RAMAN BACKSCATTER BELOW
THE ABSOLUTE THRESHOLD

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PPG-817

August, 1984

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Consider pump, backscattered, and plasma waves of the form

$$\tilde{\underline{E}}_o = \hat{y}\, E_o \cos\,(k_o x - \omega_o t) \qquad (1)$$

$$\tilde{\underline{E}}_z = \hat{y}\, E_2 \cos\,(k_2 x - \omega_2 t) \qquad (2)$$

$$\tilde{n}_1 = n_1 \sin\,(k_1 x - \omega_1 t)\ . \qquad (3)$$

Let the SRS reflectivity be small enough that E_o can be considered constant; let the amplitudes $E_2(x,t)$ and $n_1(x,t)$ vary slowly compared to the sinusoidal factors; and let (ω_o,k_o), (ω_2,k_2) and (ω_1,k_1) obey both the linear dispersion relations and the phase matching conditions $\omega_o = \omega_1 + \omega_2$, $k_o = k_1 + k_2$. Standard analysis of the parametric instability then give these coupled equations for E_2 and n_1:

$$\dot{E}_2 + \frac{c^2 k_2}{\omega_2} E_2' + \gamma_2 E_2 = \alpha n_1 \qquad (4)$$

$$\dot{n}_1 + \frac{3v_e^2 k_1}{\omega_1} n_1' + \gamma_1 n_1 = \beta E_2, \qquad (5)$$

where $$\alpha = \frac{\omega_p^2 E_o}{4\omega_o n_o} \qquad (6)$$

and $$\beta = \frac{\omega_p^2 k_1^2}{2m\omega_o \omega_1 \omega_2} \frac{E_o}{8\pi}\ . \qquad (7)$$

In an infinite plasma ($n_1' = E_2' = 0$) where the damping rates γ_1 and γ_2 are much smaller than the growth rate γ_o of E_2 and n_1, Eqs (4) and (5) give

$$\gamma_o^2 = \alpha\beta = \frac{v_o^2}{16} \frac{k_1^2 \omega_p^2}{\omega_1 \omega_2} , \tag{8}$$

where we have used Eqs. (6) and (7) and $v_o = eE_o/m\omega_o$. The homogeneous threshold is given by $\dot{n}_1 = \dot{E}_2 = n_1' = E_2' = 0$, whereupon

$$\alpha\beta = \gamma_1\gamma_2 \equiv \gamma_h^2 . \tag{9}$$

We now choose $k_o < 0$, $k_2 > 0$, and $k_1 < 0$ so that the qualitative spatial behavior of E_2 and n_1 in a finite, homogeneous plasma is given by one of the following diagrams:

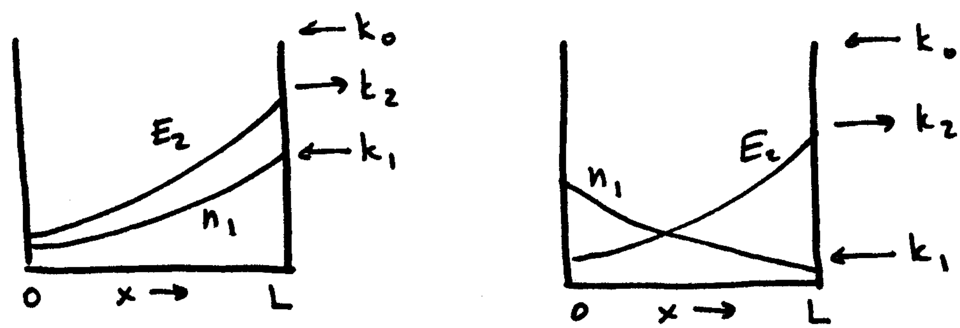

Fig. 1

We first look for steady-state solutions of Eqs. (4) and (5). Since the group velocities are

$$V_1 = 3v_e^2 \, |k_1|/\omega_1 \qquad \text{and} \qquad V_2 = c^2|k_2|/\omega_2 \tag{10}$$

