

Detection Fidelity in Distributed Wireless Sensor Networks

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Abstract

Interesting rate distortion problems arise in sensor networks. In this paper, we examine the errors of distributed sensor networks in sensing, quantization and source reconstruction. The sensing error depends on signal attenuation and measurement noise, which can be lessened by reducing source-sensor separation and increasing the number of independent observations. The rate of data transmission to the global fusion center, which incurs the most energy (capacity) cost, can be brought down by exploiting correlation among sensors by local fusion. However, the rate ultimately constrains the number of quantization levels. We model distributed phenomena as correlated point sources, and derive bounds on the total error using a localized reconstruction algorithm based on cubic splines. In particular, the interpolation error in cubic spline fitting is shown to converge at least on the same order as the sensing and quantization noise given appropriate mesh sizes.

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Index Terms

distributed sensor network, distortion bound, source/sensor distribution, sensing error, rate-distortion, quantization error, cubic spline, interpolation error.

I. INTRODUCTION

Recent advances in technology have made a broad set of applications of sensor networks possible [1] [2]. In these applications, it is often required to deploy large scale, distributed, wireless networks. One interesting set of problems is the distortion bounds of sensor networks under various constraints such as sensing noise and limited network resources. Usually, sensors monitor distributed phenomena rather than single isolated events that are considered point sources. In engineering practice, however, real distributed continuous processes are never fully observable. A typical approach in sensing is to sample the processes in time and space, in which the distributed phenomena are reasonably modelled as sets of correlated point sources. The reconstruction of the source is then done by interpolation, for example, spline fitting. In this paper, we study the distortion bounds of sensor networks based on this approach.

First of all, the distortion of sensor networks is circumscribed by the sensing capabilities of sensors. The accuracy of sensing is affected by the amount of signal attenuation and noise corruption. While noise is an unavoidable process, attenuation is strongly related to the distance between the sensor and the source, which in turn is a function of sensor coverage.

When observing a distributed phenomenon, the rate used to quantize the signal is usually constrained by the network capacity owing to the large amount of information embedded in the source. In [3], using a point-point transmission model, it was found that the capacity per node decreases as the number of nodes in the network increases, which may be attributed to the fact

that the destination is randomly chosen among all the nodes in the network. In [4], on the other hand, it was argued that the situation in sensor networks is not at all as pessimistic since the problem is often that of coding a correlated source. The minimum rate required for correlated sources under a distortion constraint, i.e. rate-distortion problem, has been studied for decades. In this case what is bounded is the rate needed to communicate to the global fusion center under a quantization distortion constraint. Although approaching this bound may entail intensive local cooperation and fusion among sensors and local fusion centers, this is still beneficial because local transmission is far less constrained due to the bounded transmission ranges [5]. Also note that in contrast to [5], where the problem is to observe distinct point sources, the perspective here is that there is one distributed process, which is modelled using correlated point sources.

For a distributed source with a continuous sample path, the source is usually reconstructed at the fusion center by interpolating from measured points. This gives rise to interpolation error. Conventionally, interpolation error has been analyzed with the assumption that the data at prescribed points has no error. Here, we consider the combined distortion of sensing, quantization and interpolation. Also we would like to point out that similar problems have been dealt with in image processing [6], and many results may be applied here except for the distributed nature of sensor networks. This results in communication cost, and hence a set of constraints on the rate. Additionally, the sensors may be irregularly deployed and heterogeneous.

The rest of the paper is organized as follows. In section II, we consider sensing, quantizing and estimating point sources. The discussion is then extended to distributed sources. In section III, the minimum rate needed for correlated sources is given based on rate-distortion theory under certain assumptions for the source distribution. In section IV, the total error due to sensing, quantization and interpolation when reconstructing continuous sources using spline fitting is

discussed. The paper concludes in section V.

II. POINT SOURCES

Point sources are useful abstractions since many phenomena can be reasonably modelled as either a single point source, or constructed via interpolation from a set of point sources. We begin with a single point source. The data sent to a fusion center by sensors is the quantized version of attenuated and noise corrupted signals radiated from the source. The sensing error, which is dependent on the strength of signal and noise at the sensor, persists even if errors due to quantization and communication are zero. In this section, we use a simple model to capture this sensing distortion.

