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"Universal" Overstability of a Resistive, Inhomogeneous Plasma

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An inhomogeneous plasma with finite resistivity in a straight or curved magnetic field is subject to instabilities of electrostatic drift waves with small but finite $k_\parallel$. The influence on these instabilities of various factors such as finite β, resistive and viscous damping, and finite Larmor radius is investigated; and explicit, experimentally useful expressions for the frequency and growth rate are given.

I. INTRODUCTION

IN a previous paper[1] [hereafter called (I)], we pointed out the existence of a "universal" instability, driven only by the pressure gradient, of a plasma with finite resistivity. This overstability has been found independently by Moiseev and Sagdeev[2] and, recently, by Jukes.[3] However, neither of these authors realized the experimental importance of this effect. We believe that this overstability is responsible for low-frequency oscillations seen in cesium and potassium plasmas[4-6] and that these experiments give the first direct verification of the theory of drift waves in fully ionized plasmas. In another paper[7] we have examined the effect of end-plates on the excitation of this overstability. Furthermore, the "universal" nature of this overstability sets a limit to the confinement time of a plasma in a nearly uniform magnetic field; we have discussed this elsewhere.[8] In view of its experimental application, we have examined this overstability in some detail; thus we have computed the effect of various perturbing factors of possible experimental importance and have shown its relation to the well-known resistive gravitational instability when the magnetic field is curved. In Sec. II we give an elementary derivation of the "universal" overstability and indicate its relation to previous work. In that derivation we make the following 19 simplifications, which we either remove or discuss in the remainder of this article: (1) $\beta = 0$; (2) the waves are electrostatic; (3) no classical diffusion ($\eta_\perp = 0$); (4) nearly perpendicular propagation ($k_\parallel^2/k^2 \ll 1$); (5) negligible ion motion along **B**; (6) infinite heat conductivity; (7) zero ion temperature; (8) zero ion Larmor radius; (9) zero ion viscosity; (10) no gravitational field; (11) uniform $\mathbf{B}_0$; (12) negligible influence of higher-frequency roots; (13) negligible x (or radial) dependence of the perturbation; (14) constant density gradient; (15) zero electron inertia; (16) no zero-order parallel current; (17) no zero-order temperature gradient; (18) no Landau damping; (19) plane geometry. In Sec. III the single-fluid equations are used to find the effect of finite β and resistive damping for the case $T_i = 0$. In Sec. IV the two-fluid equations are used to generalize the theory to include finite Larmor radius, ion viscosity, and gravitational fields. In Sec. V the remaining effects are discussed. We assume singly charged ions and employ cgs–esu throughout.

II. THE BASIC EFFECT

The "universal" resistive overstability can easily be found from the usual single-fluid equations (in standard notation):

$$Mn\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{j}\times\mathbf{B} - \nabla p - \nabla\cdot\boldsymbol{\pi}, \qquad (1)$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{j} + \frac{1}{en}(\mathbf{j}\times\mathbf{B} - \nabla p_e), \qquad (2)$$

$$\partial n/\partial t + \nabla\cdot(n\mathbf{v}) = 0, \qquad (3)$$

$$\nabla\cdot\mathbf{j} = 0. \qquad (4)$$

These equations are valid in the limit $m/M = 0$ and, with the exception of the viscosity term $\nabla\cdot\boldsymbol{\pi}$ neglected in (I), are exactly equivalent to the two-fluid equations used in (I) in this limit. To demonstrate the basic effect we may set $T_i = 0$, and therefore $\boldsymbol{\pi} \approx 0$ and $p_e = p$. In equilibrium we postulate $\mathbf{E}_0 = 0$ and $\nabla p = p'\hat{x}$, with $\mathbf{B} = B\hat{z} =$ const, so

[1] F. F. Chen, Phys. Fluids **7**, 949 (1964). [A more extensive discussion appears in Princeton Plasma Physics Laboratory Report MATT-227 (1963)].
[2] S. S. Moiseev and R. Z. Sagdeev, Zh. Eksperim. i Teor. Fiz. **44**, 763 (1963) [English transl.: Soviet Phys.—JETP **17**, 515 (1963)] and Zh. Tekhn. Fiz. **34**, 248 (1964) [English transl.: Soviet Phys.—Tech. Phys. **9**, 196 (1964)]. This work apparently preceded ours by about six months.
[3] J. D. Jukes, Phys. Fluids **7**, 1468 (1964).
[4] N. D'Angelo and R. W. Motley, Phys. Fluids **6**, 422 (1963).
[5] N. D'Angelo, D. Eckhartt, G. Grieger, E. Guilino, and M. Hashmi, Phys. Rev. Letters **11**, 525 (1963).
[6] H. Lashinsky, Phys. Rev. Letters **12**, 121 (1964), **13**, 47 (1964).
[7] F. F. Chen, Phys. Fluids **8**, 752 (1965).
[8] F. F. Chen, Phys. Fluids **8**, 912 (1965).

that $j_x^{(0)} = v_y^{(0)} = 0$, $j_y^{(0)} = p'/B$, and $v_x^{(0)} = -\eta p'/B^2$. In first order we consider isothermal, electrostatic oscillations of the form

$$\nabla p^{(1)} = KT_e \nabla n_1, \quad n_1 = \nu n_0(x) \exp i(ky + k_\parallel z - \omega t),$$

$$\mathbf{E}^{(1)} = -\nabla \phi_1, \qquad \phi_1 = \phi \exp i(ky + k_\parallel z - \omega t).$$

If we neglect the classical diffusion term $v_x^{(0)}$, the linearized form of Eqs. (1) and (2) becomes

$$-i\omega M n_0 \mathbf{v}_1 = \mathbf{j}_1 \times \mathbf{B} - KT_e \nabla n_1$$
$$= en_0(-\nabla\phi_1 + \mathbf{v}_1 \times \mathbf{B} - \eta \mathbf{j}_1). \qquad (5)$$

If η is sufficiently small, it may be neglected in the x and y components of Eqs. (5). We also assume low frequencies, so that $\omega^2/\omega_c^2 \ll 1$, where $\omega_c \equiv eB/M$, and nearly perpendicular propagation, so that $k_\parallel/k \ll 1$ and v_z may be neglected. With the x dependence of ν and ϕ also neglected, Eqs. (5) then give

$$j_x = -ik(KT_e/B)n_1, \qquad (6)$$

$$j_y = (KT_e/B)(n_1/\Lambda) - kn_0(\omega/\omega_0)(e\phi/B), \qquad (7)$$

$$e\eta j_z = ik_\parallel(KT_e\nu - e\phi), \qquad (8)$$

$$v_x = -ik\phi/B, \qquad (9)$$

$$v_y = -(\omega/\omega_c)(k\phi/B), \qquad (10)$$

where $\Lambda^{-1} \equiv n_0'/n_0$, the prime denoting $\partial/\partial x$. Substituting (9) and (10) into the linearized form of Eq. (3) and neglecting v_z, we obtain

$$\nu = -(k\phi/B)[(1/\Lambda\omega) + (k/\omega_c)]. \qquad (11)$$

The linearized form of Eq. (4) is $j_x' + ikj_y + ik_\parallel j_z = 0$. When Eqs. (6), (7), and (8) are inserted, there is a cancellation of the j_x' term with the first term of j_y, since $n_1' = n_1/\Lambda$; and we are left with

$$ik^2 n_0(\omega/\omega_c)(e\phi/B) + (k_\parallel^2/e\eta)(KT_e\nu - e\phi) = 0. \qquad (12)$$

This cancellation is *not* a fortuitous one which depends on the assumed x dependence of the perturbation. It is the cancellation of two "fictitious" electron pressure-gradient drift terms which appear in the fluid picture but which do not correspond to actual particle motions. The determinant of Eqs. (11) and (12) in ν and ϕ give the dispersion relation

$$\left(\frac{\omega}{\omega_c}\right)^2 + i\frac{\omega}{\omega_c}\frac{k_\parallel^2}{k^2}\frac{B}{n_0e\eta}\left(1 + \frac{KT_e}{eB}\frac{k^2}{\omega_c}\right) + i\frac{k_\parallel^2}{k^2}\frac{B}{n_0e\eta}\frac{KT_e}{eB}\frac{k}{\omega_c}\frac{n_0'}{n_0} = 0. \qquad (13)$$

Except for the small correction term in the coefficient of ω, this is exactly the equation found by Moiseev and Sagdeev[2] for $T_i = 0$ and always has a complex root with Im $\omega > 0$ for any finite values of $k_\parallel$ and η.