and k_1 is negative, the coupled equations become

$$V_2E_2' + \gamma_2E_2 = \alpha n_1 \tag{11}$$

$$-V_1n_1' + \gamma_1n_1 = \beta E_2 \, . \tag{12}$$

In cool plasmas, electron plasma waves have appreciable γ_1 due to electron-ion collisions and rather small V_1, while the reflected light wave has large $V_2 \simeq c$ and small γ_2 for $n << n_c$. It is then customary to neglect the γ_2 and V_1 terms to obtain

$$V_2E_2' = \alpha n_1 \; , \qquad \gamma_1 n_1 = \beta E_2$$

$$E_2' = \frac{\alpha\beta}{\gamma_1 V_2} \; E_2 \quad . \tag{13}$$

Using Eq. (8), we arrive at the familiar spatial growth rate

$$\kappa \simeq \gamma_o^2/c\gamma_1 \quad . \tag{14}$$

In this case $n_1 \alpha E_2$, so that the plasma wave peaks at the upstream end, as in Fig. 1a.

In making this approximation, however, we have lost a second root which corresponds to large n_1', such that V_1n_1' is non-negligible even if V_1 is small.

Thus we treat Eqs. (11) and (12) in full and write them as

$$E_2' + \bar{\gamma}_2 E_2 = \bar{\alpha} n_1 \tag{15}$$

$$-n_1' + \bar{\gamma}_1 n_1 = \bar{\beta} E_2 \quad , \tag{16}$$

where

$$\bar{\gamma}_1 = \gamma_1/V_1, \quad \bar{\gamma}_2 = \gamma_2/V_2. \tag{17}$$

$$\bar{\alpha} = \alpha/V_1 \quad , \quad \bar{\beta} = \beta/V_2 \quad . \tag{18}$$

Differentiating Eqs. (15) and (16) and combining, we obtain

$$n_1'' + (\bar{\gamma}_2 - \bar{\gamma}_1) n_1' + (\bar{\gamma}_o^2 - \bar{\gamma}_1 \bar{\gamma}_2) n_1 = 0 \tag{19}$$

$$E_2'' + (\bar{\gamma}_2 - \bar{\gamma}_1) E_2' + (\bar{\gamma}_o^2 - \bar{\gamma}_1 \bar{\gamma}_2) E_2 = 0 \ , \tag{20}$$

where

$$\bar{\gamma}_o^2 \equiv \bar{\alpha}\,\bar{\beta} = \gamma_o^2/V_1 V_2 \ . \tag{21}$$

Defining

$$a \equiv \frac{1}{2}(\bar{\gamma}_1 - \bar{\gamma}_2) \tag{22}$$

$$c^2 \equiv \bar{\gamma}_o^2 - \bar{\gamma}_1 \bar{\gamma}_2 = \bar{\gamma}_o^2 - \bar{\gamma}_h^2 \quad , \tag{23}$$

we have

$$n_1'' - 2a n_1' + c^2 n_1 = 0 \tag{24}$$

$$E_2'' - 2a E_2' + c^2 E_2 = 0 \ . \tag{25}$$

Solutions are of the form $Ae^{\kappa_1 x} + Be^{\kappa_2 x}$, where

$$\kappa_{1,2} = a \pm (a^2 - c^2)^{\frac{1}{2}} \ . \tag{26}$$

The quantity $a^2 - c^2$ can be written

$$d^2 \equiv a^2 - c^2 = \Gamma^2 - \bar{\gamma}_o^2 \quad , \tag{27}$$

where

$$\Gamma \equiv \tfrac{1}{2}(\bar{\gamma}_1 + \bar{\gamma}_2) \quad . \tag{28}$$

The nature of the solution depends on the sign of d^2. If $d^2 < 0$, n_1 and E_2 are of the form $A \cos bx + B \sin bx$, where

$$b = i(c^2 - a^2)^{1/2} \quad . \tag{29}$$

In terms of the boundary values n_1^o and n_1^L, we have

$$n_1(x) = e^{ax} \left[n_1^o \cos bx + \frac{n_1^L \, e^{-aL} - n_1^o \cos bL}{\sin bL} \sin bx \right] \quad , \tag{30}$$