A. Sensing Model

The signal radiated by the source X , corrupted by noise Z_i , received by sensor i is:

$$Y_i = a_i X + Z_i$$

with

$$X \sim \mathcal{N}(0, \sigma_X^2), \quad Z_i \sim \mathcal{N}(0, \sigma_Z^2), \quad a_i = \frac{1}{1 + \kappa r_i^2}.$$

The attenuation factor a_i is a function of the distance r_i between the source and the sensor. κ is a constant that controls how strongly the distance affects the attenuation. For convenience, we assume that the data sequence X^n produced by the source are i.i.d. Gaussian random variables. The noise Z_i at sensor i is assumed Gaussian and is independent of X . Also the noise is assumed i.i.d. at different sensors, which implies that they are the same type of sensor nodes and in similar environments.

B. Sensor Coverage

The sensors are deployed in a unit area to monitor a point source. We are tempted to presume that the source is uniformly distributed due to the following reasons. First, we often have no prior knowledge about the probability distribution of the source location except that it is confined to a certain region. Second, the monitored area can often be considered homogeneous or comprising homogeneous sections, in which the source appears anywhere with equal probability. Third, uniformly distributed sources are amenable to analysis. With this assumption, it is natural to distribute sensors in an equally homogeneous fashion. Two ways to deploy n static sensors are considered. One is deterministic: divide the unit area into n identical square cells, then place one sensor at the center of each cell. It should be noted that this is often an approximation since the boundaries of an arbitrary area seldom conform to the collection of square cells. Thus the approximation is more reasonable for networks with a large n . The other way is random: independently place all the sensors according to the uniform distribution assuming the area is a unit disc. If only the sensor that is closest to the source transmits its observation to the fusion center, the source-sensor separation r is R_{min} , the distance between the source and the closest sensor. For either way of deploying n sensors, the probability density functions of R_{min} are computed, and the mean values are found to decrease with \sqrt{n} as shown in Appendix I.

$$E(R_{min}) \propto \frac{1}{\sqrt{n}}$$

Placing sensors on a predetermined grid results in a lower $E(R_{min})$ than distributing sensors randomly. However, if node failures are taken into account, R_{min} may deteriorate more drastically in a deterministic scheme after a certain number of sensor nodes become defunct, but this subject is not pursued further here.

C. Optimal Distortion

The distortion measure is defined over n estimations, $d(X, \hat{X}) = \frac{1}{n} \sum_{i=1}^n (X_i - \hat{X}_i)^2$. The raw data Y_i observed by the sensor that is closest to the source is sent back to the fusion center with the quantization distortion constraint D_q . Ideally, the quantization noise Z_q is a Gaussian random variable with variance D_q , independent of X and Z_i . Received at fusion center is the signal Y . No communication error occurs as long as the source coding rate R is less than the link capacity W .

$$Z_q \sim \mathcal{N}(0, D_q)$$

$$Y = a_i X + Z_i + Z_q$$

$$R = \frac{1}{2} \log \left(\frac{\sigma_{Y_i}^2}{D_q} \right) \leq W \quad (1)$$

The optimal estimator at the fusion center is a linear estimator [7], which is also the best that can be done given any sender X and receiver Y .

$$\hat{X} = \frac{a_i \sigma_X^2 Y}{a_i^2 \sigma_X^2 + \sigma_Z^2 + D_q} \quad (2)$$

$$D(a_i) = E_{X, Z_i, Z_q} [(\hat{X} - X)^2] = \frac{\sigma_X^2 (\sigma_Z^2 + D_q)}{a_i^2 \sigma_X^2 + \sigma_Z^2 + D_q}$$

As the data from the sensor that is closest to the source is used, $r_i = R_{min}$. Thus,

$$D(n) = E_{R_{min}} \left[\frac{\sigma_X^2 (\sigma_Z^2 + D_q)}{a_i^2 \sigma_X^2 + \sigma_Z^2 + D_q} \right] \quad (3)$$

with a communication cost of $R_g = R$, Equation (1). R_g indicates global transmissions between sensors and fusion center, which is distinguished from R_l , transmissions among sensors and local fusion centers with bounded range. Note that it is usually desirable to make Z_q and Z_i about the same. Therefore, we can select

$$R \approx \frac{1}{2} \log \left(\frac{\sigma_Y^2}{\sigma_Z^2} \right)$$

D. Simulation

Considering a single point source appearing in a unit disc, we numerically evaluate Equation (3) to show the dependence of optimal distortion on the size of the sensor network and quantization rate. In addition, a practical scheme based on an optimal scalar quantizer [8] is simulated, and the resulting distortion is compared to the optimal distortion. In practice, since it is impossible to make the quantization noise behave like $\mathcal{N}(0, D_q)$, we choose to obtain a local estimate of the source at sensor, then send it to the global fusion center. The result is shown in Figure 1. It can be seen that in a relatively sparse sensor network ($n \leq 50$), the sensing error is the major contribution to the total distortion. Hence the increase in sensing accuracy (by deploying more sensors) leads to a significant drop in distortion. As the sensor network gets denser, quantization error due to insufficient rate starts to dominate, and a rate increase reduces the distortion. Besides, as the quantization rate increases the distortion gap between Max and optimal quantizers diminishes.