The first term in Eq. (12) comes from the divergence of the current j_y. This current is carried entirely by ions, since the electron Larmor radius has been taken to be zero, and electrons cannot move in the y direction in the absence of an E_x. Indeed, the second term in Eq. (7) is just n_0ev_y, given by Eq. (10). Equation (12) therefore says that the divergence of charge caused by v_y is cancelled by a flow of electrons along $\mathbf{B}$; this requires finite $k_\parallel$ and, if η is finite, requires $\nu \neq e\phi/KT_e$. This phase shift between the perturbed density and potential distributions then gives rise to the growth rate. This is the reason for the physical interpretation of the phenomenon presented in Ref. 8. This overstability did not appear in the work of Furth, Killeen, and Rosenbluth[9] on resistive instabilities because compressibility, and hence v_y, was neglected. It also did not appear in the paper of Rosenbluth, Krall, and Rostoker[10] on finite Larmor radius effects because resistivity was neglected. The velocity component v_y arises because of finite ion inertia, as can be seen from the factor ω/ω_c in Eq. (10). In the two-fluid treatment, ion inertia is simply taken into account by retaining the term $M\partial\mathbf{v}/\partial t$ in the ion equation of motion. In the equivalent one-fluid treatment, one must also retain all the terms in Ohm's law, Eq. (2). Johnson, Greene, and Coppi[11] treated ion inertia inconsistently in neglecting the last two terms of Eq. (2) while retaining the $M\,\partial\mathbf{v}/\partial t$ term in Eq. (1). With the same approximations as above, this gives rise to the same j_y as in Eq. (9), but now $v_y = 0$; hence, j_y would exist although neither ions nor electrons can move in the y direction. The equations used by Coppi[12] and Kulsrud[13] were general enough to give the "universal" overstability, but it was not pointed out because attention was focused on the modes of Ref. 9.

Note that the question of finite Larmor radius does not arise here because the ions have been assumed cold. The phenomenon depends only on ion inertia. When finite Larmor radius is taken into account via the viscosity tensor $\boldsymbol{\pi}$, we find that for $T_i = T_e$ the growth rate is twice that found from Eq (13) for $T_i = 0$ but is half that found in (I) for $T_i = T_e$ with $\boldsymbol{\pi}$ neglected. For brevity we hence-

[9] H. P. Furth, J. Killeen, and M. N. Rosenbluth, Phys. Fluids 6, 459 (1963).
[10] M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Nucl. Fusion Suppl. Pt. 1, 143 (1962).
[11] J. L. Johnson, J. M. Greene, and B. Coppi, Phys. Fluids 6, 1169 (1963).
[12] B. Coppi, Phys. Fluids 7, 1501 (1964).
[13] R. M. Kulsrud, Princeton University Plasma Physics Laboratory Report Matt-258 (1964).

forth use dimensionless units in which time is measured in units of $\omega_c^{-1} = M/eB$, velocities in units of $v_s \equiv (KT_e/M)^{\frac{1}{2}}$, and hence lengths in units of $a \equiv v_s/\omega_c$. Note that $r_L = (2\lambda)^{\frac{1}{2}}a$, where $\lambda = T_i/T_e$ and r_L is the ion Larmor radius. Wherever possible we denote normalized quantities by a suitable Greek letter; thus, κ is a normalized k, $\mathbf{v}$ a normalized $\mathbf{v}$, ϵ a normalized resistivity η, etc. Equation (13) then becomes simply

$$\Omega^2 + i(1+\kappa^2)Y\Omega + i\delta\kappa Y = 0, \qquad (14)$$

where

$$\Omega = \omega/\omega_c, \qquad \kappa = ka, \qquad \delta = an_0'/n_0,$$

$$Y = k_\parallel^2/\epsilon\kappa^2, \qquad \epsilon = n_0 e\eta/B, \qquad a_i^2 = KT_e/M\omega_c^2.$$

For $T_i = T_e$ it will be shown in Sec. IV that the corresponding equation is, with the neglect of the small term κ^2,

$$\psi^2 + (\delta\kappa + iY)\psi + 2i\delta\kappa Y = 0, \qquad (15)$$

where $\psi = \Omega - \delta\kappa$ is the frequency in the frame of the ion fluid. By introducing $x = \mathrm{Re}\,\Omega/\delta\kappa$, $z = \mathrm{Im}\,\Omega/\delta\kappa$, and $y^2 = Y/\delta\kappa$, we can obtain a parameterless equation which can be solved easily by hand computation. The resulting dispersion curves, x and z as functions of y, are shown in Fig. 1. The wave traveling in the ion drift diriction is damped, while the wave traveling with the electrons[14] is unstable for any finite value of y. The maximum growth rate is approximately 0.4 kv_D. Note that this growth rate is independent of η if $k_\parallel$ is allowed to adjust itself so that $y \approx 1.3$ is maintained. As η is decreased, the wavelength of the most unstable mode increases to keep the number of collisions per parallel wavelength, and hence the frictional drag on the electrons, sufficient to cause the instability. To avoid this instability experimentally, one must set a lower limit on $k_\parallel$ by means of end-plates or sheared magnetic fields. The value of y is then limited to the upper region of Fig. 1, where the growth rate can be found by expanding the solution of Eq. (15) to second order in $(\delta\kappa/Y)$. One then obtains, for large $Y/\delta\kappa$, a growth rate proportional to η:

$$\mathrm{Im}\,\Omega \approx 2\delta^2\kappa^2/Y, \quad \text{or} \quad \mathrm{Im}\,(\omega/\omega_c) = 2(k/k_\parallel)^2(ka)^2(an_0'/n_0)^2(n_0e\eta/B). \qquad (16)$$

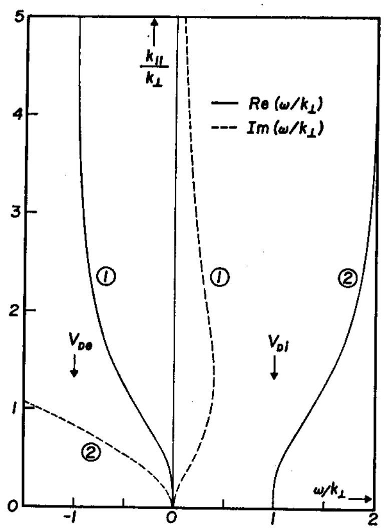

FIG. 1. Dispersion curves for low-frequency drift waves in an isothermal, inhomogeneous, resistive plasma. The "universal" overstability is labeled 1. v_{De} and v_{Di} are the electron and ion diamagnetic drift velocities. The meaning of the numbers on the scales is given in the text.

[14] This is the mode DE_θ of (I); the inclusion of finite Larmor radius effects has moved the $k_\parallel = 0$ frequency from $\omega = kv_{Di}$ to $\omega = 0$. The point $\omega = 0$, $k_\parallel = 0$ corresponds to a static density perturbation which cannot propagate because no electric fields can develop which maintain charge neutrality.

III. EFFECT OF FINITE-β AND RESISTIVE DAMPING

We now remove simplifications (1)–(6) of Sec. I for the case of zero ion temperature and straight magnetic fields. The appropriate equations are Eqs. (1) to (3), with $\pi = 0$, and Maxwell's equations with no displacement current:

$$\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t, \qquad (17)$$

$$c^2\nabla \times \mathbf{B} = 4\pi\mathbf{j}. \qquad (18)$$

To these must be added an equation of state. If heat conductivity is infinite, the plasma is isothermal; and we may replace ∇p by $KT\nabla n$. This is a good approximation in thermionic plasmas, where electrons rae in good thermal contact with the end-plates; and we shall henceforth make this simplification. On the other hand, if heat conductivity is poor, one must use the adiabatic equation

$$\frac{d}{dt}\left[\frac{p}{(Mn)^{5/3}}\right] = 0.$$

This is satisfied in equilibrium because $d/dt = \partial/\partial t + \mathbf{v}_0\cdot\nabla_0 \approx 0$. In first order, we have

$$\frac{d}{dt}\left[p_1 - \frac{5}{3}p_0\frac{n_1}{n_0}\right] = 0,$$

which is satisfied if $p_1 = \frac{5}{3}p_0n_1/n_0 = \frac{5}{3}KT_0n_1$. Hence, the adiabatic result can be obtained from the isothermal one simply by replacing Boltzmann's constant K by $\frac{5}{3}K$ in the final answer. For intermediate heat conductivities the result will lie somewhere in between, since the phenomenon is not sensitively dependent on temperature gradients; and it is not worthwhile to use the heat equation.