and similarly for $E_2(x)$. It is seen that for

$$bL = n\pi \quad , \tag{31}$$

$n_1(x) \to \infty$ inside the region $0 < x < L$, and a steady-state solution is not possible. The instability is then absolute. The condition $d^2 < 0$ is simply

$$\bar{\gamma}_o^2 > \Gamma^2 \quad \text{or} \quad \gamma_o > \frac{1}{2}\left(\frac{\gamma_1}{V_1} + \frac{\gamma_2}{V_2}\right) (V_1 V_2)^{1/2} \equiv \gamma_a \quad . \tag{32}$$

This is the usual "absolute" threshold given by a number of authors[1-3]. That an additional condition on L such as Eq. (31) is required in a finite plasma has been pointed out by Nishikawa and Liu[3]. When $\gamma_o > \gamma_a$, the growth of a pulse is described by the space-time solution of Eqs. (4) and (5). Numerical examples including pump depletion have been treated by Harvey and Schmidt[4] and by Bers et al.[5] In the absolute instability, the slope of $n_1(x)$ can be either positive, as in Fig. 1a or negative, as in Fig. 1b, depending on the parameters.

Below the absolute threshold $\gamma_o = \gamma_a$, it is well known[1-6] that convective instability is still possible. However, the nature of the instability in the range $\gamma_h < \gamma_o < \gamma_a$ has never been clarified. Consider now the case $d^2 > 0$ in Eq. (27). The solutions are exponentials with $\kappa = a \pm d$; let them be of the form

$$n_1(x) = e^{ax}(Ae^{dx} + Be^{-dx}) \qquad (33)$$

$$E_2(x) = e^{ax}(Ce^{dx} + De^{-dx}) \ . \qquad (34)$$

Substituting these into Eqs. (15) and (16) and integrating from 0 to x, we obtain

$$[(\bar{\alpha}A - (\bar{\gamma}_2 + \kappa_1)C] \frac{e^{\kappa_1 x} - 1}{\kappa_1} + [\bar{\alpha}B - (\bar{\gamma}_2 + \kappa_2)D] \frac{e^{\kappa_2 x} - 1}{\kappa_2} = 0 \qquad (35)$$

$$[(\bar{\gamma}_1 - \kappa_1)A - \bar{\beta}C] \frac{e^{\kappa_1 x} - 1}{\kappa_1} + [(\bar{\gamma}_1 - \kappa_2)B - \bar{\beta}D] \frac{e^{\kappa_2 x} - 1}{\kappa_2} = 0 \ , \qquad (36)$$

where $\kappa_1 = a + d$, $\kappa_2 = a - d$.

By virtue of Eqs. (22) and (28), we have $\bar{\gamma}_1 - \kappa_2 = \bar{\gamma}_2 + \kappa_1 = \Gamma + d$, $\bar{\gamma}_1 - \kappa_1 = \bar{\gamma}_2 + \kappa_2 = \Gamma - d$. Defining $F_1 = (e^{\kappa_1 x} - 1)/\kappa_1$, $F_2 = (e^{\kappa_2 x} - 1)/\kappa_2$, we now have

$$\bar{\alpha}F_1A + \bar{\alpha}F_2B - (\Gamma + d)F_1C - (\Gamma - d)F_2D = 0 \qquad (37)$$

$$(\Gamma - d)F_1A + (\Gamma + d)F_2B - \bar{\beta}F_1C - \bar{\beta}F_2D = 0 \qquad (38)$$

The boundary condition at x = 0 gives, from Eqs. (33) and (34),

$$A + B = n_1^o \qquad (39)$$

$$C + D = E_2^o \; . \qquad (40)$$

The last four equations determine the coefficients A - D in terms of n_1^o and E_2^o , resulting in

$$2d\, n_1(x) = [(d + \Gamma)n_1^o - \bar{\beta}E_2^o]\, e^{(a+d)x} + [(d - \Gamma)n_1^o + \bar{\beta}E_2^o]\, e^{(a-d)x} \qquad (41)$$

$$2d\, E_2(x) = [(d - \Gamma)E_2^o + \bar{\alpha}n_1^o]\, e^{(a+d)x} + [(d + \Gamma)E_2^o - \bar{\alpha}n_1^o]\, e^{(a+d)x} \; , \qquad (42)$$

