Axial eigenmodes for long -$\lambda_\parallel$ waves in plasmas bounded by sheaths

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(Received 14 May 1979)

Waves such as drift waves and lower hybrid oscillations in a plasma are sensitive to the degree to which charge neutrality can be maintained by electron flow along the magnetic field. When the plasma is bounded axially, the sheath conditions on the end plates determine the parallel wavelength. It is found that the nature of the bounded modes depends on whether the motion of electrons is resistive or inertial. If it is resistive, the sheath matching conditions can be satisfied by standing waves with the proper wavelength. If it is inertial, pure standing waves are not possible; there must also be a variation of phase along **B**. Application is made to two-ion hybrid waves in connection with isotope separation.

I. BACKGROUND

It has long been recognized[1] that the frequencies and wavelengths of waves excited in finite-sized laboratory plasmas depend on the boundaries. Plasma boundaries are covered by sheaths, and it is not always sufficient to assume that the perturbation amplitude vanishes at the wall. Here we treat a problem which apparently has not been addressed in the past. For instance, the Tonks–Dättner resonances of electron plasma waves in a finite plasma have been studied in great detail,[2] but the concern was with the density gradient and, hence, variation of ω_p, in the plasma rather than with the boundary condition at the wall. The change of sheath potential and the flux of electrons into the wall were neglected. Similarly, papers dealing with the reflection of ion waves at boundaries,[3] although treating the sheath problem adequately, do not address the effect of electron flow into the wall. In connection with rf heating of tokamak devices, there is current interest in toroidal eigenmodes. These do not encounter axial boundaries except in the region outside the aperture limiter; for that region the results presented in this paper may be of some interest.

We consider the following problem: Waves with $k_\parallel/k_\perp \ll 1$ are excited in a plasma in a strong magnetic field $\mathbf{B}_0$. The plasma is infinite in the directions perpendicular to $\mathbf{B}_0$ but is bounded in the direction of $\mathbf{B}_0$ by hot or cold end plates. Waves such as a lower hybrid or a two-ion hybrid[4] wave can exist if the fluctuating ion space charge is neutralized not by electron flow along $\mathbf{B}_0$ but by perpendicular electron or secondary ion motions. In an infinite plasma, this requires $k_\parallel/k_\perp < (m/M)^{1/2}$. In a plasma bounded by cold end plates, the electron flow is basically interrupted by the nonemitting plates, but a small neutralizing current can be provided by a modulation of the steady-state electron loss to the plates through the sheaths. If electron conduction along $\mathbf{B}_0$ is too good, the lower hybrid turns into a high-frequency electron wave and the two-ion hybrid into an electrostatic ion cyclotron wave.

Waves such as the collisional drift wave[5] are also sensitive to electron flow along $\mathbf{B}_0$. If it is too good (that is, if $k_\parallel/k_\perp$ is too large), the drift wave turns into an ion acoustic wave. If it is too poor (that is, if $k_\parallel/k_\perp$ is too small), the drift wave cannot exist, but there can be a flute instability propagating opposite to the electron diamagnetic drift direction. We have shown previously[6] that resistive drift waves in a Q machine of length $2L$ have $k_\parallel$ determined by

$$k_\parallel L \tan k_\parallel L = f(T_c)\ , \tag{1}$$

where $f(T_c)$ is a number depending on the plasma parameters and the temperature of the emitting end plates. If electron emission at the ends is large, $f(T_c)$ is $\gg 1$, so that the lowest eigenmode is given by $k_\parallel L = \frac{1}{2}\pi$. This is the usual case where the wave amplitude vanishes at the boundaries. If electron emission is small or zero, $f(T_c)$ is small, and consequently $k_\parallel L$ is small. The wave then has an effective parallel wavelength longer than the plasma. This is possible because the sheath on a cold conductor behaves like a good insulator. The validity of Eq. (1) has been checked experimentally by Rowberg and Wong.[7]

This simple result does not apply to a wave like the two-ion hybrid in a collisionless plasma. There, the electron parallel motion is controlled by electron inertia rather than by collisions with ions. Because of the 90° difference in phase between a resistivity-dominated current and an inertia-dominated current, a simple standing wave in the axial direction cannot satisfy the sheath-matching conditions on both end plates at the same time. However, it will be shown in this paper that a standing wave with an axial phase shift will satisfy the boundary conditions, giving rise to a rather unusual eigenmode of the system.

II. SHEATH MATCHING CONDITIONS

Consider a plasma that is unbounded in the x and y directions but bounded by flat conducting plates in the z direction, as shown in Fig. 1. In the unperturbed state, the plasma density n_0 and potential ϕ_0 are assumed to be uniform except in the sheaths, where ϕ changes rapidly to zero at the conducting boundaries. If the end plates are "cold" (i.e., emitting fewer electrons than they receive from the plasma), ϕ_0 will be positive, as in Fig. 1(a), so as to retard the loss of electrons from

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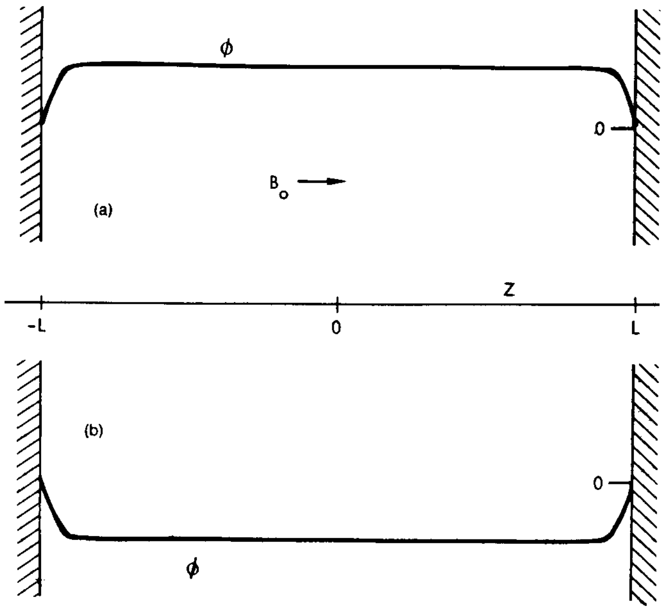

FIG. 1. Schematic of equilibrium potential variation for (a) ion-rich and (b) electron-rich plasmas.

the plasma. If the end plates are "hot" (i.e., emitting more electrons than the number escaping), ϕ_0 will be negative, as in Fig. 1(b), so as to reflect the excess electrons back to the plate.

For the case $\phi_0 > 0$ (the usual case), the assumed density profile in the plasma ($\partial n_0/\partial z = 0$) can only be satisfied approximately, because sheath formation requires that the ion fluid at the sheath edge have a directed velocity[8]

$$v_B \simeq \tfrac{1}{2}(KT_e/M)^{1/2} \tag{2}$$

(the Bohm criterion). (The factor $\frac{1}{2}$ does not appear in the usual sheath criterion. Here, it accounts approximately for the decrease of density in the presheath, so that the current computed from v_B is more nearly correct.) To produce this drift, there must be an electric field accelerating the ions in a presheath, whose scale length depends on such quantities as the ionization length or collision mean-free-path in the plasma. We make the basic assumption that the presheath gradients can be neglected compared with those in the sheath. We can then write the net flux of electrons *into* an end plate as

$$\Gamma^+ = nv_r \exp(-e\phi/KT_e) - j_t \ , \tag{3}$$

where v_r is the random electron velocity in one direction,

$$v_r \equiv (KT_e/2\pi m)^{1/2} \ , \tag{4}$$

and j_t is the thermionically emitted electron flux[9]

$$j_t = A_R T_c^2 \exp(-eW/KT_c). \tag{5}$$

Here, T_c is the cathode temperature, W is the work function, and A_R is the Richardson constant, 7.5×10^{20} $\text{cm}^{-2}\ \text{sec}^{-1}\ {}^\circ\text{K}^{-2}$. The + sign indicates $\phi_0 > 0$ (ion sheath).

If $\phi_0 < 0$ (the case of electron-rich Q machines or space-charge-limited cathodes), no presheath is needed to accelerate the ions, and the gradients in the plasma can indeed vanish. The net electron flux into the end plates can be written

$$\Gamma^- = nv_r - j_t \exp(e\phi/KT_c) \ . \tag{6}$$

If a wave is present in the plasma, the flux Γ^+ or Γ^- will fluctuate as the values of n and ϕ at the sheath edge fluctuate. In the steady state, the electron flux is balanced by an equal ion flux into the end plates. A wave upsets this balance and causes a net flow of electrons into or out of the end plates. We make a further assumption that the ion flow does not fluctuate. This flow can change only if the presheath changes, but the waves under consideration have periods much shorter than the transit time of an ion through the presheath or a parallel wavelength. In equilibrium, there is ambipolar flow of electrons and ions into the end plates, and Eqs. (3) and (6) read

$$\Gamma_0^+ = n_0 v_r \exp(-e\phi_0/KT_e) - j_t = n_0 v_B \ , \tag{7}$$

$$\Gamma_0^- = n_0 v_r - j_t (e\phi_0/KT_c) = n_0 v_{ir} \exp(e\phi_0/KT_i) \ , \tag{8}$$

where the + and − signs stand for $\phi > 0$ and $\phi < 0$, respectively, and $n_0 v_{ir}$ is the random ion flux. Note that when $j_t = 0$ (cold end plates), only the case $\phi > 0$ is possible, and Eq. (7) then states

$$\frac{e\phi_0}{KT_e} = \ln\frac{v_r}{v_B} = \frac{1}{2}\ln\left(\frac{2}{\pi}\frac{M}{m}\right) \ . \tag{9}$$