Equilibrium

Instead of the usual approximation that the sources required to maintain a steady state in the presence of collisions may be neglected because the rate of classical diffusion is much slower than the frequencies of interest, we prefer to imagine that the steady state is maintained not by sources of particles, but by an electric field E_y, produced in an unspecified way, which keeps $v_x^{(0)} = 0$. If, for instance, E_y is produced by a slowly increasing magnetic field, the neglect of this increase is equivalent to the usual two-time-scale formalism. In dimensionless notation the zero-order equations are:

$$\mathbf{v}_0\cdot\nabla\mathbf{v}_0 = \boldsymbol{\iota}_0\times(\hat{\mathbf{z}}\times\mathbf{h}_0) - \nabla n_0/n_0, \qquad (19)$$

$$\boldsymbol{\mathcal{E}}_0 + \mathbf{v}\times(\hat{\mathbf{z}}+\mathbf{h}_0) = \epsilon\boldsymbol{\iota}_0 + \mathbf{v}_0\cdot\nabla\mathbf{v}_0, \qquad (20)$$

$$\nabla\cdot(n_0\mathbf{v}_0) = 0, \qquad (21)$$

$$\nabla\times\boldsymbol{\mathcal{E}}_0 = -\partial\mathbf{h}_0/\partial\tau, \qquad (22)$$

$$\nabla\times\mathbf{h}_0 = \tfrac{1}{2}\beta\boldsymbol{\iota}_0, \qquad (23)$$

where

$$\mathbf{B}_0 = B_{00}[\hat{\mathbf{z}} + \mathbf{h}_0(\mathbf{x})],$$

$$\mathbf{v}_0 = \mathbf{V}_0/v_s, \qquad \boldsymbol{\iota}_0 = \mathbf{j}_0/n_0ev_s,$$

$$\boldsymbol{\mathcal{E}}_0 = \mathbf{E}_0/B_{00}v_s, \qquad \epsilon = n_0e\eta/B_{00},$$

$$\beta = 8\pi n_0KT_e/c^2B_{00}^2,$$

and ∇ is in (ξ, η, ζ) space, so that $\nabla_\xi = a\nabla_x$. The quantities v_s, ω_c, and a are as before, but with B_{00} replacing B. In the steady state we assume $\nabla n_0/n_0 = \delta\hat{\mathbf{x}}$, $\nabla = \hat{\mathbf{x}}\partial/\partial\xi$, $v_{x0} = v_{z0} = \iota_{z0} = 0$, so that $\mathbf{v}_0\cdot\nabla\mathbf{v}_0 = 0$. From Eqs. (19) and (23) we then obtain $\iota_{x0} = h_{x0} = h_{y0} = 0$ and

$$h_{z0} = -\tfrac{1}{2}\beta\delta\xi + O(\beta^2),$$

$$\iota_{y0} = \delta/(1+h_{z0}).$$

From Eq. (20) we obtain

$$\mathcal{E}_{z0} = 0,$$

$$\mathcal{E}_{x0} = -v_{y0}(1+h_{z0}),$$

$$\mathcal{E}_{y0} = \epsilon\iota_{y0} + v_{x0}(1+h_{z0}) = \epsilon\iota_{y0}.$$

This gives the E_y necessary to maintain a steady state; the time-varying B field necessary to produce this is given by Eq. (22). We shall work in the frame in which $\mathcal{E}_{x0}$ vanishes, so that $v_{y0} = 0$. Since all components of $\mathbf{v}_0$ vanish, Eq. (21) is satisfied identically.

Perturbation

Since $\mathbf{v}_0$ is zero and $\mathbf{h}_0$ is small, Eqs. (1), (2), (17), (18), and (3) in first order become

$$-i\Omega\mathbf{v} = \boldsymbol{\iota}\times\hat{\mathbf{z}} + \boldsymbol{\iota}_0\times\mathbf{h} - \nabla\nu - \nu\delta\hat{\mathbf{x}}, \qquad (24)$$

$$\boldsymbol{\mathcal{E}} + \mathbf{v}\times\hat{\mathbf{z}} = \epsilon\boldsymbol{\iota} - i\Omega\mathbf{v}, \qquad (25)$$

$$\nabla\times\boldsymbol{\mathcal{E}} = i\Omega\mathbf{h}, \qquad (26)$$

$$\nabla\times\mathbf{h} = \tfrac{1}{2}\beta\boldsymbol{\iota}, \qquad (27)$$

$$-i\Omega\nu + \delta v_x + \nabla\cdot\mathbf{v} = 0, \qquad (28)$$

where $\nu = n_1/n_0$, $\mathbf{h} = \mathbf{B}_1/B_{00}$, $\mathbf{v} = \mathbf{v}_1/v_s$, $\boldsymbol{\iota} = \mathbf{j}_1/n_0ev_s$, $\boldsymbol{\mathcal{E}} = \mathbf{E}_1/B_{00}v_s$, and we have used $\nabla\nu = \nabla n_1/n_0 - \nu\delta\hat{x}$. Note that $\boldsymbol{\mathcal{E}}_0$ has dropped out, that ϵ and β are functions of ξ, and that $\nabla\cdot\boldsymbol{\iota} = -\delta\iota_x \neq 0$ since $\nabla\cdot(\beta\boldsymbol{\iota}) = 0$. We look for solutions ν, $\mathbf{v}$, $\boldsymbol{\iota}$, $\boldsymbol{\mathcal{E}}$ of the form $\exp i(\kappa\eta + \kappa_\parallel\zeta - \Omega\tau)$ with a negligible dependence on ξ. Then for consistency we must give $\mathbf{h}$ the same ξ dependence as n_0, so that $\mathbf{h}' = \delta\mathbf{h}$. The first four equations above in the four unknowns $\boldsymbol{\iota}$, $\mathbf{v}$, $\boldsymbol{\mathcal{E}}$, and $\mathbf{h}$ may be solved to give $\mathbf{v}(\nu)$. This can then be inserted into Eq. (28) to give the dispersion relation. After some algebra, we obtain without further approximation

$$v_x = \Omega^{-1}[iv_y(1 - i\Omega D\epsilon) - D\epsilon\kappa\nu],$$

$$v_z = \theta(C\Omega)^{-1}[\kappa\nu + \Gamma(v_x - i\Omega v_y)],$$

$$v_y = -\Omega\nu \cdot \frac{\delta\theta^2 + i\kappa CGD\epsilon + i\kappa Y^*(1 + iD\epsilon\Omega^{-1})}{G\Omega + \delta\kappa^{-1}\Omega^2 + iY^*(1-\Omega^2) + D\epsilon\Omega(Y^* - iCG\Omega)},$$

where

$$\theta = \kappa_\parallel/\kappa, \qquad G = 1 + \theta^2, \qquad Y^* = \theta^2/CD\epsilon,$$

$$C = 1 - i\Gamma, \qquad \Gamma = \beta\iota_0/2GD\epsilon\kappa,$$

$$D = 1 - i\Delta, \qquad \Delta = \beta\Omega/2G\epsilon\kappa^2.$$

Inserting this into Eq. (28), we obtain the exact local dispersion relation

$$\begin{aligned}&[i\Omega(C\Omega^2 - \kappa^2\theta^2) + D\epsilon\kappa(C\Omega\delta + i\kappa\Gamma\theta^2)]\\ &\quad\cdot[G\Omega + \delta\kappa^{-1}\Omega^2 + iY^*(1-\Omega^2) + D\epsilon\Omega(Y^* - iCG\Omega)]\\ &= -i\Omega[\delta\theta^2 + i\kappa CGD\epsilon + i\kappa Y^*(1 + iD\epsilon\Omega^{-1})]\\ &\quad\cdot[\Omega^2\kappa(C - i\Gamma\theta^2) + (1 - i\Omega D\epsilon)(\delta C\Omega + i\kappa\Gamma\theta^2)]. \end{aligned} \qquad (29)$$

Results

This dispersion relation can be studied in various limits. If we set $\beta = \epsilon = \theta^2 = 0$ but let Y^* remain finite, so that $C = D = G = 1$ and $Y^* = Y$, we recover the elementary equation (14) if high-frequency roots are neglected ($\Omega^2 \ll 1$, $\Omega \ll |\kappa/\delta|$). The effect of finite β [effects (1) and (2) of Sec. I] can be seen by setting $\epsilon = \theta^2 = 0$ and again retaining parallel resistivity in Y^* and neglecting high-frequency roots. Equation (29) then becomes

$$\Omega(\Omega + iY^*) + i\kappa Y^*(\Omega\kappa + \delta) = 0. \quad (30)$$

This is identical with Eq. (14) except that Y is replaced by Y^*. Thus the only effect of finite β is to shift the ordinate of Fig. 1 by a factor CD. Since Γ is of order 0.02 for typical stellarator conditions and 0.04 for cesium plasma conditions even for the smallest value of κ, and since Δ is of the same order of magnitude, the effect of β is entirely negligible in these devices. In the upper region of Fig. 1 the effect of β is even smaller, since $\Omega \approx -\delta\kappa$, and hence $\Gamma \approx -\Delta$; then CD differs from unity only by a term of $O(\beta^2)$.