E. Local Fusion to Reduce Sensing Error

In general, the error due to sensing is much larger than the achievable quantization error D_q . Therefore, more than one sensor's observations should be used to estimate X such that a lower distortion is achieved. Consider N_m sensors that are in the vicinity of the source. Each sensor has an observation of the source. Instead of sending all the observations to the global fusion center, we obtain a refined estimate locally, and transmit this value. The resulting estimate, mean square error and transmission cost are as follows assuming that the local fusion center knows the a_i 's of the surrounding sensors. R_g is the rate required to transmit to the global fusion center with distortion constraint D_g , and R_l is the total rate for transmitting to the local fusion center with

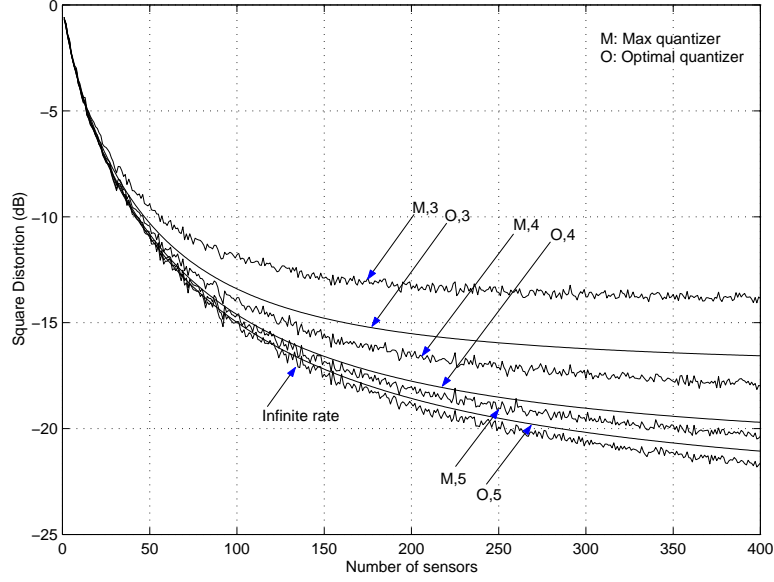


Fig. 1. Comparison of the distortion of a Max quantizer with the optimal quantizer. $\sigma_X^2 = 1$, $\sigma_Z^2 = 0.0032$, $\kappa = 500$. Labels are interpreted in the following way. “M/O, r ”: Max/optimal quantizer with quantization rate r bits per sample; “Infinite rate”: quantizer with infinite rate.

distortion constraint D_l . Notice that no further optimal estimator is used at the global fusion center because it may not know the individual sensor who has contributed to the estimation. Also, we assumed every Y_i is corrupted by the quantization noise Z_{li} . For convenience, we assume that all observations are corrupted by the quantization error D_l . The distortion can be a little smaller if the local fusion center is one of the sensing nodes.

$$\hat{X} = \frac{\sigma_X^2 \sum_{i=1}^{N_m} a_i (Y_i + Z_{li})}{D_l + \sigma_Z^2 + \sigma_X^2 \sum_{i=1}^{N_m} a_i^2} \quad (4)$$

$$D = \frac{(D_l + \sigma_Z^2) \sigma_X^2}{D_l + \sigma_Z^2 + \sigma_X^2 \sum_{i=1}^{N_m} a_i^2} + D_g \quad (5)$$

$$R_g = \frac{1}{2} \log \left(\frac{\sigma_X^2}{D_g} \right) \quad (6)$$

$$R_l = \sum_{i=1}^{N_m} \frac{1}{2} \log \left(\frac{\sigma_{Y_i}^2}{D_l} \right) \quad (7)$$

The average D here is related to the distances between the source and its N_m neighboring sensors. In principle, we can find the probability density function of all r_i 's and evaluate the mean distortion. Figure 2 compares the distortions using single and multiple observations. As shown, the distortion can be effectively reduced by using more independent observations.