When j_t is at all appreciable, the right-hand sides of Eqs. (7) and (8) can be neglected. In the Q-machine case $T_e = T_c$, they both become

$$e\phi_0/KT_e = \ln(n_0 v_r/j_t). \tag{10}$$

When this equilibrium is perturbed by a wave, the density and potential at the sheath edge become $n_0 + n_1$ and $\phi_0 + \phi_1$, respectively. Inserting these into Eqs. (3) and (6), linearizing, subtracting the equilibrium values of Eqs. (7) and (8), and assuming that j_t and the ion fluxes are unperturbed, we obtain the following first-order electron fluxes *into* the sheaths:

$$\Gamma_1^+ = n_0 v_r \exp\left(-\frac{e\phi_0}{KT_e}\right)\left(\frac{n_1}{n_0} - \frac{e\phi_1}{KT_e}\right) \ , \tag{11}$$

$$\Gamma_1^- = n_0 v_r (n_1/n_0) - (n_0 v_r - \Gamma_i)(e\phi_1/KT_c) \ . \tag{12}$$

Here Γ_i is the right-hand side of Eq. (8) and can safely be neglected relative to $n_0 v_r$. Defining

$$\nu \equiv n_1/n_0 \text{ and } \chi \equiv e\phi_1/KT_e \, , \tag{13}$$

we may write Eqs. (11) and (12), with the help of Eq. (7), as

$$\Gamma_1^+ = (n_0 v_B + j_t)(\nu - \chi) \tag{14}$$

and

$$\Gamma_1^- = n_0 v_r \left[\nu - \chi(T_e/T_c)\right] . \tag{15}$$

These fluxes must now be matched to the electron fluxes in the wave. Treating the electrons as a fluid, we may write the linearized electron flux in the z direction as

$$\Gamma_z = n_0 v_z + n_1 v_{z0} = n_0 v_z \ , \tag{16}$$

where the zero-order drift v_{z0} is assumed to vanish, and the first-order fluid velocity v_z is given by the equation of motion

$$mn_0 \frac{\partial v_z}{\partial t} = en_0 \frac{\partial \phi}{\partial z} - KT_e \frac{\partial n_1}{\partial z} - mn_0 \nu_{ei} v_z \ . \tag{17}$$

Here, ν_{ei} is the electron-ion collision frequency, and we have assumed low frequencies such that the oscillations are both electrostatic and isothermal. Since the plasma is assumed to be initially homogeneous between the sheath edges, we may Fourier analyze in the z direction:

$$n_1, \phi_1 \sim \exp[i(k_z z - \omega t)] \ . \tag{18}$$

With the help of Eq. (13), the equation of motion (17) becomes

$$(\nu_{ei} - i\omega) v_z = (KT_e/m) i k_z (\chi - \nu) \ , \tag{19}$$

and the flux (16) becomes

$$\Gamma_z = i k_z n_0 (KT_e/m)(\nu_{ei} - i\omega)^{-1}(\chi - \nu) \ . \tag{20}$$

If this flux is matched to the sheath flux of Eq. (14) or (15) at the sheath edges, the proper neutralizing electron currents will be provided by the sheaths to sustain the wave motions pertinent to an infinite, homogeneous plasma. The matching determines the allowable values of k_z. When there is either an ion sheath ($\phi_0 > 0$) or a thermally ionized plasma ($T_e = T_c$), the analysis is particularly simple. Since both expressions for Γ_z then depend only on $\nu - \chi$, it is not necessary to find ν and χ separately. Otherwise, one would have to solve the ion equations of motion and continuity to obtain a relation

$$\nu = g(\omega, \mathbf{k}) \chi \ , \tag{21}$$

and solve the corresponding electron equations to obtain another relation

$$\nu = h(\omega, \mathbf{k}) \chi \ . \tag{22}$$

Setting $g(\omega, \mathbf{k})$ equal to $h(\omega, \mathbf{k})$ then gives the dispersion relation $\omega(\mathbf{k})$ and the relation $\nu(\chi)$. The sheath matching procedure, however, can be carried out without solving for $\omega(\mathbf{k})$ since it depends only on the z component of the electron equation of motion and only on the difference $\nu - \chi$. Thus, the results to be given are quite general and apply to any low-frequency electrostatic wave which does not involve resonant particles. The case $\phi_0 < 0$, $T_e > T_c$ is an exception.

III. PRELIMINARY DISCUSSION

A. Results for collisional electron motion

If resistivity rather than inertia dominates the electron motion, Eq. (20) is approximated by

$$\Gamma_z = i k_z n_0 (KT_e / m\nu_{ei})(\chi - \nu) \ . \tag{23}$$

1. Ion sheath, $\phi_0 > 0$

In this case we set Γ_1^+ equal to Γ_z at $z = L$ and to $-\Gamma_z$ at $z = -L$. Eqs. (14) and (23) then give

$$(n_0 v_B + j_t)(\nu - \chi) = \mp \left(\frac{n_0 KT_e}{m\nu_{ei}}\right) \frac{\partial}{\partial z} (\nu - \chi) \tag{24}$$

at $z = \pm L$. By symmetry, we choose the even function

$$\nu - \chi \sim \cos k_z z \quad (k_z > 0), \tag{25}$$

whereupon Eq. (24) becomes

$$k_z L \tan k_z L = L (m\nu_{ei}/n_0 KT_e)(n_0 v_B + j_t) \ . \tag{26}$$

This can be simplified by defining a mean-free-path λ_m,

$$\lambda_m = \frac{\bar{v}}{\nu_{ei}} = \left(\frac{2KT_e}{\pi m}\right)^{1/2} \frac{1}{\nu_{ei}} = \frac{2v_r}{\nu_{ei}} \ . \tag{27}$$

With Eqs. (2), (4), and (27), the boundary condition (26) can be written

$$k_z L \tan k_z L = \frac{L}{\pi\lambda_m}\left(\frac{v_B}{v_r} + \frac{j_t}{n_0 v_r}\right) = \frac{L}{\pi\lambda_m}\left[\left(\frac{\pi m}{2M}\right)^{1/2} + \frac{j_t}{n_0 v_r}\right] . \tag{28}$$

This is identical to the result given previously for resistive drift waves,[6] except that the ion current term has been added to allow a smooth transition to the case of completely cold plates.

2. Electron sheath, $\phi_0 < 0$

In this case, we set Γ_1^- equal to $\pm\Gamma_z$ at $z = \pm L$. Equations (15) and (23) yield

$$n_0 v_r \left[\nu - \chi\left(\frac{T_e}{T_c}\right)\right] = \mp \frac{n_0 KT_e}{m\nu_{ei}} \frac{\partial}{\partial z} (\nu - \chi) \ . \tag{29}$$

If $T_e = T_c$, use of Eqs. (25) and (27) leads to

$$k_z L \tan k_z L = L/\pi\lambda_m \ . \tag{30}$$

Except for notation, this is identical to the result given previously[6] for drift waves in a Q machine. It is clear that this simple result will not hold for discharge plasmas with electron emitting end plates, where $T_e > T_c$. The fact that the emitted electrons are at a different temperature from the plasma electrons not only complicates the solution of Eq. (29), but also calls for reexamination of the equilibrium for the possibility of a nonmonotonic (double) sheath. This is deferred to Sec. VI.

3. Numerical example

Let $L = 31.4$ cm, $n = 10^{11}$ cm^{-3}, $KT_e = 0.2$ eV. Then, $\lambda_m \simeq 1$ cm, and Eq. (30) gives, for the lowest mode,

$$k_z L \tan k_z L = 10, \quad k_z L = 1.43 = 82°, \quad \cos k_z L = 0.14.$$

Thus, the perturbation is quite effectively tied to the emitting end plate by the electron sheath, and the value of λ_z ($\simeq 2L$) is nearly what it would be if the sheath had been neglected altogether. On the other hand, if cold plates are used, Eq. (28) gives, for $M = 40\, M_H$,

$$k_z L \tan k_z L = (10)(4.6 \times 10^{-3}) = 0.05, \quad k_z L = 0.21 = 12° \ .$$

Now the parallel wavelength is about $30L$, and the fluctuation in potential occurs almost entirely in the sheath drop. The ion sheath acts like a good insulator. For a discharge plasma with $KT_e \simeq 2$ eV, the factor $(L/\pi\lambda_m)$ is decreased by a factor 100, since $\lambda_m \propto T_e^2$, so that $k_z L \tan k_z L \leqslant 0.1$. The sheath acts like an insulator even for electron sheaths. However, λ_m is now larger than L, and our neglect of electron inertia breaks down.

B. Results for inertial electron motion

If collisions can be neglected, Eq. (20) becomes, in the opposite limit,

$$\Gamma_z = (n_0 k_z/\omega)(KT_e/m)(\nu - \chi) . \tag{31}$$

1. Ion sheath, $\phi_0 > 0$

Equating Γ_1^+ [Eq. (14)] to Γ_z, we obtain

$$\omega/k_z = (KT_e/m)(v_B + j_t/n_0)^{-1} . \tag{32}$$

Thus, it appears that the length of the plasma does not matter as long as the wave has the proper parallel phase velocity. However, if $k_z > 0$, Eq. (32) ensures only that the boundary condition at $z = L$ is satisfied. At $z = -L$, we must set Γ_1^+ equal to $-\Gamma_z$

$$\omega/k_z = -(KT_e/m)(v_B + j_t/n_0)^{-1} . \tag{33}$$

A traveling wave cannot satisfy the sheath condition at both ends at the same time. Superposing two traveling waves to make a standing wave does not help, since each component has a time-varying mismatch at one end or the other.