The effect of resistive damping is found by setting $\beta = \theta^2 = 0$ in Eq. (29). We also neglect terms in ϵ^2 since ϵ is usually less than 10^{-4}. For low frequencies we obtain

$$-i\epsilon\Omega^3 + (1 + \epsilon Y)\Omega^2 + i\Omega[(1 + \kappa^2)Y + \epsilon\kappa^2] + i\delta\kappa Y(1 + i\epsilon\kappa\delta^{-1}) = 0. \quad (31)$$

For large Y, the growth rate can be approximated by setting $\Omega = -\delta\kappa$ in the first two terms. For $\delta^2 \ll 1$, $\kappa^2 \ll 1$, we then obtain approximately

$$\operatorname{Im}\Omega = \delta^2\kappa^2/Y - \epsilon\kappa^2. \quad (32)$$

Comparing this with half of Eq. (16) (because $T_i = 0$ here), we see that perpendicular diffusion decreases the growth rate by a factor $(1 - \epsilon Y/\delta^2)$.

The effect of ion motion v_z along $\mathbf{B}$ and of large angles of propagation θ is found by setting $\beta = \epsilon = 0$ in Eq. (29). We then obtain approximately

$$iY[\Omega^2(\Omega^2 - 1 - \kappa^2) - \delta\kappa\Omega + \kappa^2\theta^2(1 - \Omega^2)] = \Omega^3(1 + \delta\kappa^{-1}\Omega), \quad (33)$$

which is essentially Eq. (68) of (I). The effect of the extra term in θ^2 is to turn the drift waves into ordinary ion acoustic waves ($\Omega^2 = \kappa^2\theta^2 = \kappa_{\parallel}^2$) in the limit of large-angle propagation in a homogeneous plasma ($\delta \to 0$, $Y \to \infty$). A spurious root at $\Omega = 0$ [mode DI_θ of (I) corrected for finite r_L] also appears at small Y. For v_z to be negligible, the third term in the square brackets must be smaller than the second term; thus the drift-wave approximation is valid for $\kappa^2\theta^2 \ll |\delta\kappa\Omega| \approx \delta^2\kappa^2$, or $\theta^2 \ll \delta^2$. In order for this to hold near the condition for maximum growth rate, which, from Fig. 1, is $y = 1.3$ or $Y = 1.7\delta\kappa$ we require $1.7\epsilon \ll \delta/\kappa$. This is usually well satisfied experimentally, except possibly near $\kappa = 1$, where the theory breaks down anyway. Equation (33) also contains the effects of high-frequency roots near $\Omega^2 = 1$ and $\Omega = -\kappa/\delta$. In (I) these effects, as well as the effect of finite v_z, were included exactly by numerical solution of Eq. (33); these effects [Nos. (4), (5), and (12) of Sec. I] are generally negligible.

Note that two other low-frequency roots which turn into torsional Alfvén waves at large Y appear in other analyses but do not appear here because these roots correspond to velocity fluctuations without perturbations in density. Since density fluctuations are always found experimentally, we have used a method which depends on the finiteness of ν.

IV. EFFECT OF ION TEMPERATURE, VISCOSITY, AND CURVED B

The Two-Fluid Equations

We now wish to remove simplifications (7) to (11) of Sec. I. For simplicity, we assume $\beta = 0$, $\mathbf{E} = -\nabla\phi$ at the outset; in this case it is somewhat more convenient to use the two-fluid equations:

$$Mn(\partial\mathbf{v}_i/\partial t + \mathbf{v}_i\cdot\nabla\mathbf{v}_i) = en(-\nabla\phi + \mathbf{v}_i\times\mathbf{B}) - \gamma_i KT_i\nabla n - \nabla\cdot\boldsymbol{\pi}_i + n^2e^2\eta(\mathbf{v}_e - \mathbf{v}_i) + (n/R)(KT_i + Mv_{zi}^2)\hat{\mathbf{x}}, \quad (34)$$

$$0 = -en(-\nabla\phi + \mathbf{v}_e\times\mathbf{B}) - \gamma_e KT_e\nabla n + n^2e^2\eta(\mathbf{v}_i - \mathbf{v}_e) + (n/R)KT_e\hat{\mathbf{x}}. \quad (35)$$

We have assumed quasi-neutrality and neglected the electron mass. As discussed in Sec. III, we consider only the isothermal case and set $\gamma_i = \gamma_e = 1$. The last term on the right-hand side is the effective gravitational force, taken to be in the positive x direction, due to a curvature of radius R in $\mathbf{B}_0$. Note that in addition to the terms in $KT_{i,e}$ due to the outward component of pressure in a U-bend we have included a term nMv_{zi}^2/R due to centrifugal force of mass motion around a U-bend. The corresponding term for electrons, as well as the $m_e\mathbf{v}_e\cdot\nabla\mathbf{v}_e$ term are negligible unless a large v_{ez} exists in equilibrium; this will be considered in Sec. V. As shown in (I), the $m_e\,\partial\mathbf{v}_e/\partial t$ term is negligible if $m_e/M \ll \epsilon$.

Since we now consider finite ion thermal energies, we must include the effects of finite Larmor radius. These effects have a simple physical interpretation[8] and can be included in a macroscopic theory by the use of the viscosity tensor $\boldsymbol{\pi}_i$, as shown by Roberts and Taylor.[15] The simplest form of $\boldsymbol{\pi}_i$ consistent with the accurate computations[16] of the transport

[15] K. V. Roberts and J. B. Taylor, Phys. Rev. Letters 8, 197 (1962).

[16] I. P. Shkarovsky, I. B. Bernstein, and B. B. Robinson, Phys. Fluids 6, 40 (1963).

coefficients is that given by Bernstein and Trehan[17] with the coefficient given by Kaufman.[18] To put π_i into dimensionless units, we define

$$\Pi = \pi_i/nKT_i, \qquad \alpha = \omega_c\tau_{ii},$$
$$\Upsilon = \tfrac{1}{2}(\nabla \mathbf{v}_i + \nabla \mathbf{v}_i^T) - \tfrac{1}{3}(\nabla\cdot\mathbf{v}_i)\mathbf{I},$$
$$\tau_{ii} = \frac{5}{8\pi^{\frac{1}{2}}}\frac{M^{\frac{1}{2}}(KT_i)^{\frac{3}{2}}}{ne^4 \ln \Lambda},$$

where Υ is the symmetric, traceless velocity gradient tensor in units of v_s/a, as defined preceding Eq. (14). For $\alpha^2 \gg 1$, Π takes the comparatively simple form

$$\begin{aligned} -\Pi_{xx} &= \alpha(\Upsilon_{xx} + \Upsilon_{yy}) + \Upsilon_{xy}, \\ -\Pi_{yy} &= \alpha(\Upsilon_{xx} + \Upsilon_{yy}) - \Upsilon_{xy}, \\ -\Pi_{zz} &= 2\alpha\Upsilon_{zz}, \\ -\Pi_{xy} &= -\Pi_{yx} = \tfrac{1}{2}(\Upsilon_{yy} - \Upsilon_{xx} + \alpha^{-1}\Upsilon_{xy}), \\ -\Pi_{xz} &= -\Pi_{zx} = 2(\Upsilon_{yz} + \alpha^{-1}\Upsilon_{xz}), \\ -\Pi_{yz} &= -\Pi_{zy} = 2(\alpha^{-1}\Upsilon_{yz} - \Upsilon_{xz}). \end{aligned} \tag{36}$$

The terms not containing α are independent of collisions and contain the finite Larmor radius effects. The terms containing α are due to collisional viscosity. For the last three components in Eq. (36), the collisionless limit is correctly given by taking $\alpha \to \infty$, as shown by Thompson.[19] The first three components diverge, however, for $\alpha \to \infty$ because the method of computation of π breaks down for long mean free paths in the z direction. Roberts and Taylor[15] obtain the collisionless limit by simply ignoring the α terms without proof; Stringer[20] has shown by including inertia in the derivation of π that this procedure indeed gives the correct result. We avoid any uncertainty by considering large but finite values of α^2 and keeping all the terms in Eq. (36) including the collisional ones.

Dividing Eqs. (34) and (35) by $Mn\omega_c v_s$, we obtain their dimensionless forms

$$\begin{aligned} \partial \mathbf{v}/\partial\tau + \mathbf{v}\cdot\nabla\mathbf{v} = &-\nabla\chi - \lambda\nabla n/n \\ &+ \mathbf{v}\times(\hat{\mathbf{z}} + \mathbf{h}) + (\lambda + v_z^2)\rho\hat{\mathbf{x}} \\ &+ \epsilon(\mathbf{v}_e - \mathbf{v}) - \lambda(\nabla\cdot\Pi + \Pi\cdot\nabla n/n), \end{aligned} \tag{37}$$

$$\begin{aligned} 0 = \nabla\chi - \nabla n/n - \mathbf{v}_e\times(\hat{\mathbf{z}} + \mathbf{h}) \\ + \rho\hat{\mathbf{x}} + \epsilon(\mathbf{v} - \mathbf{v}_e), \end{aligned} \tag{38}$$

[17] I. B. Bernstein and S. K. Trehan, Nucl. Fusion 1, 3 (1960). This paper contains several misprints in π.

