small by the end-plates, ν is also small, since it is produced by $\nabla\chi \times \mathbf{B}$ drifts.

Experiments with caesium plasmas are interesting from the point of view of thermonuclear fusion because the Larmor radius in these experiments is about the same as in a thermonuclear reactor with $KT = 4$ keV, $B = 50$ kG; and the growth rates of microinstabilities would therefore be comparable. It is fortunate that end-plate stabilization in thermionic plasmas allows us to study the onset of these instabilities once the boundary effects are understood. The methods we have used to treat the end-effects may also have relevance to instabilities in other machines with end-plates, such as mirror machines.

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