To find a solution, we return to the original differential equation, Eq. (17), and write it for $\nu_{ei} = 0$ as follows:

$$\frac{\partial v_z}{\partial t} = -\frac{KT_e}{m}\frac{\partial}{\partial z}(\nu - \chi) . \tag{34}$$

The sheath condition, Eq. (14), requires v_z, at $z = \pm L$, to have the value

$$v_z = \pm(v_B + j_t/n_0)(\nu - \chi) , \tag{35}$$

or

$$\frac{\partial v_z}{\partial t} = \pm\left(v_B + \frac{j_t}{n_0}\right)\frac{\partial}{\partial t}(\nu - \chi) . \tag{36}$$

Define

$$\psi \equiv \nu - \chi, \quad a \equiv (v_B + j_t/n_0)(m/KT_e) . \tag{37}$$

Equations (34) and (36) require that ψ satisfy

$$\frac{\partial \psi}{\partial z} = \mp a \frac{\partial \psi}{\partial t} \quad at\ z = \pm L . \tag{38}$$

Consider the qualitative behavior of the wave variable ψ assuming that it varies as cos ωt at the sheath edges, as if driven by an oscillator on one of the end plates. The lowest symmetric mode would fluctuate in the manner shown in Fig. 2, with the slope large at the ends when $\partial\psi/\partial t$ is large, and hence ψ is small; and vice versa.

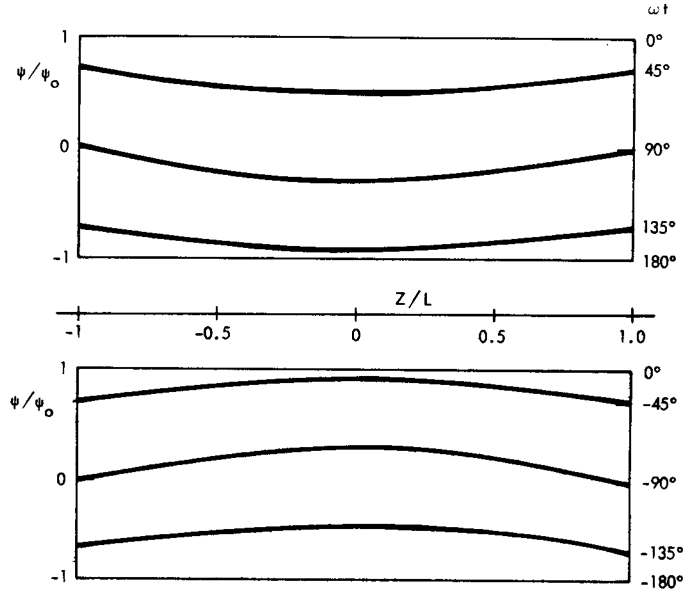

FIG. 2. Axial variation of perturbed potential as a function of time for the case of inertia-limited electron motion.

Each of the curves in Fig. 2 is a section of a cosine, since the equations governing the wave motion between the sheaths are the same as for an infinite, homogeneous plasma. The cosine describing ψ can have wavelength, amplitude, and zero-point varying with time. Let ψ have the general form

$$\psi(z,t) = A(t) + B(t)\cos[\gamma(t)z] . \tag{39}$$

The boundary conditions are

$$\psi(\pm L, t) = \psi_0 \cos\omega t , \tag{40}$$

and

$$\psi'(\pm L, t) = \mp a\dot{\psi}(\pm L, t) , \tag{41}$$

where the prime indicates $\partial/\partial z$ and the overdot indicates $\partial/\partial t$. Substituting Eq. (39) into these yields, for $\gamma \equiv |\gamma|$,

$$A(t) + B(t)\cos\gamma L = \psi_0 \cos\omega t , \tag{42}$$

$$\pm B(t)\gamma(t)\sin\gamma L = \mp a(\dot{A} + \dot{B}\cos\gamma L - BL\dot{\gamma}\sin\gamma L) . \tag{43}$$

The first of these shows that if γ varies with time, harmonics of ω will be generated, and it will be impossible to solve these equations without some method of truncation. If $\dot{\gamma} = 0$, however, we can find a simple solution if $\cos\gamma L = 0$, $\gamma = \pi/2L$. Equations (42) and (43) then become

$$A(t) = \psi_0 \cos\omega t , \tag{44}$$

$$\mp B(t)\gamma = \pm a\omega\psi_0 \sin\omega t , \tag{45}$$

or

$$B(t) = -(2La\omega/\pi)\psi_0 \sin\omega t , \tag{46}$$

$$\psi(z,t) = \psi_0[\cos\omega t - (2L\omega/\pi)a\cos(\pi z/2L)\sin\omega t] . \tag{47}$$

Antisymmetric solutions varying as sin γz can easily be shown to be incompatible with the boundary conditions.

We can write Eq. (47) in terms of a spatially varying phase shift δ

$$\psi(z,t)=\psi_0 \sec\delta \cos(\omega t+\delta) \ , \tag{48}$$

where

$$\tan\delta = \frac{2\omega L}{\pi n_0}(n_0 v_B + j_t)\left(\frac{m}{KT_e}\right)\cos\frac{\pi z}{2L} \ . \tag{49}$$

Since δ is normally small, the eigenmode in the ion sheath case has little variation in amplitude along z but must have a phase shift varying with z in order to satisfy the sheath boundary conditions.

2. *Electron sheath, $\phi_0 < 0$*

Again, we must defer discussion of the difficult case $T_e \neq T_c$ and assume that the emitted electrons have the same temperature as the plasma electrons. Equation (15) then reads

$$\Gamma_1^- = n_0 v_r(\nu-\chi), \quad v_r \equiv (KT_e/2\pi m)^{1/2} \ . \tag{50}$$

Setting this equal to $\pm n_0 v_z$ at $z=\pm L$, we have

$$\frac{\partial v_z}{\partial z} = \pm v_r \frac{\partial}{\partial t}(\nu-\chi) \ . \tag{51}$$

The equation of motion (34) then yields

$$\frac{\partial \psi}{\partial t} = \mp \frac{1}{2\pi v_r}\frac{\partial \psi}{\partial t} \qquad (z=\pm L), \tag{52}$$

which is identical to Eq. (38), except that a is now given by

$$a=(2\pi \nu_r)^{-1} \ . \tag{53}$$

The axial eigenmode is again of the form of Eq. (48), with δ now given by

$$\tan\delta = (\omega L / v_r \pi^2)\cos(\pi z/2L) \ . \tag{54}$$

The phase shift δ is larger than in the ion sheath case by about $(m/M)^{1/2}$.

IV. GENERAL SOLUTION

The unperturbed plasma is assumed to have uniform density n_0 and potential ϕ_0 relative to the end plates. In the presence of linear oscillations, the sheath edge conditions are $n=n_0+n_1$, $\phi=\phi_0+\phi_1$. We consider only the case $T_e=T_c$. The electron fluid velocity in the z direction is given by Eq. (17):

$$\frac{\partial v_z}{\partial t} = \frac{KT_e}{m}\frac{\partial}{\partial z}(\chi-\nu) - \nu_{ei}v_z = -\frac{KT_e}{m}\frac{\partial \psi}{\partial z} - \nu_{ei}v_z , \tag{55}$$

where ν, χ, and ψ are defined in Eqs. (13) and (17). The sheath conditions at $z=\pm L$ are given by Eqs. (14) and (15):

$$\phi_0>0: \ n_0 v_z^+ = \pm(n_0 v_B + j_t)\psi, \tag{56}$$

$$\phi_0<0: \ n_0 v_z^- = \pm n_0 v_r \psi \qquad (T_e=T_c) \ , \tag{57}$$

where v_B, v_r, and j_t are defined in Eqs. (2), (4), and (5). Taking the time derivative and substituting into Eq. (55), we obtain for either case the sheath matching equation

$$\frac{\partial \psi}{\partial z} \pm a\frac{\partial \psi}{\partial t} \pm a\nu_{ei}\psi = 0 \qquad (z=\pm L), \tag{58}$$

where, for $\phi_0>0$,

$$a=a^+ \equiv (v_B + j_t/n_0)(KT_e/m)^{-1} \ , \tag{59}$$

and, for $\phi_0<0$,

$$a=a^- \equiv v_r(KT_e/m)^{-1} = (m/2\pi KT_e)^{1/2} \ . \tag{60}$$

We assume that a wave is symmetrically excited in the body of the plasma (for instance, by grids or an induction coil), so that the wave variable $\psi=\nu-\chi$ fluctuates sinusoidally at the sheath edges. Alternatively, we may consider the wave to be excited by a sinusoidal voltage applied to one or both end plates. The latter may be segmented, so as to permit excitation of a finite-$k_\perp$ mode with $\mathbf{k}_\perp = k\hat{y}$. The only way energy can be supplied by the end plates is if a current is drawn through the sheath; this implies that a difference $\nu-\chi$ must exist at the sheath edge, and therefore $\psi(\pm L)$ must fluctuate sinusoidally. Thus, for either method of excitation, we may assume

$$\psi(\pm L) = \psi_0 \cos(\omega t - ky) \ . \tag{61}$$

The problem, then, is the simultaneous solution of Eqs. (58) and (61).