One expects the initial noise level n_1^o to be above the thermal level, since a spectrum of plasma waves is usually excited in the plasma creation process. The initial level of E_2 then arises from Thomson scattering off the n_1^o oscillations, and E_2^o is larger than the bremsstrahlung level. The relation between n_1^o and E_2^o can be obtained by considering Eqs. (15) and (16) at the homogeneous threshold $\gamma_o = \gamma_1\gamma_2$. There being no growth or damping at this intensity, the derivatives E_2' and n_1' vanish, and these equations become

$$\bar{\gamma}_2 E_2^o = \bar{\alpha}_h n_1^o \tag{43}$$

$$\bar{\gamma}_1 n_1^o = \bar{\beta}_h E_2^o \ , \tag{44}$$

where $\bar{\alpha}_h$ and $\bar{\beta}_h$ are evaluated with E_o such that $\bar{\gamma}_o = \bar{\gamma}_h$. Thus we have

$$E_2^o/n_1^o = \bar{\alpha}_h/\bar{\gamma}_2 = \bar{\gamma}_1/\bar{\beta}_h \ . \tag{45}$$

The last equality shows that $\bar{\gamma}_1\bar{\gamma}_2 = \bar{\alpha}_h\bar{\beta}_h = \bar{\gamma}_h^2$, which is self-consistent. Since α and β are proportional to E_o and γ_o , we can write α_h and β_h in terms of α and β as follows:

$$\alpha_h = (\gamma_h/\gamma_o)\alpha \quad , \qquad \beta_h = (\gamma_h/\gamma_o)\beta \ . \tag{46}$$

Eq. (45) then can be written

$$E_2^o/n_1^o = (\bar{\gamma}_1/\bar{\gamma}_2)^{\frac{1}{2}} (\bar{\alpha}/\bar{\gamma}_o) = (\bar{\gamma}_1/\bar{\gamma}_2)^{\frac{1}{2}} (\bar{\gamma}_o/\bar{\beta}) \ . \tag{47}$$

Substituting into Eqs. (41) and (42), we obtain the final result

$$n_1(x) = \frac{n_1^o}{2d} \left\{ [d + \Gamma - \bar{\gamma}_o(\bar{\gamma}_1/\bar{\gamma}_2)^{\frac{1}{2}}] e^{(a+d)x} + [d - \Gamma + \bar{\gamma}_o(\bar{\gamma}_1/\bar{\gamma}_2)^{\frac{1}{2}}] e^{(a-d)x} \right\} \tag{48}$$

$$E_2(x) = \frac{E_2^o}{2d}\left\{[d - \Gamma + \bar{\gamma}_o(\bar{\gamma}_2/\bar{\gamma}_1)^{\frac{1}{2}}]e^{(a+d)x} + [d + \Gamma - \bar{\gamma}_o(\bar{\gamma}_2/\bar{\gamma}_1)^{\frac{1}{2}}]e^{(a-d)x}\right\} , \quad (49)$$

or equivalently,

$$n_1(x) = n_1^o e^{ax}\left\{\cosh dx + [\Gamma - \bar{\gamma}_o(\bar{\gamma}_1/\bar{\gamma}_2)^{\frac{1}{2}}]\sinh dx/d\right\} \quad (50)$$

$$E_2(x) = E_2^o e^{ax}\left\{\cosh dx - [\Gamma - \bar{\gamma}_o(\bar{\gamma}_2/\bar{\gamma}_1)^{\frac{1}{2}}]\sinh dx/d\right\} . \quad (51)$$

Before discussing the SRS reflectivity, we first demonstrate that these expressions give reasonable results in two limits. For $\bar{\gamma}_o = 0$, we have $\Gamma = d > a$, and Eqs. (48) and (49) become

$$n_1(x) = n_1^o e^{(a+d)x} = n_1^o e^{\gamma_1 x/V_1} \quad (52)$$

$$E_2(x) = E_2^o e^{(a-d)x} = E_2^o e^{-\gamma_2 x/V_2} . \quad (53)$$

This shows that, in the absence of a pump, both n_1 and E_2 damp at the expected rate as they propagate to the left and to the right, respectively.