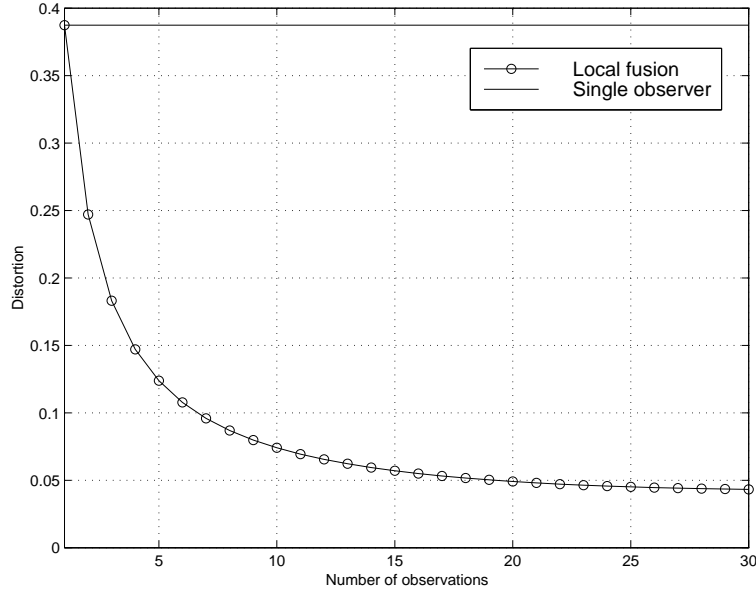


Fig. 2. Comparing distortions using single and multiple observations. $a_i = [0.1 - 0.001(i - 1)]$, ($i = 1, 2, \dots, N_m$), $\sigma_X^2 = 1$, $\sigma_Z^2 = 0.003162$, $D_l = D_g = \sigma_Z^2$.

The local transmission cost can be reduced by noticing that the observations at separate sensors are correlated (Section IV, [5]). This is hence a problem of source coding with side information [9]. Specifically, for a Gaussian source [10], the following local transmission rate is achievable assuming node 1 serves as the local fusion center.

$$R_l = \sum_{i=2}^{N_m} \frac{1}{2} \log \left[\frac{\sigma_{Y_i}^2}{D} (1 - \rho_i^2) \right] \quad (8)$$

$$\rho_j = \frac{a_1 a_j \sigma_X^2}{\sqrt{(a_1^2 \sigma_X^2 + \sigma_Z^2)(a_j^2 \sigma_X^2 + \sigma_Z^2)}}$$

Note that the rate can be further reduced, if received information from other sensors is used as side information as well. As an example, consider $a_1 = a_j = 0.1, (j = 2, \dots, M), \sigma_X^2 = 1$ and $\sigma_Z^2 = 0.003162$. This results in $\rho_j = 0.756$ and an approximate rate reduction of $1.231(M - 1)$ bits per sample.

III. DISTRIBUTED SOURCE: TRANSMITTING DATA ACROSS THE NETWORK

We now look at the distortion due to quantization when transmitting the sensor observations made for a distributed source to the fusion center. When observing a single point source, the sensing error often dominates since a relatively small amount of information needs to be transmitted to the global fusion center. However, there is a great amount of information embedded in a distributed source. Thus a compromise will often need to be made between the reduction of quantization error and the constrained network resource. On the other hand, due to the spatial correlations of a distributed source, it is possible to significantly reduce the rate at which sensors transmit to the global fusion center, compared to the raw information rate accumulated by sensors. Usually attaining this reduction entails intensive local interactions among sensors. This kind of tradeoff is desirable because the energy (capacity) constraint on global transmissions (between sensors and global fusion center) is far more severe than local transmissions (among sensors and local fusion centers), whose range is bounded [5]. The minimum rate required for distributed sources under certain error constraints, i.e. the rate-distortion problem, has received a fair amount of research focus in the image processing literature [11] [12]. A number of results can be borrowed from there. However, we should point out that the minimum rates are harder to achieve in sensor networks because of their distributed nature. That is, communication rates are limited and have a cost.

In engineering practice, real continuous distributed sources are never fully observable. The

physical world sensed by sensors and processed by computers is always discrete, and comprises a finite collection of points. In this section, we consider the problem of coding N correlated point source observations, which can be considered as the individual observations of a distributed source made by N sensors, given certain distortion constraints.