Since the unperturbed plasma is uniform, the solution may be expressible in an even series of the form $\psi(z) = b_0 + \sum b_n \cos(n\pi z/L)$. However, we must allow the coefficients to be time dependent. We also limit the series to the first term, since the modes being considered favor small $k_\parallel$. Thus, we assume a solution of the form

$$\psi(y,z,t) = A(y,t) + B(y,t)\cos\gamma z \quad (\gamma>0). \tag{62}$$

Substituting this into Eqs. (61) and (58), we obtain

$$A(y,t) + B(y,t)\cos\gamma L = \psi_0 \cos(\omega t - ky) \ , \tag{63}$$

and

$$\mp B\gamma \sin\gamma L \pm a\frac{\partial}{\partial t}(A + B\cos\gamma L) \pm a\nu_{ei}(A + B\cos\gamma L) = 0 \ . \tag{64}$$

Using Eq. (63), the latter becomes

$$B(t)\gamma \sin\gamma L = a\nu_{ei}\psi_0\cos(\omega t - ky) - a\omega\psi_0 \sin(\omega t - ky) \ . \tag{65}$$

A solution incorporating both inertial and resistive effects is, therefore, Eq. (62) with

$$B(t) = a\psi_0(\nu_{ei}\cos\phi - \omega\sin\phi)(\gamma\sin\gamma L)^{-1} , \tag{66}$$

$$A(t) = \psi_0\cos\phi - a\psi_0(\gamma\tan\gamma L)^{-1}(\nu_{ei}\cos\phi - \omega\sin\phi) , \tag{67}$$

where

$$\phi \equiv \omega t - ky \ . \tag{68}$$

A. Resistive limit

When resistivity dominates the electron parallel motion, the terms proportional to ω in Eqs. (66) and (67) can be neglected. The solution then becomes

$$\psi=\psi_0\cos(\omega t-ky)\left[1+\frac{a\nu_{ei}}{\gamma\tan\gamma L}\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right)\right]. \tag{69}$$

The choice of the parallel wavenumber γ is apparently arbitrary. This is to be expected, since the assumed solution, Eq. (62), had three constants, A, B, and γ, to be determined by the two boundary conditions (58) and (61). The third boundary condition, of course, comes from the matching of the *ion* fluxes through the sheaths. Physically, we would expect that the solution would not depend greatly on the ion motions. We shall now show that this is indeed true, since $\psi(z)$ is not sensitive to the value of γ.

Define

$$g\equiv R/\gamma L\tan\gamma L,\quad R\equiv a\nu_{ei}L\,. \tag{70}$$

Equation (69) reads

$$\psi=\psi_0\cos(\omega t-ky)\left[1+g\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right)\right]. \tag{71}$$

A particularly simple solution is $g=1$, giving

$$\psi=\psi_0\sec\gamma L\cos\gamma z\cos(\omega t-ky)\,. \tag{72}$$

The condition $g=1$ is equivalent to

$$\gamma L\tan\gamma L=R=a\nu_{ei}L\,. \tag{73}$$

With the values of a given in Eqs. (59) and (60), this is identical to the special solutions found earlier [Eqs. (28) and (30)]. If γL takes on other values than that specified by Eq. (73), nonetheless $\psi(z)$ is not greatly affected. This is seen in Fig. 3, where $\psi(z)/\psi_0$ is plotted against z/L for fixed R and the entire range of γL. The value $R=10/\pi$ is chosen to correspond to $L/\lambda_m=10$ in the electron sheath case, where $R=L/\pi\lambda_m$. For fixed R, if g is large, γL must be small. For $g\gg1$ and $\gamma L\ll1$, Eq. (71) gives approximately

$$\left|\frac{\psi_{\max}}{\psi(L)}\right|\simeq1+\frac{R}{\gamma^2L^2}\left(1+\frac{1}{2}\gamma^2L^2-1\right)=1+\frac{R}{2}\,. \tag{74}$$

If g is small, then $\gamma L\simeq\frac{1}{2}\pi$, and this ratio becomes

$$\left|\frac{\psi_{\max}}{\psi(L)}\right|\simeq1+\frac{2R}{\pi}\left(\frac{1}{\sin\frac{1}{2}\pi}-\frac{1}{\tan\frac{1}{2}\pi}\right)=1+\frac{2R}{\pi}\,. \tag{75}$$

Thus, the line-tying effect of the sheath boundaries is insensitive to γL and depends only on R. For ion sheaths, R is much smaller than $10/\pi$, and one has $|\psi_{\max}/\psi(L)|\simeq1$.

B. Inertial limit

In this case the ν_{ei} terms in Eqs. (66) and (67) can be neglected, and Eq. (62) becomes

$$\psi=\psi_0\left[\cos\phi-\frac{a\omega L}{\gamma L\tan\gamma L}\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right)\sin\phi\right], \tag{76}$$

$$\psi=\psi_0\left[\cos\phi-g'\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right)\sin\phi\right], \tag{77}$$

where

$$g'\equiv R'/\gamma L\tan\gamma L,\quad R'\equiv a\omega L,\quad \phi=\omega t-ky\,. \tag{78}$$

This can be written in terms of a phase shift $\delta(z)$:

$$\psi=\psi_0\sec\delta\cos(\phi+\delta)\,,\qquad \tan\delta=g'\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right). \tag{79}$$

The nature of this mode is illustrated in Fig. 4, where the constant amplitude contours are shown in comparison with the case of an ordinary standing wave (the collisional case). The special solution previously discussed [Eqs. (49) and (54)] corresponds to $\gamma L=\frac{1}{2}\pi$, or

$$\psi=\psi_0[\cos\phi-(2L/\pi)a\omega\cos(\pi z/2L)\sin\phi], \tag{80}$$

and

$$\tan\delta=(2L/\pi)a\omega\cos(\pi z/2L)\,. \tag{81}$$

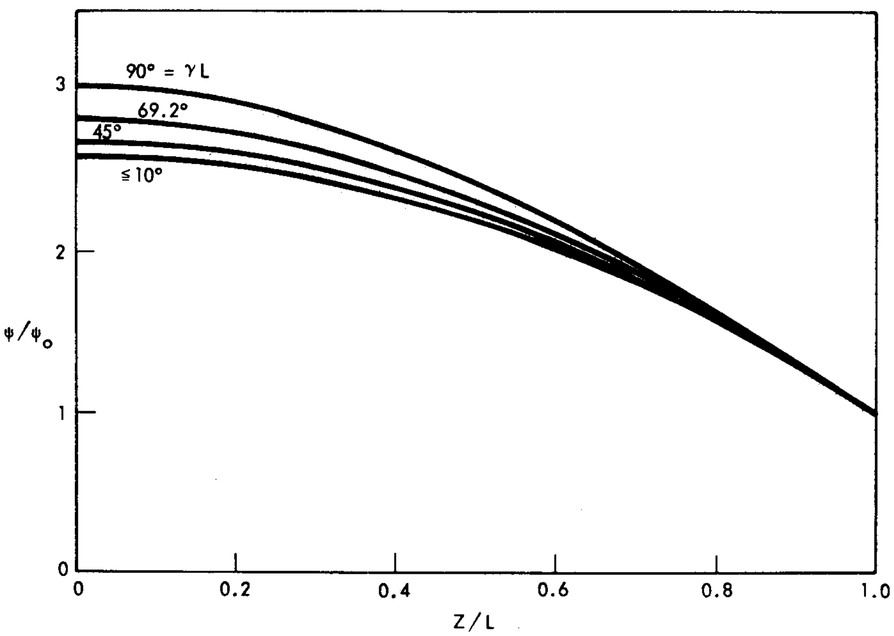

FIG. 3. Axial variation of wave amplitude between the midplane and the sheath edge, for the case of collision-limited electron motion in a plasma 20 mean-free-paths long. The curve $\gamma L=69.2°$ corresponds to the approximate solution of Eq. (72), but the solution is insensitive to the assumed value of γL.