[18] A. N. Kaufman, Phys. Fluids 3, 610 (1960).

[19] W. B. Thompson, *Reports on Progress in Physics* (The Physical Society, London, 1961), Vol. 24, p. 363.

[20] T. E. Stringer, private communication.

where

$$\lambda = T_i/T_e, \qquad \chi = e\phi/KT_e, \qquad \rho = a/R,$$
$$\mathbf{v} = \mathbf{v}_i/v_s, \qquad \mathbf{v}_e = \mathbf{v}_e/v_s, \qquad \tau = \omega_c t,$$
$$\mathbf{B} = B_{00}(\hat{\mathbf{z}} + \mathbf{h}),$$

and the other quantities are as previously defined in terms of the field B_{00} at $\xi = 0$.

Equilibrium

In a steady state with no driving current we assume $v_z = v_{ze} = 0$, $\chi_0 = 0$, and $\nabla n_0/n_0 = \delta\hat{\mathbf{x}}$. The field $\mathbf{B}_0$ is assumed to be slightly curved but uniform in the y and z directions. The curvature implies a small inhomogeneity in $\mathbf{B}_0$,

$$\mathbf{B}_0 = \hat{\mathbf{z}}B_{00}/(1 + x/R), \qquad \mathbf{h}_0 \cong -\rho\xi\hat{\mathbf{z}}.$$

Neglecting $\mathbf{v}\cdot\nabla\mathbf{v}$ and Π for the time being, we obtain in zero-order

$$\lambda(\delta - \rho)\hat{\mathbf{x}} = \mathbf{v}\times\hat{\mathbf{z}}b + \epsilon(\mathbf{v}_e - \mathbf{v}), \tag{39}$$
$$(\delta - \rho)\hat{\mathbf{x}} = -\mathbf{v}_e\times\hat{\mathbf{z}}b + \epsilon(\mathbf{v} - \mathbf{v}_e), \tag{40}$$

with

$$b \equiv (1 + \rho\xi)^{-1}.$$

As in Sec. III we neglect the inhomogeneity in B_0 due to plasma diamagnetism, but we retain the factor b to study the effect of inhomogeneity due to curvature. Equations (39) and (40) give

$$v_{ey} = -b^{-1}(\delta - \rho) \equiv v_0, \qquad v_y = -\lambda v_0, \tag{41}$$
$$v_x = v_{ex} = -\epsilon b^{-2}(1 + \lambda)(\delta - \rho). \tag{42}$$

For simplicity we consider density distributions such that v_0 is a positive constant; deviation from this will be discussed in Sec. V. For small $\rho\xi$, constant v_0 implies a nearly exponential density distribution:

$$n_0 = n_{00} \exp \xi[-v_0 + \rho(1 + \tfrac{1}{2}v_0\xi)].$$

Neglecting the small curvature ρ and noting that $\epsilon' \equiv \partial\epsilon/\partial\xi = \delta\epsilon$, we see from Eq. (42) that $v_x v_x' = O(\epsilon^2\delta^3)$ and hence, the $\mathbf{v}\cdot\nabla\mathbf{v}$ term is of $O(\delta^2)$ smaller than the resistence term. Similarly, our neglect of Π_0 is justified since the dominant term in Π_0 is $\alpha\Upsilon_{xx} = O(\alpha v_x') = O(\alpha\epsilon\delta^2)$, which gives a contribution of $O(\delta^2)$ smaller than the pressure gradient term if α is not as large as ϵ^{-1}. The steady state is supposed to be maintained by a small source term in the equation of continuity.

Perturbation

In applying perturbations of the form ν, χ, $\mathbf{v}$, $\mathbf{v}_e \propto$

$\exp i(ky + k_{\parallel}z - \omega t)$, where $\nu = n_1/n_0(x)$, to Eqs. (37) and (38), we make use of the fact that v_{0y} is constant and v_{0x} is small, so that $\nabla \mathbf{v}_0$ can be neglected. We note, however, that ϵ is proportional to n, so that $\epsilon_1 = \nu\epsilon_0$. Similarly, α is proportional to n^{-1}, so that $\mathbf{\Pi}_1$ contains terms of the form $\alpha_1 \mathbf{\Upsilon}_0$ as well as $\alpha_0 \mathbf{\Upsilon}_1$. The former vanish with the neglect of $\nabla \mathbf{v}_0$, so that $\mathbf{\Pi}_1$ is given by Eq. (36) with $\alpha = \alpha_0$ and $\mathbf{v} = \mathbf{v}_1$. This approximation amounts to neglecting cross terms involving both ion-ion and ion-electron collisions. With the subscripts on α_0, ϵ_0, χ_1, $\mathbf{v}_1$ and $\mathbf{\Pi}_1$ suppressed, the linearized equations become

$$-i\psi \mathbf{v} = -\nabla\chi - \lambda\nabla\nu + \mathbf{v}\times\hat{\mathbf{z}}b + \epsilon(\mathbf{v}_e - \mathbf{v}) + \nu\epsilon(1+\lambda)v_0\hat{\mathbf{y}} - \lambda(\nabla\cdot\mathbf{\Pi} + \mathbf{\Pi}\cdot\delta\hat{\mathbf{x}}), \tag{43}$$

$$0 = \nabla\chi - \nabla\nu - \mathbf{v}_e\times\hat{\mathbf{z}}b + \epsilon(\mathbf{v} - \mathbf{v}_e) - \nu\epsilon(1+\lambda)v_0\hat{\mathbf{y}}. \tag{44}$$

The Doppler-shifted frequencies ψ and ψ_e are defined as

$$\psi = \Omega - \boldsymbol{\kappa}\cdot\mathbf{v}_{0i} = \Omega + \lambda\kappa v_0, \qquad \psi_e = \Omega - \boldsymbol{\kappa}\cdot\mathbf{v}_{0e} = \Omega - \kappa v_0, \tag{45}$$

with the use of Eq. (41). Note that the terms in $\rho\hat{\mathbf{x}}$ are unperturbed if $v_{x0} = 0$ and drop out in first order; therefore the gravitational overstability is trivial to include, and ρ makes it appearance only in the evaluation of ψ and v_0 in the final answer.

We can now write out the separate components of Eqs. (43) and (44) with the help of Eq. (36). Because of the complexity introduced by the coupling of these components via the viscosity tensor, we make the approximation $v_z = 0$ at the outset, so that the z components of Eq. (43) need not be solved. The effect of v_z is well understood from (I) and from Sec. III. It is important only in the mode DI_θ of (I), and we can expect that the effect of $\mathbf{\Pi}$ in the equation for v_z is to damp this physically unimportant mode. With $v_z = 0$, the components $\mathbf{\Pi}_{zx}$ and $\mathbf{\Pi}_{zy}$ appearing in the x and y components of $\nabla\cdot\mathbf{\Pi}$ are multiplied by θ^2 compared to $\mathbf{\Pi}_{yx}$ and $\mathbf{\Pi}_{yy}$ and therefore can be neglected. Thus none of the z components of $\mathbf{\Pi}$ needs to be considered. For the z components we thus have

$$v_z \approx 0, \qquad v_{ze} \approx i\kappa_{\parallel}\epsilon_{\parallel}^{-1}(\chi - \nu). \tag{46}$$

If for simplicity we assume $\mathbf{v}' = \mathbf{v}_e' = 0$, so that the ξ derivatives in $\mathbf{\Pi}$ are zero, the use of Eq. (36) yields the following perpendicular components of Eqs. (43) and (44):

$$-i\psi v_x = -(\chi + \lambda\nu)' + bv_y + \epsilon(v_{xe} - v_x) + i\lambda\delta\kappa(\tfrac{1}{2}v_x + \tfrac{1}{3}\alpha v_y) - \tfrac{1}{2}\lambda\kappa^2(v_y + v_x/2\alpha), \tag{47}$$

$$-i\psi v_y = -i\kappa(\chi + \lambda\nu) - bv_x + \epsilon(v_{ye} - v_y) + \epsilon(1+\lambda)v_0\nu + \tfrac{1}{2}i\lambda\delta\kappa(v_y + v_x/2\alpha) - \lambda\kappa^2(\tfrac{1}{3}\alpha v_y - \tfrac{1}{2}v_x), \tag{48}$$

$$0 = (\chi - \nu)' - bv_{ye} + \epsilon(v_x - v_{xe}), \tag{49}$$

$$0 = i\kappa(\chi - \nu) + bv_{xe} + \epsilon(v_y - v_{ye}) - \epsilon(1+\lambda)v_0\nu. \tag{50}$$