A. Correlated Point Sources

Consider N point sources $\mathbf{X} = \{X_i, i = 1, 2, \dots, N\}$ in the space. The sensor measurements $\mathbf{Y} = \{Y_i : a_i X_i + Z_i, i = 1, 2, \dots, N\}$ are attenuated signals plus noise. At discrete times, the measurements $Y_i^1, Y_i^2, Y_i^3 \dots$ at each sensor are zero-mean i.i.d. random variables. The samples $Y_1^n, Y_2^n, \dots, Y_N^n$ measured at the same time are correlated with covariance matrix \mathbf{Q}_N , whose positive eigenvalues are given by $\lambda_1, \lambda_2, \dots, \lambda_N$. We assume nothing else is known of the probability distribution of \mathbf{Y} . Therefore, we are facing the problem of jointly coding the i.i.d. blocks of N samples generated from a class of random variables with the distortion requirement D_q such that

$$E[d(\mathbf{Y}, \hat{\mathbf{Y}})] \leq D_q.$$

For the distortion measure defined as $d(\mathbf{Y}, \hat{\mathbf{Y}}) = \frac{1}{N} \sum_{i=1}^N (Y_i - \hat{Y}_i)^2$, it can be shown that the superior of the minimum rate of this class of random variables is attained when \mathbf{Y} is Gaussian [13] [14]. Thus, to be able to code this whole class of random variables and satisfy the given constraint, the minimum rate required is [12] [13]:

$$R_g = \sum_{i=1}^N \frac{1}{2} \log \left(\frac{\lambda_i}{\min[\lambda_i, D^*]} \right); \quad (9)$$

$$D_q = \frac{1}{N} \sum_{i=k}^N \min(\lambda_i, D^*) \quad (10)$$

Consider the case when the distortion is small, i.e. $D_q \leq \lambda_i$, ($i = 1, \dots, N$). We define $\mathbf{D}_N = D_q \mathbf{I}$. This is to impose the same quantization distortion constraint at all the sensors.

$$R_g = \sum_{i=1}^N \frac{1}{2} \log \frac{\lambda_i}{D_q} = \frac{1}{2} \log \frac{|\mathbf{Q}_N|}{|\mathbf{D}_N|} \quad (11)$$

$$\mathbf{Q}_N = E(\mathbf{Y}\mathbf{Y}^t), \quad \mathbf{Y} = \begin{bmatrix} a_1 X_1 + Z_1 \\ a_2 X_2 + Z_2 \\ \vdots \\ a_N X_N + Z_N \end{bmatrix}.$$

B. One Dimensional Brownian Field

For a special case, we consider that the source is a one dimensional Brownian field $\mathbf{X}_u(k)$ defined on $u \in [0, 1]$, the same as the setting in [4]. For fixed k , $\mathbf{X}_u(k)$ is a Brownian motion with σ_X^2 ; for fixed u , $\mathbf{X}_u(k)$ are i.i.d. Gaussian random variables $\sim \mathcal{N}(0, \sigma_X^2 u)$. The N measuring points are uniformly placed on $[0, 1]$, and the attenuation factor is assumed to be unity.

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_3 \end{bmatrix} = \mathbf{H}\mathbf{W} + \mathbf{Z}.$$

$$w_i \sim \mathcal{N}\left(0, \frac{\sigma_X^2}{N}\right), \quad Z_i \sim \mathcal{N}(0, \sigma_Z^2) \quad \text{i.i.d.}$$

$$\mathbf{Q}_N = \frac{\sigma_X^2}{N} \mathbf{H}\mathbf{H}^t + \sigma_Z^2 \mathbf{I}$$

In this case, if we require $D_q \leq \sigma_Z^2$, which means the quantization error is no more than the noise variance, we have $D_q < \lambda_i$. The determinant of the covariance matrix can be evaluated to be the following.

$$|\mathbf{Q}_N| = \left(\frac{\sigma_X^2}{N}\right)^N (Ar_1^N + Br_2^N) \quad \text{for } N \geq 1. \quad (12)$$

where

$$A = \frac{1}{r_1} \left(\frac{1 + \nu}{2} + \frac{1 + 3\nu}{2\sqrt{1 + 4\nu}} \right), \quad B = \frac{1 + \nu - A}{r_2},$$

$$r_{1,2} = \frac{(1 + 2\nu) \pm \sqrt{1 + 4\nu}}{2}, \quad \nu = \frac{N}{\sigma_X^2 / \sigma_Z^2}.$$

So the minimum distortion due to quantization is:

$$D_q = \frac{\sigma_X^2}{N} (Ar_1^N + Br_2^N)^{1/N} 2^{-2R_g/N} \quad (13)$$

To achieve $D_q \leq \sigma_Z^2$, the rate R_g needs to grow at least linearly with N , the number of measurements. On the other hand, N should be appropriately chosen according to the sensing noise level σ_Z^2 . R_g 's linear growth with N is expected because at each sensor, at least one bit is required to transmit the noise Z_i , given $D_q \leq Z_i$. If such R_g is not available, appropriate D_q should be designed to satisfy the capacity constraint based on Equation (9).