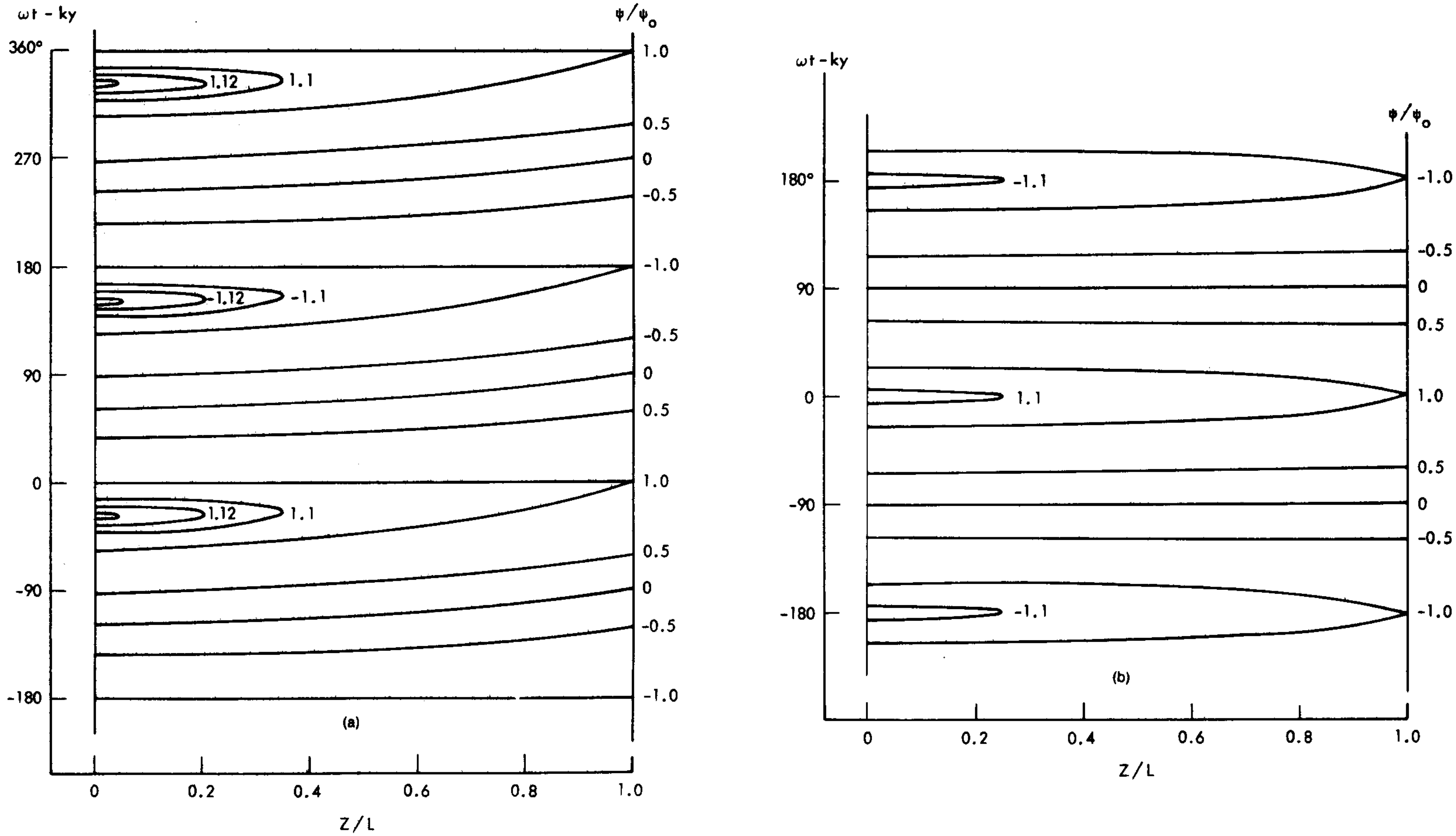

FIG. 4. Contours of constant wave amplitude in the y-z plane for (a) an inertia-dominated wave propagating in the y direction with $\gamma L = \frac{1}{4}\pi$ and $R' = 0.2$, and (b) an ordinary standing wave, corresponding to a collision-dominated wave with $\gamma L = \frac{1}{4}\pi$ and $R = 0.2$.

In general, other values of γL are possible; but as in the resistive case, the factor

$$\frac{R'}{\gamma L \tan\gamma L}\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right)$$

is insensitive to γL and depends only on R'.

The question arises as to what is the effective value of k_z in this mode. In the resistive case, where the mode is a standing wave in the axial direction [Eq. (72)], one cannot go far wrong by taking $k_z = \pm\gamma$, where γ is the solution of Eq. (73). The other amplitude profiles of Fig. 3 are so similar that the E_z seen by the electrons is about the same in all cases, and the resonant frequency ω can be found from the wave dispersion relation using this value of k_z. In the inertial case, however, the solution of Eq. (80) is a superposition of two modes 90° out of phase: one with $k_z=0$, and one with $k_z=\pi/2L$. These may have different eigenfrequencies; yet we have assumed, because of the small amplitude, that only one frequency is present at a time. The difficulty is that the sheath matching conditions have caused linear mode coupling, and one should use the frequency of the coupled modes rather than the frequencies calculated for each individual mode in an infinite plasma. In practice, there is no problem as long as one is dealing with waves of small parallel dispersion (ω a weak function of k_z), such as drift waves, lower hybrid waves, or two-ion hybrid waves. The finite bandwidth of the system, as set by some damping or lifetime consideration, should be large enough to encompass the ω's of both the $k_z=0$ and the $k_z=\pi/2L$ modes.

C. Summary of results

We have found the following symmetric solutions where ψ has an amplitude ψ_0 at the sheath edges $z=\pm L$:

collisional limit:

$$\frac{\psi}{\psi_0}=\left[1+\frac{R}{\gamma L\tan\gamma L}\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right)\right]\cos\phi\,; \qquad (82)$$

collisionless limit:

$$\frac{\psi}{\psi_0}=\cos\phi-\frac{R'}{\gamma L\tan\gamma L}\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right)\sin\phi\,; \qquad (83)$$

where

$$R\equiv a\nu_{ei}L,\quad R'\equiv a\omega L,\quad \phi\equiv\omega t-ky\,, \qquad (84)$$

and

$$a\equiv(v_B+j_t/n_0)/(KT_e/m) \qquad (85a)$$

for ion sheaths, and

$$a\equiv(m/2\pi KT_e)^{1/2} \qquad (85b)$$

for electron sheaths. The solution with both resistivity and inertia is given by Eqs. (62), (66), and (67). Antisymmetric solutions will be given in Sec. VI.

The collisional limit will be valid when $\nu_{ei}\gg\omega$, and vice versa. For Q-machine plasmas with $KT\simeq 0.2$ eV and $B\simeq 4$ kG, collisions dominate above $n\simeq 3\times10^{9}$ cm^{-3} for waves with ω less than the potassium cyclotron frequency. For discharge plasmas with $KT\simeq 3$ eV and $B\simeq 4$ kG, inertia dominates below $n\simeq 2\times10^{11}$ cm^{-3} for waves with ω above the argon cyclotron frequency. For fusion plasmas the collisionless limit is almost always valid.

V. APPLICATION TO DRIFT WAVES

The case of resistive drift waves in a Q machine is particularly simple because the ions, electrons, and end plates are all at the same temperature, and no potentials are applied. The sheath conditions can then be written exactly[6] without resorting to an approximate treatment of the presheath. We shall make use of the known dispersion relation to see in detail what would be involved if one were to match ion fluxes as well as electron fluxes at the end plates.

In the collisional limit, the simplest nontrivial symmetric solution satisfying the end plate conditions is given by Eq. (69)

$$\psi=\psi_0\cos(\omega t-ky)\left[1+\frac{a\nu_{ei}L}{\gamma L\tan\gamma L}\left(\frac{\cos\gamma z}{\cos\gamma L}-1\right)\right], \qquad (86)$$

where γL is arbitrary. Although $\psi(z)$ is not sensitive to γL (cf. Fig. 3), we wish to see whether the use of ion matching conditions will specify the exact value of γL, at least in principle. For definiteness consider the case of electron-rich sheaths ($\phi<0$). Plasma electrons then reach the end plate at their random velocity, and emitted electrons are repelled by a Coulomb barrier. The value of a is given by Eq. (60). As for the ions, those striking the end plate reach it only after overcoming a Coulomb barrier; there they recombine into neutrals and have a finite probability p of being contact ionized again. To supply the unspecified losses in the plasma (by radial diffusion or volume recombination), a source of neutral atoms is directed at the end plates, and the neutrals are contact ionized to create an ion flux $\mathbf{j}_0$. This flux and the reionized flux flow down a potential hill into the plasma. The net ion flux into the sheaths in equilibrium is therefore[6]:

$$\Gamma_{i0}=(1-p)n_0v_{ir}\exp(\chi_0)-j_0\,, \qquad (87)$$

where

$$v_{ir}\equiv(KT_i/2\pi M)^{1/2}\,. \qquad (88)$$

In the presence of a wave, the values of n_0 and χ_0 at the sheath edge change to n_0+n_1 and $\chi_0+\chi_1$ while j_0 remains fixed. The perturbed ion flux is then

$$\Gamma_{i1}=j_0(\chi+\nu)\,. \qquad (89)$$

Ion motion in a drift wave is mostly perpendicular to B_0; nonetheless, the flux Γ_{i1} can be matched only to the small ion flux in the z direction. This is specified by the $\mathbf{z}$ components of the perturbed ion and electron equations of motion (for $v_{z0}=0$):

$$Mn_0\frac{\partial v_{iz}}{\partial t}=-en_0\frac{\partial\phi_1}{\partial z}-KT_i\frac{\partial n_1}{\partial z}+mn_0\nu_{ei}(v_{ez}-v_{iz})\,, \qquad (90)$$

$$mn_0\frac{\partial v_{ez}}{\partial t}=en_0\frac{\partial\phi_1}{\partial z}-KT_e\frac{\partial n_1}{\partial z}-mn_0\nu_{ei}(v_{ez}-v_{iz})\simeq 0\,, \qquad (91)$$

where the ion collisional term has been set equal and opposite to the electron term by conservation of momentum. Addition yields

$$Mn_0 \frac{\partial v_{iz}}{\partial t} = -(KT_i + KT_e) \frac{\partial n_1}{\partial z}. \tag{92}$$