From Eqs. (48) and (50) it is seen that if $\mathbf{v}$ and $\mathbf{v}_e$ are independent of ξ, χ and ν must vary with ξ because b, ϵ, and α depend on ξ. However, since b, ϵ, and α represent small effects which we wish to estimate, no great error in the local (in ξ) dispersion relation will be made by neglecting χ' and ν', even though this is strictly consistent with $\mathbf{v}' = 0$ only in the absence of these small effects. With this approximation, Eqs. (47) to (50) can be solved to lowest order in ϵ to give

$$C_3 v_x = -i\kappa C_2(\chi + \lambda\nu) + \epsilon C_2\bar{\lambda}v_0\nu - \epsilon b^{-1}\kappa\psi_2(\chi - \nu), \tag{51}$$

$$C_3 v_y = -\kappa\psi_1(\chi + \lambda\nu) - i\epsilon\psi_1\bar{\lambda}v_0\nu + i\epsilon b^{-1}C_1\kappa(\chi - \nu), \tag{52}$$

$$bv_{xe} = -i\kappa(\chi - \nu) + \epsilon\bar{\lambda}v_0\nu + \epsilon\kappa\psi_1 C_3^{-1}(\chi + \lambda\nu), \tag{53}$$

$$bv_{ye} = i\kappa b^{-1}\epsilon(\chi - \nu) - i\epsilon\kappa C_2 C_3^{-1}(\chi + \lambda\nu), \tag{54}$$

where

$$C_1 = b - \tfrac{1}{2}\lambda\kappa^2 - i\lambda\delta\kappa/4\alpha, \tag{55}$$

$$C_2 = b - \tfrac{1}{2}\lambda\kappa^2 + \tfrac{1}{3}i\lambda\delta\kappa\alpha, \tag{56}$$

$$\psi_1 = \psi + \tfrac{1}{2}\lambda\delta\kappa + i(\epsilon + W_1), \tag{57}$$

$$\psi_2 = \psi + \tfrac{1}{2}\lambda\delta\kappa + i(\epsilon + W_2), \tag{58}$$

$$\bar{\lambda} = 1 + \lambda, \quad W_1 = \tfrac{1}{4}\lambda\kappa^2/\alpha, \quad W_2 = \tfrac{1}{3}\lambda\kappa^2\alpha, \tag{59}$$

$$C_3 = C_1C_2 - \psi_1\psi_2. \tag{60}$$

In Eq. (53) the last term is generally negligible with respect to the preceding term.

The equations of continuity will contain terms in $v_x^{(0)}$ and $v_{xe}^{(0)}$. If one uses a simplified form of Eq. (42) in these terms, namely, $v_x^{(0)} = v_{xe}^{(0)} = -\epsilon\bar{\lambda}\delta$, the linearized continuity equations become

$$\psi\nu - 2i\epsilon\bar{\lambda}\delta^2\nu + i\delta v_x - \kappa v_y = 0, \tag{61}$$

$$\psi_e\nu - 2i\epsilon\bar{\lambda}\delta^2\nu + i\delta v_{xe} - \kappa v_{ye} - \kappa_{\parallel}v_{ze} = 0. \tag{62}$$

We now insert Eqs. (46) and (51) to (54) into (61) and (62), setting $b = C_1 = C_2 = C_3 = 1$, $\psi_1 =$

$\psi_2 = \psi$ in all terms which already contain ϵ. We then obtain two equations for χ and ν:

$$C_3\psi\nu + (C_2\delta\kappa + \kappa^2\psi_1)(\chi + \lambda\nu) - i\epsilon[3\delta^2\bar{\lambda}\nu + \kappa^2(\chi - \nu)] = 0, \quad (63)$$

$$[\psi_e + i\epsilon\bar{\lambda}(\kappa^2 - 3\delta^2)]\nu + \kappa(\delta b^{-1} - i\kappa Y)(\chi - \nu) = 0, \quad (64)$$

where $Y = \epsilon_\parallel^{-1}\theta^2$. The determinant of these equations, with ψ_e and ψ given by Eqs. (45) and (41), then gives the required dispersion relation. The ϵ which appears explicitly is $\epsilon_\perp$; parallel resistivity is contained only in Y.

Results

(*1*). *Effects of* T_i *and* ρ. Simplifications (7), (8), and (10) of Sec. I can be removed by setting $b = 1$, $\epsilon_\perp = W_1 = W_2 = 0$ in Eqs. (63) and (64), and thus neglecting the magnetic field inhomogeneity and resistive and viscous damping. With the use of Eqs. (55) to (57), the continuity equations take the form

$$C_3\psi\nu + (\delta\kappa + \kappa^2\psi)(\chi + \lambda\nu) = 0, \quad (65)$$

$$\psi_e\nu + (\delta\kappa - i\kappa^2 Y)(\chi - \nu) = 0. \quad (66)$$

Expressing ψ_e in terms of ψ by Eq. (45) (thus reintroducing ρ) and using Eq. (60) for C_3, we obtain the determinantal equation

$$\psi^2[1 + \lambda\delta^2 + \delta\kappa^{-1}\psi - iY(\lambda\delta\kappa + \psi)] + \psi[\lambda\delta\kappa - \bar{\lambda}\rho\kappa + i(1 + \kappa^2)Y] + \bar{\lambda}\delta(i\kappa Y - \rho) = 0. \quad (67)$$

For $\psi \ll |\kappa\delta^{-1}|$, $|\delta\kappa Y| \ll 1$, $\kappa^2 \ll 1$, this simplifies to our principal dispersion equation

$$\psi^2 + (\lambda\delta\kappa - \bar{\lambda}\rho\kappa + iY)\psi + \bar{\lambda}\delta(i\kappa Y - \rho) = 0. \quad (68)$$

This is just Eq. (15) if $\rho = 0$, $\lambda = 1$. Comparing this with Eq. (14) for $T_i = 0$, we see that finite T_i has three effects: A Doppler shift $\Omega \to \psi$, a finite Larmor radius term $\lambda\delta\kappa\psi$, and an increased growth rate through the factor $\bar{\lambda} \equiv 1 + T_i/T_e$. For $\rho = 0$ the first two of these effects cancel each other to maintain $\Omega = 0$ at $Y = 0$.

For $Y \gg |\delta\kappa|$, $|\delta| \gg \rho$, $Y \gg 2|\delta\rho|^{\frac{1}{2}}$, we can expand the square root in the solution of Eq. (68) to obtain the "large"-angle frequency for the unstable root:

$$\psi = -\bar{\lambda}\delta\kappa + i\bar{\lambda}Y^{-1}(\delta^2\kappa^2 - \delta\rho) \text{ (large } \theta^2/\epsilon). \quad (69)$$

The "gravitational" overstability occurs only for $\delta\rho < 0$ and is given by the last term in Im ψ. The "universal" overstability, given by the first term, is independent of the sign of δ. The frequency is given by Re $\Omega = -\kappa(\delta + \lambda\rho) \approx \kappa v_0$. Note that the $\delta^2\kappa^2$ term comes from second order in the expansion of ψ and may pass unnoticed if one is concerned only with curved systems. It is clear that the "universal" growth rate dominates for $\rho < |\delta|\kappa^2$. In contrast to "screw" instabilities which are driven by longitudinal currents, these overstabilities are independent of the sign of the pitch angle.

Setting $\lambda = 1$, $Y = 0$ in Eq. (68), we recover identically the result of Roberts and Taylor[15] and Rosenbluth *et al.*[10] for finite Larmor radius stabilization of the ordinary ($k_\parallel = 0$) gravitational instability. In this case the frequency is given by

$$\psi = -\tfrac{1}{2}\kappa(\lambda\delta - \bar{\lambda}\rho) \pm [\tfrac{1}{4}\kappa^2(\lambda\delta - \bar{\lambda}\rho)^2 + 4\bar{\lambda}\delta\rho]^{\frac{1}{2}}. \quad (70)$$

The finite-r_L stabilization term in the square root makes ψ real even for $\delta\rho < 0$ if κ is large enough; this does not happen if $Y \neq 0$. The maximum growth rate, occurring at small κ, is Im $\Omega = (\bar{\lambda}|\delta|\rho)^{\frac{1}{2}}$, and the wave travels at approximately half the ion diamagnetic drift velocity:

$$\text{Re } \Omega = \tfrac{1}{2}\lambda\kappa(\delta - \rho) + \tfrac{1}{2}\kappa\rho \approx \tfrac{1}{2}\lambda\kappa\delta.$$

In the intermediate range of Y, the solution of Eq. (68) can easily be computed from the equations

$$z = y^2\frac{1 + x - \lambda r}{\lambda + (\lambda - 1)r - 2x}, \quad (71)$$

$$y^4 = \frac{[-(x+r)(\lambda+\lambda r-x)+\bar{\lambda}r\kappa^{-2}][\lambda+(\lambda-1)r-2x]^2}{(1+x-\lambda r)(1+\lambda-x-r)}, \quad (72)$$

where $y^2 = Y/\delta\kappa$, $x = \text{Re }\Omega/\delta\kappa$, $z = \text{Im }\Omega/\delta\kappa$, and $r = -\rho/\delta$. Representative results for x and z as functions of y for $\lambda = 1$ are shown in Figs. 2 and 3. Note that κ is now a parameter in addition to ρ because the effect of finite-r_L stabilization depends on κ. Because of this effect, the gravitational instability has a larger growth rate for small κ, as seen in Fig. 2. By contrast, the overstability at large Y has a larger growth rate for large κ, as seen from Eq. (69). A further point of experimental interest is that the wave travels in the electron-drift direction when $\rho = 0$ or when $Y \gg |\delta\kappa|$, but in the ion-drift direction when $\rho > \rho_0$ and $Y \ll |\delta\kappa|$, where ρ_0 is extremely small.