C. A Two Dimensional Isotropic Random Field

The rate-distortion function is evaluated for a two dimensional isotropic random field with correlation function $e^{-|r|/d_c}$, where d_c is the coherence distance. Nine sensors are placed in a square grid, and s is defined as in Figure 3. The minimum total rates required for transmitting the data collected at these nine sensors to the global fusion center are plotted in Figure 4. It can be seen that as the sensors become closer (with s decreasing), the correlation among sensors increases. As a result, the data rate needed to transmit to the global fusion center decreases. Moreover, this rate drop can be accomplished by local fusion.

IV. DISTRIBUTED SOURCE: RECONSTRUCTION BY CUBIC SPLINE

Assume the distributed source in space to be observed is continuous or at least piecewise continuous (for the latter, we consider one continuous piece of the sample path). Consider spline

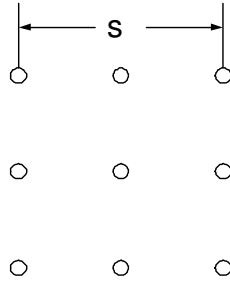


Fig. 3. Nine sensors in a random field.

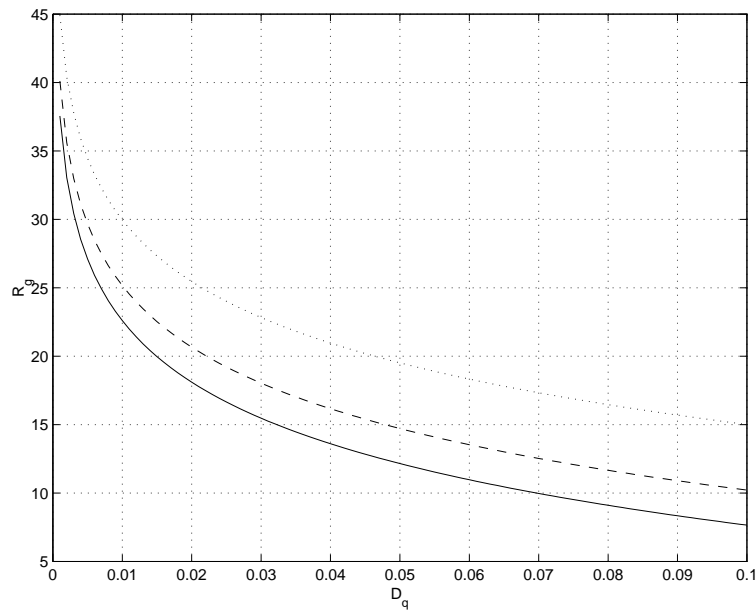


Fig. 4. The comparison of rate-distortion with different coherence distance. $a_i = 1$, $\sigma_X^2 = 1$, $\sigma_Z^2 = 0.01$, solid line: $s/d_c = 0.4$, dashed line: $s/d_c = 0.7$, dotted line: $s/d_c = \infty$.

fitting the source based on the sensing data on discrete points. While it is usually assumed that no error occurs for the measurements at prescribed points, in this section, we examine how sensing and quantization errors affect the result of interpolation.

By reconstructing the source using spline fitting, we implicitly treat the source as deterministic. However, when seeking the limits of source coding rate in the previous section, we considered

the source to be a random process. The reconciliation of these two models is reached as follows. First, minimum source rates are obtained only by considering jointly coding long blocks of i.i.d. realizations of sources, while source reconstruction is performed for one particular realization of the source. Second, evaluating the minimum rate demands certain knowledge about the distribution of the source field. However spline fitting is able to take advantage of the correlation among local observations embedded in the continuity of the source. Third, information embedded in a distributed source is never completely conveyed to the fusion center. For a continuous sample path, spline fitting makes reasonable estimates of the missing data, provided that the sample points are closely spaced.

The derivation of this section is based on the cubic spline theory in [15]. Two types of distortion measure are considered here: $d(X, \hat{X}) = |X - \hat{X}|$ and $d(X, \hat{X}) = (X - \hat{X})^2$.