The perturbed ion flux in the z direction is then given by

$$\dot{\Gamma}_{iz} = n_0 \dot{v}_{iz} = -n_0 v_s^2 \frac{\partial \nu}{\partial z}, \tag{93}$$

where $\nu \equiv n_1/n_0$ and

$$v_s^2 \equiv (KT_i + KT_e)/M = 2KT/M; \tag{94}$$

at $z = \pm L$, Γ_{iz} must match $\pm\Gamma_{i1}$, given by Eq. (89)

$$n_0 v_s^2 \frac{\partial \nu}{\partial z} = \mp j_0 \frac{\partial}{\partial t}(\chi + \nu). \tag{95}$$

Since χ and ν no longer occur in the combination $\psi = \chi - \nu$, we must find ν and χ separately. The electron equations of motion and continuity in a linear density gradient give the following relation[10] between ν and χ:

$$\chi = \Lambda\nu \equiv \frac{\omega + ib\sigma_\parallel}{\omega_* + ib\sigma_\parallel}\nu, \tag{96}$$

where ω_* is the electron diamagnetic drift frequency, b is a finite Larmor radius parameter, and $\sigma_\parallel$ is a parallel diffusion frequency proportional to k_z^2/ν_{ei}. (The same result can be found in Ref. 5, but the notation in this earlier work was not adopted by subsequent authors.) In terms of ψ, we have

$$\nu = (1 - \Lambda)^{-1}\psi, \quad \chi + \nu = [(1 + \Lambda)/(1 - \Lambda)]\psi. \tag{97}$$

Equation (95) becomes

$$n_0 v_s^2 \frac{\partial \psi}{\partial z} \pm j_0(1 + \Lambda)\frac{\partial \psi}{\partial t} = 0 \quad \text{at } z = \pm L. \tag{98}$$

This has the same form as Eq. (38) for inertia-limited motion and cannot be satisfied at both ends by the standing wave of Eq. (86). Ion matching, therefore, requires an axial phase shift. This is because electron drag is unlikely to dominate over the inertia term for the ions. The effect of ion motion is not simply to determine the value of γL in Eq. (86); in fact, since Λ depends on ω and k_z^2, Eq. (98) is a rather complicated differential equation in t and z. An exact analysis of ion matching is of academic interest only, and we have not attempted it.

In experiments, drift waves have been found[10] to be stabilized by ion viscosity if their intrinsic growth rate is too low. Since the growth rate depends on $|k_z|$, the effective value of $|k_z|$ as given by the sheath conditions determines whether or not a mode of given k_y will be stable. A numerical example was given in Sec. III. The tendency for magnetized plasmas bounded by ion sheaths to be noisy can be explained by the fact that small $|k_z|$ values corresponding to the maximum growth rate for drift waves are then allowed.

VI. APPLICATION TO TWO-ION HYBRID WAVES

A. Dispersion relation

Interest in the two-ion hybrid resonance[4] has arisen in connection with its use in isotope separation[11] and heating of DT plasmas.[12] The resonance exists only when $k_z = 0$; as $|k_z/k_y|$ is increased, the two-ion hybrid mode turns first into an electrostatic ion cyclotron wave and then into an ion acoustic wave. To see this, one can consider electrostatic waves of the form $\exp[i(k_y y - \omega t)]$ in a uniform, magnetized plasma consisting of electrons of temperature T_e and two species of ions of zero temperature. The linearized electron and ion equations of motion and continuity and Poisson's equation then yield the following dispersion relation for $\omega^2 \ll \omega_c^2$ and $k^2\lambda_D^2 \ll 1$:

$$\left[1 + \left(\frac{k_y^2 v_{th}^2}{\omega_c^2} - \frac{k_z^2 v_{th}^2}{\omega^2}\right)^{-1}\right]\left[\alpha_1 c_{s1}^2\left(\frac{k_y^2}{\omega^2 - \Omega_1^2} + \frac{k_z^2}{\omega^2}\right) + \alpha_2 c_{s2}^2\left(\frac{k_y^2}{\omega^2 - \Omega_2^2} + \frac{k_z^2}{\omega^2}\right)\right] = 1. \tag{99}$$

Here

$$v_{th}^2 \equiv KT_e/m, \quad \omega_c \equiv eB_0/m, \quad \Omega_j \equiv Z_j eB_0/M_j$$
$$c_{sj}^2 \equiv Z_j KT_e/M_j, \tag{100}$$

and

$$\alpha_j \equiv n_{0j}/n_{0e} \qquad (\alpha_1 + \alpha_2 = 1)$$

is the partial density of the ion species j. If $k_z = 0$, Eq. (99) reduces to

$$\frac{\alpha_1\Omega_1}{\omega^2 - \Omega_1^2} + \frac{\alpha_2\Omega_2}{\omega^2 - \Omega_2^2} = \frac{1}{\omega_c}. \tag{101}$$

The three terms are, respectively, proportional to the charge densities of species 1, species 2, and electrons. When $\alpha_2 = 0$, the charge fluctuation of species 1 is canceled by electron polarization drift, resulting in the lower hybrid frequency

$$\omega^2 = \Omega_1^2 + \omega_c\Omega_1 \simeq \omega_c\Omega_1. \tag{102}$$

When $\alpha_2 \neq 0$, it is possible for the two-ion space charges to cancel each other at a lower frequency, where the electron drift would be negligible. Setting the right-hand side of Eq. (101) to zero yields the two-ion hybrid frequency.

$$\omega^2 = \Omega_1\Omega_2 \frac{\alpha_1\Omega_2 + \alpha_2\Omega_1}{\alpha_1\Omega_1 + \alpha_2\Omega_2}. \tag{103}$$

We shall let α_2 designate the "minor" species and α_1 the "major" species, so that $\alpha_2 < \alpha_1$. If $\alpha_2 \ll \alpha_1$, Eq. (103) shows that ω lies near Ω_2, the minor species cyclotron resonance, as is clear from the second term of Eq. (101).

At very large $|k_z|$, the k_z terms in Eq. (99) dominate, and one obtains a simple ion acoustic wave in a two-species plasma. Now consider small but finite $|k_z|$ and frequencies near the ion cyclotron frequencies, so that the k_z can be neglected in the ion terms. In the electron term (first bracket) of Eq. (99), the term $k_y^2 v_{th}^2/\omega_c^2$ can be neglected for $|k_z/k_y| \gg m/M$ and $\omega \simeq \Omega$. If, furthermore, $\omega^2/k_z^2 \ll v_{th}^2$, the electron term becomes 1; this is the limit of Boltzmann electrons. In this range of k_z, Eq. (99) becomes

$$\alpha_1 k_y^2 c_{s1}^2(\omega^2 - \Omega_2^2) + \alpha_2 k_y^2 c_{s2}^2(\omega^2 - \Omega_1^2) = (\omega^2 - \Omega_1^2)(\omega^2 - \Omega_2^2). \tag{104}$$

For $\alpha_2 \ll \alpha_2$, the two roots of this biquadratic are approximately

$$\omega^2 = \Omega_1^2 + \alpha_1 k_y^2 c_{s1}^2 \left(1 + \frac{\alpha_2 k_y^2 c_{s2}^2}{\alpha_1 k_y^2 c_{s1}^2 + \Omega_1^2 - \Omega_2^2}\right) \tag{105}$$

and

$$\omega^2 = \Omega_2^2 + \alpha_2 k_y^2 c_{s2}^2 \frac{\Omega_1^2 - \Omega_2^2}{\alpha_1 k_y^2 c_{s1}^2 + \Omega_1^2 - \Omega_2^2} . \tag{106}$$

The first of these is an electrostatic ion cyclotron wave in the major species, with a small correction due to the minor species. The second is a minor-species cyclotron wave whose frequency shift from Ω_2 is always toward Ω_1. In isotope separation, where $\alpha_2 \ll \alpha_1$, both Eq. (103) and Eq. (106) are close to Ω_2, but the frequency shifts are different, and the two waves lie on different branches of the full dispersion relation (99), shown in Fig. 5. The transition occurs near $k_z^2/k_y^2 \simeq m/M_2$. The branch which is excited depends on the spatial structure of the exciter and the sheath matching conditions on the end plates.

B. Antisymmetric solutions

If the excitation mechanism is symmetric in the z direction, one would expect that the solution given in Sec. IV would be the longest parallel wavelength mode obtainable. If a charge imbalance should arise in the plasma, due to, say, a density inhomogeneity, electrons could stream in from the end plates to cancel the charge imbalance. For this purpose, one would want to use electron-emitting end plates, since ion sheaths are basically insulators. Even then, the electron current available from the end plates may be insufficient: In the inertia-dominated case we saw that $k_z = 0$ component is almost always dominant. To increase the space-charge canceling capability of the end plates one can excite higher-order longitudinal modes using asymmetric or periodic drivers. In this section we give possible solutions which are antisymmetric about the midplane.