At the homogeneous threshold $\bar{\gamma}_o = \bar{\gamma}_1\bar{\gamma}_2$, we have c=0 , d=a , and $\bar{\gamma}_o(\bar{\gamma}_1/\bar{\gamma}_2)^{\frac{1}{2}} = \bar{\gamma}_1$. Eq. (48) then becomes

$$n_1(x) = \frac{n_1^o}{2a}[(a - \bar{\gamma}_1 + \Gamma)e^{2ax} + (a + \bar{\gamma}_1 - \Gamma)]$$

$$= \frac{n_1^o}{2a}[(-\Gamma + \Gamma)e^{2ax} + 2a] = n_1^o . \quad (54)$$

Similarly,

$$E_2(x) = \frac{E_2^o}{2a}[(-\bar{\gamma}_2 + \bar{\gamma}_2)e^{2ax} + 2a] = E_2^o . \qquad (55)$$

Thus, n_1 and E_2 have the initial noise amplitude everywhere.

As γ_o increases slightly beyond γ_h, the exact cancellation of the coefficients of e^{2ax} in Eqs. (54) and (55) does not occur; and the instability grows at the extremely fast (spatial) rate e^{2ax}, while the dependence on I_o is linear, not exponential. As we shall show, the damping length 1/a can be much shorter than the interaction length L, so that 2ax can represent as many as 100 e-foldings. Therefore, either nonlinear saturation or the finite growth rate is needed to limit the exponentiation, and in this sense the instability resembles an absolute instability.

If one neglects the $V_1 n_1'$ term in Eq. (12) or assumes a spatial dependence of the form of Fig. 1b, one obtains[7] only the second terms of Eqs. (48) and (49), terms with the slower growth rate $e^{(a-d)x}$. To see this, assume $\gamma_o >> \gamma_h$, so that $c^2 \simeq \bar{\gamma}_o^2$ [Eq. (23)], and write a-d as $(a^2-d^2)/(a+d) \simeq c^2/2a \simeq \bar{\gamma}_o^2/2a$ [Eq. (27)]. The exponentiation is then $(a-d)L \simeq \gamma_o^2 L/2aV_1V_2 \simeq \gamma_o^2 L/c\gamma_1$, as in Eq. (14). Since $d < a$ for $\gamma_o > \gamma_h$, both terms in Eqs. (48) and (49) grow in the $+\hat{x}$ direction, so that the waves in the convective regime behave as in Fig. 1a.

At the absolute threshold $\bar{\gamma}_o = \bar{\gamma}_c = \Gamma$, Eq. (27) gives d=0. Eqs. (50) and (51) then become

$$n_1(x) = n_1^o e^{ax}\left\{1 + \Gamma x[1 - (\bar{\gamma}_1/\bar{\gamma}_2)^{1/2}]\right\} \qquad (56)$$

$$E_2(x) = E_2^o e^{ax}\left\{1 - \Gamma x[1 - (\bar{\gamma}_2/\bar{\gamma}_1)^{1/2}]\right\} . \qquad (57)$$

of e-foldings estimated form Eq. (14) is only a few. If this picture is correct, the presence of even a small amount of dissipation will cause the standard formula for the convective threshold in an inhomogeneous plasma, $v_o^2/c^2 = 2/k_oL_n$, to be inapplicable.

This work was supported by NSF under grant no. ECS 83-10972, Lawrence Livermore National Laboratory under P.O. #3446905, and DOE under contract no. DE-AS08-84DP40196.