(*2*). *Physical interpretation.* The picture of the finite Larmor radius effect given in Ref. 8 can be justified by considering Eqs. (65) and (66). For simplicity let $\rho = 0$. The deviation of C_3 from unity contains the effect of the collision-independent terms in π. The terms in $\delta\kappa\nu$ come from the pressure-gradient drift due to $\partial n_1/\partial x$; these terms are cancelled

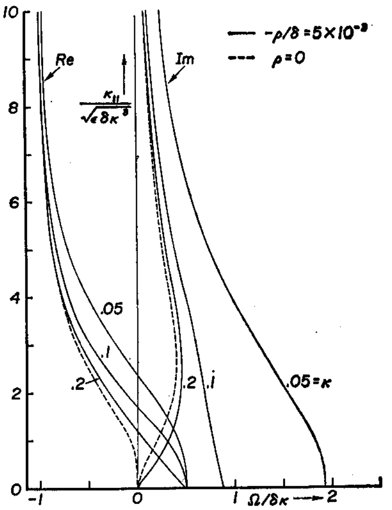

FIG. 2. Dispersion curves for the electron drift mode in a field with curvature ρ, for various values of the wavelength κ. The effect of finite Larmor radius stabilization on the gravitational instability is clearly shown with the increase in κ. The dashed curve is the $\rho = 0$ curve of Fig. 1; this is not stabilized by an increase in κ.

by the Doppler-shift terms in ψ and ψ_e and therefore disappear when Ω is substituted for ψ and ψ_e. The terms in $\delta\kappa\chi$ give the density change due to the **E×B** drifts along ∇n_0; these give no separation of charge because the same term occurs in both the ion and the electron equation. The term in $\kappa^2\psi$ in the ion equation is essentially the density change due to v_y, as can be seen from Eq. (52) for v_y. This term is due to ion inertia and has no counterpart in the electron equation; therefore, the terms in Y, representing electron motion along **B**, are necessary to balance the effect of v_y and maintain quasineutrality.

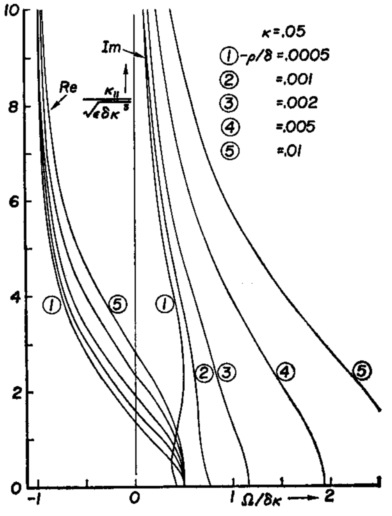

FIG. 3. Dispersion curves for various values of the curvature ρ for a fixed value of κ. Symbols are defined in the text.

The ν-dependent part of the v_y term is a pressure-gradient drift which only appears in the fluid velocity but does not represent a guiding-center drift; this part is cancelled by a term in π, as can be seen when C_3 is approximated by $C_3 \approx 1 - \lambda\kappa^2$. When $\psi = \Omega - \lambda\delta\kappa$, $\psi_e = \Omega + \delta\kappa$ is substituted and all the "fictitious" pressure-gradient drifts cancelled out, one obtains from Eqs. (65) and (66) two equations essentially describing the motion of guiding centers,

$$\Omega\nu + \kappa\delta(1 - \lambda\kappa^2)\chi + \kappa^2\Omega\chi = 0, \qquad (73)$$

$$\Omega\nu + \kappa\delta\chi - i\kappa^2 Y(\chi - \nu) = 0. \qquad (74)$$

The effect of finite Larmor radius appears in the factor $(1 - \lambda\kappa^2)$, which shows that the **E×B** drift in an inhomogeneous electric field is smaller than that for $r_L = 0$, as can easily be verified for a single particle.[21] This effect adds to that of the v_y term $\kappa^2\Omega\chi$, since $\Omega \approx -\delta\kappa$; and therefore, a larger value of $\chi - \nu$ is necessary in the electron equation to cancel the increased charge separation. This increased phase shift between χ and ν ultimately leads to an increase in Im Ω over the $\lambda = 0$ value. Note that although the guiding center drift is slowed down by an inhomogeneity in **E**, the fluid velocity v_x is actually speeded up, as can be seen from Eq. (51). This is because of the way the complicated orbits are averaged to give the fluid velocity. When $\rho \neq 0$, there is an additional drift and charge separation due to the effective gravitational field; the physical interpretation of the ρ term is trivial.

(3). *Effect of viscous damping.* We now include collisional viscosity by retaining the W terms in C_1, C_2, ψ_1, and ψ_2 while setting $\epsilon_\perp = 0$ and $b = 1$. With the assumptions $\kappa^2 \ll 1$, $\delta^2 \ll 1$, $W_2 \gg W_1$, Eq. (60) gives the following approximate form for C_3:

$$C_3 \approx 1 - \lambda\kappa^2 + i\delta\kappa^{-1}W_2. \qquad (75)$$

Using this in Eqs. (63) and (66), we obtain the dispersion relation

$$\psi^2 + \psi[\lambda\delta\kappa - \bar{\lambda}\rho\kappa + i(Y + W_1) - Y\delta\kappa^{-1}W_2] + \bar{\lambda}\delta(i\kappa Y - \rho)[1 + i(\kappa\delta^{-1}W_1 + \delta\kappa^{-1}W_2)] = 0. \qquad (76)$$

If $\alpha^2 > \delta^{-2}$, still more terms must be taken into account in C_3. The effect of W on the growth rate can be estimated by making the usual "large"-Y expansion. Keeping up to second order in W and Y^{-1}, we obtain, after another bout with algebra,

21 Actually, the correction term is twice as large as one obtains for a single particle because the distributions of particles and of guiding centers are different, as pointed out to the author by G. Schmidt.

$$\psi = -\bar{\lambda}\delta\kappa\left[1 - \frac{\lambda}{Y}\left(\frac{\kappa^2}{4\alpha} + \frac{\delta\rho\alpha}{3}\right)\right] + i\bar{\lambda}\left[\frac{\delta^2\kappa^2}{Y} - \frac{\delta\rho}{Y} - \frac{\lambda\kappa^4}{4\alpha}\right]. \quad (77)$$

As might have been expected, the terms in W_2, due to the terms proportional to α in $\mathbf{\Pi}_{xx}$ and $\mathbf{\Pi}_{yy}$, largely cancel out; and the damping comes from the small term W_1, proportional to α^{-1}, coming from $\mathbf{\Pi}_{xy}$. For $\delta^{-2} > \alpha^2 \gg 1$, which we have assumed, the collisional terms in $\mathbf{\Pi}$ generally have less effect on the growth rate than the collision-independent terms. Extension of these results to $\kappa \geq 1$ has recently been reported by Mikhailovskii and Pogutse.[22]

(*4*). *Effect of resistive damping.* As a check on the approximations made in this section, we can compute the effect of $\epsilon_\perp$ by setting $W_1 = W_2 = 0, b = 1$ in Eqs. (63) and (64) and comparing with the result of Sec. III. In Sec. III it was imagined that v_{0x} was kept zero by a slowly increasing **B**. In this section, v_{0x} was allowed to be finite; and equilibrium was provided by sources, which were neglected. These two approaches would give the same result if the time scale of resistive diffusion were long compared to frequencies of interest; however, resistive damping has the former time scale, and one would expect some difference in the two ways of computing the damping. Putting $C_1 = C_2 = 1 - \frac{1}{2}\lambda\kappa^2$, $C_3 \approx 1 - \lambda\kappa^2$, $\psi_1 = \psi + \frac{1}{2}\lambda\delta\kappa$ into Eqs. (63) and (64), we obtain

$$\psi^2 + [\lambda\delta\kappa - \bar{\lambda}\rho\kappa + i(Y + H)]\psi + \bar{\lambda}\delta(i\kappa Y - \rho) - HY = 0, \quad (78)$$

where $H = \bar{\lambda}\epsilon(\kappa^2 - 3\delta^2)$. Comparing with Eqs. (31) and (68), we see that aside from the effects of λ and ρ, the resistive damping is the same as in Sec. III except that κ^2 is replaced by $\kappa^2 - 3\delta^2$. This gives rise to the possibility of an instability if $\kappa^2 < 3\delta^2$. This is not a real instability but simply a density perturbation which is carried outward by v_{0x}, and it results from the difference in model used for the equilibrium.