1. Resistive limit

Imagine a wave excited by external coils or electrodes that are driven out-of-phase on either side of the midplane in such a way that $\psi = 0$ at $z = 0$ and $\psi = \pm\psi_0 \cos(\omega t - ky)$ at the sheath edges. From Eq. (58) for $\omega/\nu_{ei} - 0$, we find that the wave amplitude ψ must satisfy

$$\frac{\partial \psi}{\partial z} \pm a\nu_{ei}\psi = 0 \quad \text{at } z = \pm L . \tag{107}$$

Since

$$\psi(\pm L) = \pm\psi_0 \cos(\omega t - ky) , \tag{108}$$

we have

$$\frac{\partial \psi}{\partial z} + a\nu_{ei}\psi_0 \cos(\omega t - ky) = 0 \quad \text{at } z = \pm L . \tag{109}$$

The simplest solution satisfying this boundary condition is

$$\psi = \psi_0 \csc\gamma L \sin\gamma z \cos(\omega t - ky) , \tag{110}$$

where γL is the solution of

$$\gamma L \operatorname{ctn}\gamma L = - a\nu_{ei} L \equiv -R . \tag{111}$$

This is the antisymmetric equivalent of Eqs. (72) and (73). Equation (111) has no solution for small γL; the smallest parallel wavenumber possible is $\gamma \simeq \pi/2L$. For small R, expansion around $\gamma L \simeq \frac{1}{2}\pi$ yields

$$\gamma L \simeq \pi/2 + 2R/\pi . \tag{112}$$

Higher modes are possible with $\gamma L \simeq \frac{3}{2}\pi, \frac{5}{2}\pi$, etc.

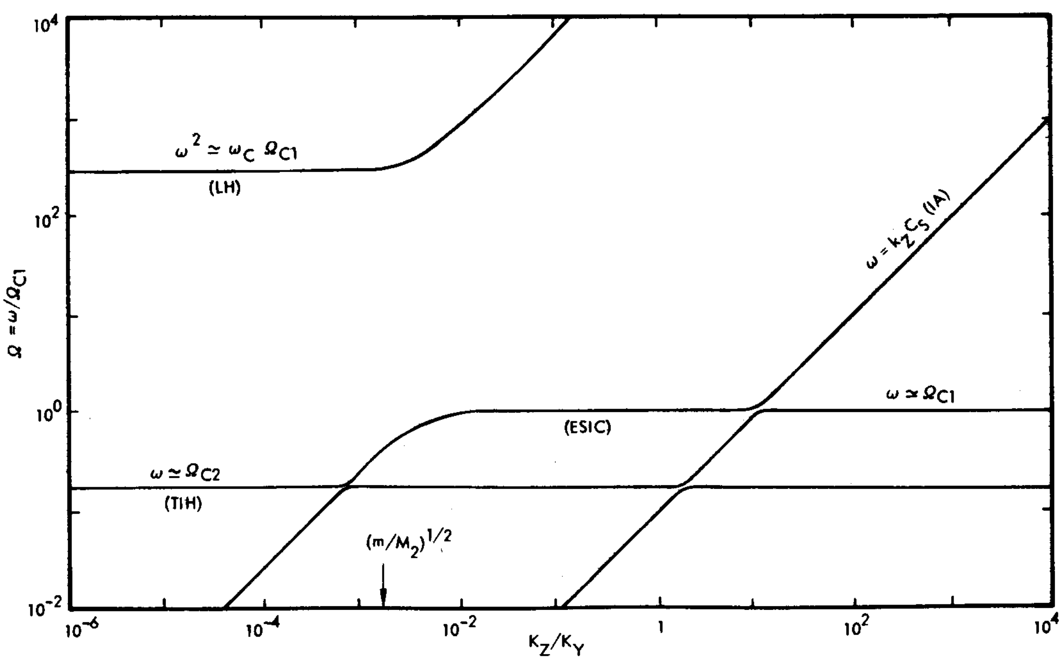

FIG. 5. Dispersion curves for electrostatic waves in a two-species plasma, drawn for a case where $n_2 \ll n_1$ and $M_2 > M_1$.

2. *Inertial limit*

For $\nu_{ei} \ll \omega$, Eq. (58) requires

$$\frac{\partial \psi}{\partial z} \pm a \frac{\partial \psi}{\partial t} = 0 \quad \text{at } z = \pm L . \tag{113}$$

An antisymmetric solution of the form of Eq. (108) must then satisfy the boundary condition

$$\frac{\partial \psi}{\partial z} = a\omega\psi_0 \sin(\omega t - ky) \quad \text{at } z = \pm L . \tag{114}$$

Surprisingly, this simple condition cannot be met by solutions proportional to $\sin\gamma z$, even if an axial phase shift is allowed. The simplest solution we have been able to find includes a term linear in z:

$$\psi = \psi_0[\sin\gamma z \cos(\omega t - ky) - a\omega I(\sin\gamma z - z/L) \times \sin(\omega t - ky)] , \tag{115}$$

where $\gamma = \pi/2L$. In terms of a phase shift δ, this can be written

$$\psi = \psi_0 \sec\delta \sin\gamma z \cos[\omega t - ky + \delta(z)] , \tag{116}$$

where $\gamma = \pi/2L$ and

$$\tan\delta = a\omega L\left(1 - \frac{2}{\pi}\frac{\gamma z}{\sin\gamma z}\right) . \tag{117}$$

The appearance of this solution is shown in Fig. 6, computed for $R' \equiv a\omega L = 0.2$. In this antisymmetric mode, $|\partial\psi/\partial z|$ is small near the sheath edge at all times, either because the amplitude is small or because the sine wave is near an extremum. This means that the sheath is not required to supply a large electron current. Most of the axial electron current flows from one-half of the machine to the other without having to leave the machine.

C. End plate excitation

A simple way to excite waves is to apply a sinusoidal voltage to one or both end plates. In this section we give the boundary conditions for this mechanism and exhibit the simplest modes that satisfy them. No attempt at generality is made.

1. *Sheath conditions with electrostatic drive*

Let a potential $\phi_p(t)$ be applied to an end plate. The

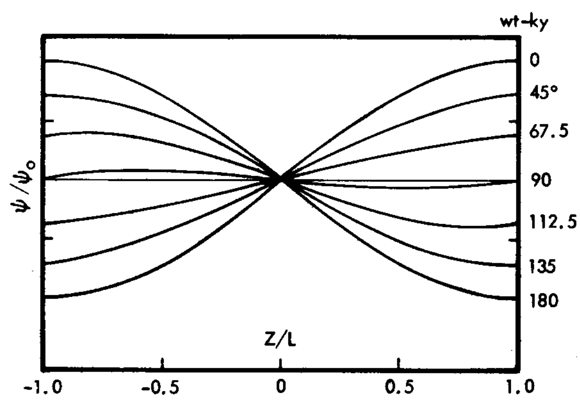

FIG. 6. Axial variation of wave amplitude for the simplest antisymmetrically excited mode in the inertia-dominated case.

electron fluxes into the end plate for ion-rich (+) and electron-rich (−) sheaths are then given by the following modification of Eqs. (3) and (6):

$$\Gamma^+ = nv_r \exp[-e(\phi - \phi_p)/KT_e] - j_t , \tag{118}$$

$$\Gamma^- = nv_r - j_t \exp[e(\phi - \phi_p)/KT_c] . \tag{119}$$

Treating ϕ_p as a perturbation, we obtain for the first-order fluxes

$$\Gamma_1^+ = (n_0 v_B + j_t)(\nu - \chi + \chi_p) \tag{120}$$

$$\Gamma_1^- = n_0 v_r[\nu - (T_e/T_c)(\chi - \chi_p)] , \tag{121}$$

where $\chi_p \equiv e\phi_p/kT_e$. Matching to the electron current in the wave motion yields (for $T_e = T_c$) the following matching condition, the analog of Eq. (58):

$$\frac{\partial \psi}{\partial z} \pm a\left(\frac{\partial \psi}{\partial t} + \frac{\partial \chi_p}{\partial t}\right) \pm a\nu_{ei}(\psi + \chi_p) = 0 . \tag{122}$$

2. *Resistive limit*

In this limit we require

$$\frac{\partial \psi}{\partial z} \pm a\nu_{ei}(\psi + \chi_p) = 0 \quad \text{at } z = \pm L . \tag{123}$$

a. Symmetric drive. At both ends, let

$$\chi_p = \chi_m \cos\phi , \tag{124}$$

where $\phi \equiv \omega t - ky$. Equations (123) and (124) are satisfied nontrivially by

$$\psi = \frac{a\nu_{ei}}{\gamma \tan\gamma L - a\nu_{ei}} \frac{\cos\gamma z}{\cos\gamma L} \chi_m \cos\phi . \tag{125}$$

This is resonant for $\gamma L \tan\gamma L = R \equiv a\nu_{ei} L$, the same as Eq. (73). When γ has the right value for this normal mode, the excitation amplitude χ_m can be infinitesimally small, since wave damping has been neglected. In practice, damping will determine the χ_m required to excite a given amplitude ψ. Here, we usually have $R \ll 1$ and $\gamma L \ll 1$.

b. Antisymmetric drive. If the end plates are driven 180° out-of-phase, we have

$$\chi_p = \pm\chi_m \cos\phi \quad \text{at } z = \pm L . \tag{126}$$

A possible solution of Eqs. (123) and (126) is

$$\psi = -\frac{a\nu_{ei}}{\gamma \operatorname{ctn}\gamma L + a\nu_{ei}} \frac{\sin\gamma z}{\sin\gamma L} \chi_m \cos\phi . \tag{127}$$

This is resonant for $\gamma L \operatorname{ctn}\gamma L = -R \equiv -a\nu_{ei} L$, the same as Eq. (111). Here, the normal mode has $\gamma \simeq \pi/2L$, and $\psi/\chi_m \to \infty$ for this wavenumber.

c. Asymmetric drive. In single-ended drive, let the end plate at $z = -L$ be grounded while that at $z = L$ be driven. We then have

$$\chi_p = \chi_m \cos\phi \quad \text{at } z = L , \qquad \chi_p = 0 \quad \text{at } z = -L . \tag{128}$$