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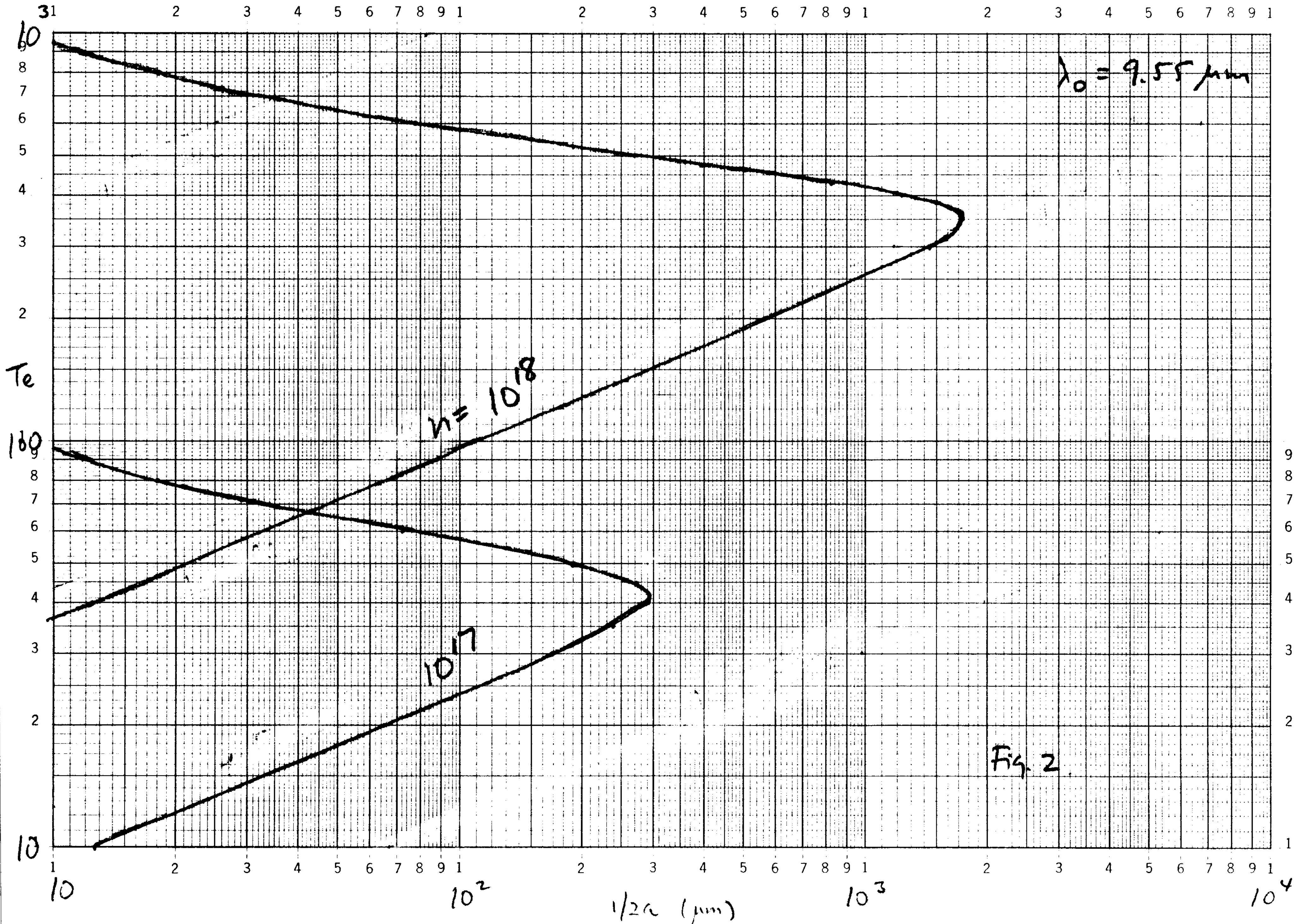
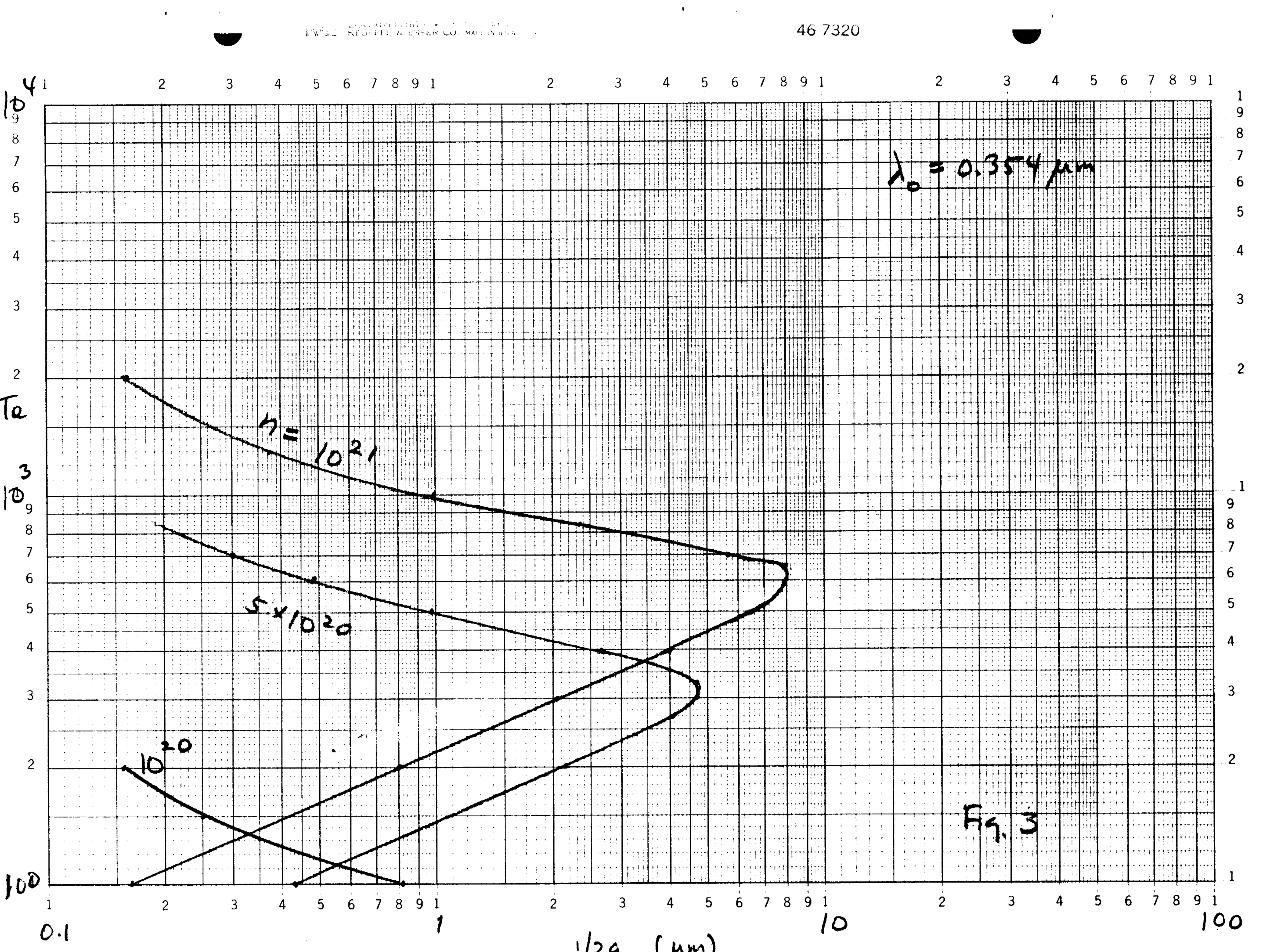


Fig. 2



46 7320
$\lambda_0 = 0.354\ \mu m$
$n = 10^{21}$
5×10^{20}
10^{20}
Te
10^4
10^3
100
0.1
1
10
100
1/2a (μm)

Fig. 3