(*5*). *Effect of field inhomogeneity.* Finally we evaluate the effect of the inhomogeneity in $\mathbf{B}_0$ arising from the curvature. We set $\epsilon_\perp = W = 0, b = 1 - \rho\xi, C_3 \approx b(b - \lambda\kappa^2)$, and neglect second and higher order in $\rho\xi$. From Eqs. (63) and (64) we then obtain approximately

$$\psi^2 + \psi[\lambda\delta\kappa - \bar{\lambda}\rho\kappa(1 + \delta\xi) + iY(b^2 + \kappa^2)] + ib\bar{\lambda}\delta\kappa Y - \bar{\lambda}\delta\rho(1 + \delta\xi) = 0. \quad (79)$$

Comparing with Eq. (68), we see that Y is shifted a negligible amount by the factor b or b^2 but that ρ is multiplied by a factor $(1 + \delta\xi)$, $\delta\xi$ being negative when $\delta\rho$ is negative. Hence, the "universal" overstability is relatively unaffected, but the growth rate of the gravitational mode can be appreciably affected by the field inhomogeneity. In obtaining this result we have neglected a term v_x' in the equation of continuity, as well as in $\mathbf{\Pi}$; these terms can change the magnitude and even the sign of the correction given in Eq. (79), but they do not affect our conclusion in the preceding sentence. Our purpose here has been to show the relation between the "universal" and gravitational overstabilities; more accurate treatments of the resistive-g mode in curved and sheared systems have been given by others.[3,9,11–13]

V. OTHER EFFECTS

(*1*). *Radial dependence.* We have made the approximation $k_x = 0$ throughout this work. When the x or radial dependence is taken into account, one obtains a second-order complex differential equation for $\nu(x)$ or $\chi(x)$, whose eigenvalues give the dispersion relation. This problem has been solved by the phase-integral method by Moiseev and Sagdeev[2] and by Jukes,[3] and we have not repeated this calculation. Inasmuch as their results are identical with ours, we have used the simpler algebraic method. However, it is not obvious from the electron continuity equation (66) why the x dependence should be so weak, since in Eq. (66) all quantities are independent of x except Y, which is proportional to n_0^{-1}, so that the equation apparently cannot be satisfied at all x by perturbations ν and χ which are constant with x. The reason is that in Eq. (66) the $\delta\kappa$ term is generally much larger than the $i\kappa^2 Y$ term, so that the variation of Y with x does not greatly affect the solution. The growth rates given here are local ones valid in a region of x in which n_0 and n_0'/n_0 do not change appreciably; as long as r_L is small compared with such a region, the exact treatment gives essentially the same result. If v_0 is a function of x, or if there is shear in $\mathbf{B}_0$ so that $k_\parallel$ is a function of x, then the radial equation must be solved. By doing this, Stringer[23] has recently shown that the "universal" overstability cannot be localized and therefore will be slowed down by shear. In effect, an average $k_\parallel(x)$ must be taken; and if this is made large enough by shear, the growth rate will take on the small value associated with a large value of Y.

[22] A. B. Mikhailovskii and O. P. Pogutse, Dokl. Akad. Nauk SSSR 156, 64 (1964) [English transl.: Soviet Phys.—Doklady 9, 379 (1964)].

[23] T. E. Stringer, Princeton Plasma Physics Laboratory Report Matt-320 (1965).

(*2*). *Electron inertia.* If the resistivity ϵ is reduced sufficiently, the motion of electrons will be controlled by inertia rather than by collisions with ions. As shown in (I), this occurs when $\epsilon \ll |\mu\psi_e|$, where $\mu = m_e/M$. To see whether or not the electron drift wave is unstable in the collisionless limit, we use Eq. (65) for the ions and Eq. (67) of (I) for the electrons, which amounts to replacing Y in Eq. (66) by iY_μ/ψ_e, where $Y_\mu \equiv \theta^2/\mu$. With the usual approximations we then obtain the dispersion relation

$$\psi_e^3 - (1 + \bar{\lambda})\delta\kappa\psi_e^2 + (\bar{\lambda}\delta^2\kappa^2 - Y_\mu)\psi_e + \bar{\lambda}\delta\kappa^3 Y_\mu = 0. \quad (80)$$

It can be shown that this cubic has three real roots for any value of λ, and hence, the wave is stable in the absence of collisions. However, it is not consistent to include electron inertia without also considering electron viscosity. By including the components of π_e parallel to $\mathbf{B}$, one can recover the so-called inertial instability.

(*3*). *Longitudinal current.* When a uniform zero-order field E_{0z} exists, so that v_{0z} and v_{0ze} are nonzero, a number of additional terms in Eqs. (37) and (38) must be taken into account. We have considered the case $T_i = 0$ neglecting perpendicular resistivity. Then the centrifugal terms give first-order contributions $2\rho v_{0z} v_z \hat{\mathbf{x}}$ and $2\mu\rho v_{0ze} v_{ze} \hat{\mathbf{x}}$ to Eqs. (43) and (44), respectively; the latter term has a larger effect, since $v_{0z} \approx -\mu v_{0ze}$, and $v \ll v_{ze}$. A term in $\pm\epsilon_1(v_{0z} - v_{0ze})\hat{z}$ must also be added to both equations, where $\epsilon_1 = \nu\epsilon_0$. Furthermore, ψ will now contain a Doppler shift $-\kappa_\parallel v_{0z}$. It can be shown that the $\mathbf{v}_e \cdot \nabla \mathbf{v}_e$ term is still negligible. Proceeding as before, we find that the dispersion relation is again given by Eq. (60) for $\lambda = 0$, but with Y replaced by $Y(1 - 2\mu\rho\iota_0\theta^{-1})$ and ρ replaced by $\rho(1 - 2\mu\iota_0^2)$, where $\iota_0 = -v_{0ze} = \mathcal{E}_{0z}/\epsilon = j_0/n_0 v_s$. The change in ρ is a stabilizing effect, and the change in Y may be stabilizing or destabilizing, depending on the sign of ι_0/θ. Since both corrections are extremely small, we conclude that a parallel current has no appreciable effect on the resistive overstabilities. This is in contrast to the collisionless case, where a longitudinal current has a great effect on the growth rate[24] on account of the resonant particles. In a resistive plasma no such resonant interaction is possible.

(*4*). *Temperature gradient.* A zero-order perpendicular temperature gradient also does not greatly alter our results, since this would only change the x dependence of ϵ, a dependence which is negligible. On the other hand, when such a gradient is coupled to a longitudinal current, instabilities with large growth rates are known to be excited.[25] A temperature gradient is also maintained by longitudinal currents. These effects would dominate over those considered in this paper when a large current is present.

(*5*). *Landau damping.* In order for Landau damping to be negligible, the ion thermal velocity $\lambda^{\frac{1}{2}}v_s$ must be much smaller than the parallel phase velocity $\omega/k_\parallel \approx v_D/\theta$. Hence, we require $\lambda^{\frac{1}{2}}\theta \ll v_D/v_s = \delta$. For $\lambda = 1$ this is the same condition as that obtained in Sec. III for the validity of the drift wave approximation: $\theta^2 \approx 0$, $v_z \approx 0$.

(*6*). *Cylindrical geometry.* These calculations can be extended easily to the case of cylindrical geometry; however, the result will be unaffected as long as the radius r is $\ll r_L$, as shown in (I). The main effect is that of the centrifugal force nMv_θ^2/r. When v_θ is due entirely to the pressure-gradient drift, Rosenbluth *et al.*[10] have shown that no instability occurs. When v_θ is due to a large zero-order radial electric field, the centrifugal force can cause a slow instability in the presence of resistivity or viscosity.[26]

(*7*). *Energy considerations.* We have seen that the growth rate of the universal overstability is not greatly affected (only a factor of 2) when T_i is varied from $T_i = 0$ to $T_i = T_e$. On the other hand, the wave does not exist when $T_e = 0$ and $T_i \neq 0$, although one would expect that the ion pressure would then drive the instability. The reason for this is given by Fowler,[27] who shows that in the frame moving with the ion fluid no free energy is available to drive any instability if $n_e = T_e = \beta = 0$.

(*8*). *Resonant particles.* It is well known that resonant electrons can also cause a "universal" instability in a collisionless plasma. The relation between the resistive and collisionless instabilities has been discussed by Stringer,[23] who shows that they are limiting cases of the same phenomenon. Since a resonant electron must be in phase with the wave for at least one oscillation before it makes a collision with either an ion or an end-plate, we know of no experimental application of the resonant-particle theory.

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