A. Cubic Spline Fitting

We first consider a one dimensional cubic spline. Given the locations of $(N + 1)$ points and a set of associated ordinates

$$\Delta : \quad a = x_0 < x_1 < \cdots < x_N = b.$$

$$Y : \quad y_0, y_1, \cdots, y_N.$$

the spline function on $[x_{j-1}, x_j]$, ($j = 1, 2, \dots, N$) is defined as follows

$$S_{\Delta} = M_{j-1} \frac{(x_j - x)^3}{6h_j} + M_j \frac{(x - x_{j-1})^3}{6h_j} + \left(y_{j-1} - \frac{M_{j-1}h_j^2}{6} \right) \frac{x_j - x}{h_j} + \left(y_j - \frac{M_j h_j^2}{6} \right) \frac{x - x_{j-1}}{h_j}. \quad (14)$$

in which $h_j = x_j - x_{j-1}$. Also $M_j = S''_{\Delta}(x_j)$, the moments of the spline, satisfy the following equations [15]:

$$\begin{bmatrix} 2 & \lambda_0 & 0 & \dots & 0 \\ \mu_1 & 2 & \lambda_1 & \dots & 0 \\ 0 & \mu_2 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_N \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} \quad (15)$$

Given appropriate end conditions and converging meshes Δ_k ($\lim_{k \rightarrow \infty} \|\Delta_k\| = 0$), if the curve belongs to $C^n[a, b]$, $n = 0, 1, 2$ and 3 , having n th continuous derivative (by 0 , we mean the curve is continuous), and satisfying Hölder's condition to the order of α ($0 < \alpha \leq 1$), the interpolation error uniformly converges with respect to x in $[a, b]$ as follows (Theorem 2.3.1, 2, 3 and 4 [15]):

$$e_1 = |f(x) - S_{\Delta}(x)| \leq K_1 \|\Delta_k\|^{n+\alpha}, \quad \text{for some constant } K_1$$

However, the spline reconstructed at the fusion center is not $S_{\Delta}(x)$ but a shifted spline $S_{\Delta}^e(x)$ due to the sensing and quantization error at measured points. It remains to be seen how much the interpolation deteriorates given that the noise at prescribed points is bounded by: $E|\delta y_i| \leq e_{sq}$ and $E(\delta y_i)^2 \leq D_{sq}$. Consider the absolute error first.

$$e = E|f(x) - S_{\Delta}^e(x)| \leq |f(x) - S_{\Delta}(x)| + E|S_{\Delta}(x) - S_{\Delta}^e(x)| = e_1 + e_2 \quad (16)$$

Note that since both $f(x)$ and $S_{\Delta}(x)$ are considered deterministic, the mean operation disappears for e_1 . As for e_2 , we have the following:

$$e_2 = E \left| \sum_{i=0}^N \frac{\partial S_{\Delta}}{\partial y_i} \delta y_i \right| \leq e_{sq} \sum_{i=0}^N \left| \frac{\partial S_{\Delta}}{\partial y_i} \right| \quad (17)$$

In Appendix II, it is shown that for proper end conditions $\lambda_0, \mu_N < 2$ and evenly distributed meshes (with bounded h_j/h_{j+1}), the following holds.

$$\beta = \sum_{i=0}^N \left| \frac{\partial S_\Delta}{\partial y_i} \right| \leq K_2, \quad \text{for some finite number } K_2$$

Hence the total absolute error is bounded by

$$e \leq e_1 + \beta e_{sq} \quad (18)$$

Next we consider the mean square error. Since data is locally fused before it is transmitted to the global fusion center, the correlation between the noise δy_i at different sensors is not necessarily zero, but it is bounded by the following:

$$E(\delta y_i \delta y_j) \leq \sqrt{E(\delta y_i)^2 E(\delta y_j)^2} \leq D_{sq}$$

We first find a bound on the mean square error between the original and noise corrupted splines with respect to $x \in [a, b]$.

$$\begin{aligned} E [S_\Delta(x) - S_\Delta^e(x)]^2 &= E \left(\sum_i \frac{\partial S_\Delta}{\partial y_i} \delta y_i \right)^2 \\ &= \sum_i \left(\frac{\partial S_\Delta}{\partial y_i} \right)^2 E(\delta y_i)^2 + 2 \sum_{i \neq j} \left(\frac{\partial S_\Delta}{\partial y_i} \frac{\partial S_\Delta}{\partial y_j} \right) E \delta y_i \delta y_j \\ &\leq \left(\sum_i \frac{\partial S_\Delta}{\partial y_i} \right)^2 D_{sq} \\ &\leq \beta^2 D_{sq} \end{aligned}$$

Now, we evaluate the total mean square error.