There are now several "simplest" solutions satisfying Eqs. (123) and (128). There is no solution with a $\cos\gamma z$ dependence, but we can expect spatial variations going as $\sin\gamma z$ or sine or cosine of $(z \pm L)$. With the abbreviation

$$\epsilon \equiv a\nu_{ei}/\gamma , \tag{129}$$

these solutions are

$$\psi_1 = -\chi_m \cos\phi \left(1 + \epsilon \frac{\cos\gamma(z-L) - 1}{\sin 2\gamma L + \epsilon(1-\cos 2\gamma L)}\right), \tag{130}$$

$$\psi_2 = -\chi_m \cos\phi \left(\epsilon \frac{1-\cos\gamma(z+L)}{\sin 2\gamma L + \epsilon(1-\cos 2\gamma L)}\right), \tag{131}$$

$$\psi_3 = -\chi_m \cos\phi \left(1 + \frac{\epsilon \sin\gamma(z-L) - 1}{\epsilon \sin 2\gamma L + 1 + \cos 2\gamma L}\right), \tag{132}$$

$$\psi_4 = -\chi_m \cos\phi \left(\frac{\epsilon \sin\gamma(z+L) + 1}{\epsilon \sin 2\gamma L + 1 + \cos 2\gamma L}\right), \tag{133}$$

$$\psi_5 = -\frac{1}{2}\chi_m \cos\phi \left(1 + \frac{\epsilon \sin\gamma z}{\cos\gamma L + \epsilon \sin\gamma L}\right). \tag{134}$$

The first two of these resonate for

$$\gamma \sin 2\gamma L = -a\nu_{ei}(1-\cos 2\gamma L) , \tag{135}$$

while ψ_3 and ψ_4 resonate for

$$\gamma(1+\cos 2\gamma L) = -a\nu_{ei} \sin 2\gamma L , \tag{136}$$

which turns out to be identical to Eq. (135). The final solution ψ_5 resonates for

$$\gamma \cos\gamma L = -a\nu_{ei} \sin\gamma L , \tag{137}$$

which is the same as Eq. (111). All of these normal modes have $\gamma \simeq \pi/2L$, or $\lambda_z \simeq 4L$; the value of γL for small $R = a\nu_{ei} L$ is given in Eq. (112).

3. *Inertial limit*

In this limit, Eq. (122) becomes

$$\frac{\partial\psi}{\partial z} \pm a\left(\frac{\partial\psi}{\partial t} + \frac{\partial\chi_p}{\partial t}\right) = 0 \quad \text{at } z = \pm L . \tag{138}$$

a. Symmetric drive. A mode satisfying Eqs. (124) and (138) is

$$\psi = -\chi_m \sec\gamma L \left(\frac{\gamma \sin\gamma L}{a\omega} \sin\phi + \cos\gamma z \cos\phi\right), \tag{139}$$

which resonates for $\gamma L = \frac{1}{2}\pi$. Since $\chi_p \propto \cos\phi$, this means that sheath edge potential oscillates 90° out-of-phase with the end plate drive.

b. Antisymmetric drive. The following mode satisfies Eqs. (126) and (138) in this case:

$$\psi = \chi_m a\omega L \frac{\gamma L \cos\gamma L(\sin\gamma z + z/L)\sin\phi - a\omega L(1+\sin\gamma L)\sin\gamma z \cos\phi}{(\gamma L \cos\gamma L)(1+\gamma L\cos\gamma L) + (a\omega L)^2 \sin\gamma L(1+\sin\gamma L)} , \tag{140}$$

The denominator vanishes, for small $a\omega L$, at

$$\gamma L \simeq \pi/2 + (4/\pi)(a\omega L)^2 , \tag{141}$$

which defines the wavelength of the normal mode.

c. Asymmetric drive. With single-ended excitation, Eqs. (128) and (138) must be satisfied. Substituting $\psi(z)$'s of the form

$$(A\cos\phi + B\sin\phi + C)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\gamma(z \pm L) + D\cos\phi + E\sin\phi ,$$

as well as $\sin\gamma z$ and $\sin\gamma z + (z/L)$ dependences, we have found the following solutions:

$$\psi_1 = -a\omega\chi_m[\cos\gamma(z+L) - 1]\frac{a\omega(\cos 2\gamma L - 1)\cos\phi + \gamma \sin 2\gamma L \sin\phi}{[a\omega(\cos 2\gamma L - 1)]^2 + (\gamma \sin 2\gamma L)^2} , \tag{142}$$

$$\psi_2 = -\chi_m \cos\phi + a\omega\chi_m[\cos\gamma(z-L) - 1]\frac{a\omega(\cos 2\gamma L - 1)\cos\phi + \gamma \sin 2\gamma L \sin\phi}{[a\omega(\cos 2\gamma L - 1)]^2 + (\gamma \sin 2\gamma L)^2} , \tag{143}$$

$$\psi_3 = \frac{-\chi_m}{[\gamma(\cos 2\gamma L + 1)]^2 + (a\omega \sin 2\gamma L)^2}$$

$$\times\{[\gamma^2(\cos 2\gamma L + 1) + a^2\omega^2 \sin 2\gamma L \sin\gamma(z+L)]\cos\phi + \gamma a\omega[\sin 2\gamma L - (\cos 2\gamma L + 1)\sin\gamma(z+L)]\sin\phi\} , \tag{144}$$

$$\psi_4 = -\chi_m\cos\phi + \frac{\chi_m}{[\gamma(\cos 2\gamma L + 1)]^2 + [a\omega \sin 2\gamma L]^2}$$

$$\times\{[\gamma^2(\cos 2\gamma L + 1) - a^2\omega^2 \sin 2\gamma L \sin\gamma(z-L)]\cos\phi + \gamma a\omega[\sin 2\gamma L - (\cos 2\gamma L + 1)\sin\gamma(z-L)\sin\phi\} , \tag{145}$$

$$\psi_5 = -\frac{1}{2}\chi_m\cos\phi + \frac{1}{2}\chi_m \frac{a\omega}{\gamma} \sec\gamma L \frac{\sin\phi - (a\omega/\gamma)\tan\gamma L\cos\phi}{1 + (a\omega/\gamma)^2\tan^2\gamma L} \sin\gamma z , \tag{146}$$

$$\psi_6 = \frac{1}{2}\chi_m\cos\phi\left(\frac{(\tan\gamma L + \gamma L)(\gamma L \operatorname{ctn}\gamma L/a\omega L)\sin\gamma z}{(\gamma^2L^2/a\omega L)\cos\gamma L + a\omega L\tan\gamma L\sin\gamma L} - \frac{\sin\gamma z}{\sin\gamma L} - \frac{z}{L}\right)$$

$$+\frac{1}{2}\chi_m\sin\phi\left(\frac{\tan\gamma L + \gamma L}{(\gamma^2L^2/a\omega L)\cos\gamma L + a\omega L\tan\gamma L\sin\gamma L}\right)\sin\gamma z . \tag{147}$$

The first two solutions have denominators which vanish at $2\gamma L = 0,\ 2\pi,\ 4\pi, \ldots$, so that the lowest nontrivial mode has $\gamma = \pi/L$. The solutions ψ_3 and ψ_4 have denominators which vanish at $2\gamma L = \pi,\ 3\pi, \ldots$, so that the lowest mode has $\gamma = \pi/2L$. The last two solutions, ψ_5 and ψ_6, do not resonate for any value of γ. These appear to be driven oscillations that have finite amplitude for finite χ_m.

D. Double sheaths

We finally discuss the case of emitting end plates which are colder than the plasma electrons, as in any discharge plasma which is not a Q machine. The emitted electrons are then at a temperature T_c, while the plasma electrons are at a temperature $T_e > T_c$. The most obvious adverse effect of this is to require the retention of the factor T_e/T_c in Eq. (15). The first-order quantities ν and χ then no longer occur only in the combination $\psi \equiv \nu - \chi$, and a relation must be found between ν and χ. To do this, one has to specify the wave, or at least the approximations used to derive the wave, as was done in Eq. (96). More troublesome than this is the possibility of double-sheath formation. It is not our intent to analyze this effect in this paper; we shall only make a few qualitative remarks.

If the plasma is very positive, as with a cold plate, and one begins to emit a few electrons, the ion sheath is not sensitive to the emitter temperature, since the electrons are accelerated into the plasma and leave the sheath with an energy much larger than their initial energy. As emission increases and the plasma potential falls toward zero, a space-charge problem develops. Plasma electrons leave in great number because the Coulomb barrier has been depressed. To replace these electrons with emitted electrons requires a large density of the latter, since they move at slow speeds. The large negative space charge causes a potential dip next to the end plate surface. When the plasma potential is negative relative to the end plate, there will be a potential dip that is further negative, caused by the pile-up of emitted electrons. The sheath is nonmonotonic and presents a Coulomb barrier to electrons coming from both sides. Ions are trapped in this potential well and tend to obliterate it. Whether a double sheath in fact forms in steady state depends on the ion lifetime. In the oscillatory state considered here, the ions may not have time to move, and double-sheath formation may occur. In general, one may expect the space-charge effects caused by $T_c < T_e$ to decrease the flow of electrons into the plasma and make the sheath a better insulator than in the isothermal case.

ACKNOWLEDGMENT

This work was supported by the U. S. Department of Energy under Contract No. EN 77CO31461.

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