$$\begin{aligned} D &= E [(f(x) - S_\Delta(x)) + (S_\Delta(x) - S_\Delta^e(x))]^2 \\ &= [f(x) - S_\Delta(x)]^2 + 2[f(x) - S_\Delta(x)] E [S_\Delta(x) - S_\Delta^e(x)] + E [S_\Delta(x) - S_\Delta^e(x)]^2 \\ &\leq e_1^2 + 2\beta e_1 e_{sq} + \beta^2 D_{sq} \end{aligned}$$

In the following simulation, we use a cubic spline to fit a sinusoid function based on noise corrupted data. Figure 5 shows error e_2 plotted against the noise due to sensing and quantization e_{sq} . The relation in this case appears to be approximately linear. In simulation, we kept mesh size relatively small, so $e \approx e_2$.

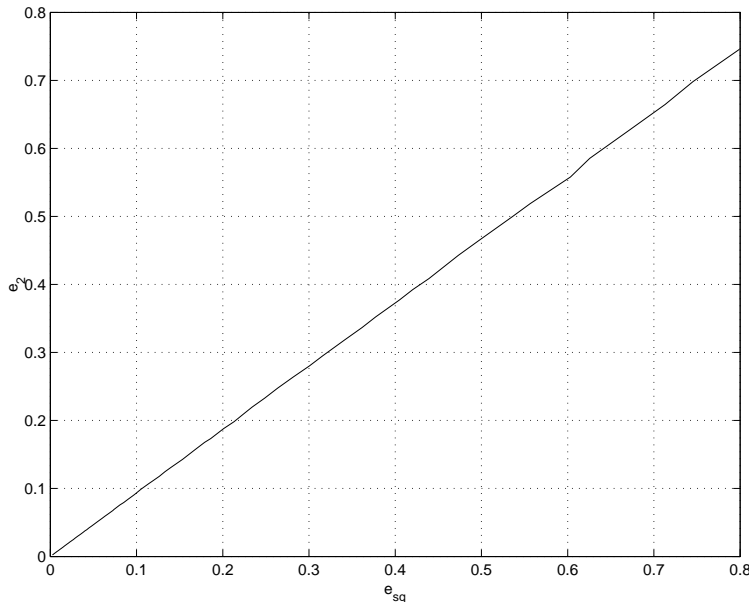


Fig. 5. e_2 versus sensing and quantizing error e_{sq} .

This result is readily extensible to a two-dimensional doubly cubic spline defined on a rectangular grid, $\Delta_t : a = t_0 < t_1 < \dots < t_N = b$, $\Delta_s : c = s_0 < s_1 < \dots < s_M = d$, noticing that a doubly cubic spline can be obtained by partial splines on t and s (p. 238 [15]). The resulting error after twice one-dimensional interpolation is thus bounded by:

$$e \leq e_{1s} + \beta_s(e_{1t} + \beta_t e_{sq}) \quad (19)$$

where, $\beta_i, e_{1i}, (i = s, t)$ are the corresponding parameters on s and t coordinates. A similar derivation applies to the mean square error.

V. CONCLUSION

In this paper, we discussed the distortions due to sensing, quantization and interpolation in sensor networks by modelling distributed phenomena as correlated point sources. Sensing error is determined by measurement noise and signal attenuation, which can be reduced by improving sensor coverage and increasing the number of observations. When observing a distributed source, network capacity may become strained due to the large raw information rate. In this case, local cooperation and fusion based on correlations among nearby observations can be used to bring down the quantization rate, which is bounded by rate-distortion theory. Here, we considered cubic splines as the local algorithm to reconstruct the continuous sources. The total error was found to converge at least on the same order of sensing and quantization error given appropriate mesh sizes.

Many topics for future research in this general area suggest themselves. Different sensing models can be proposed, which will lead to different behaviors for the sensing error. Alternative practical local fusion algorithms can be designed and compared to the rate bounds. In this paper, we considered only source coding, but channel coding enters the picture either by setting the limits on quantization rate or joint source-channel coding. The convergence of distortion with other interpolation schemes (besides the cubic spline) in the presence of sensing and quantization noise can also be studied.

APPENDIX I

R_{min} FOR UNIFORMLY DISTRIBUTED SOURCE AND SENSORS

First consider dividing the unit area into n identical square cells, and placing one sensor at the center of each cell. This is equivalent to fixing the sensor at $(0, 0)$ and placing the source in

the area $[0, \frac{1}{2\sqrt{n}}] \times [0, \frac{1}{2\sqrt{n}}]$ uniformly.

$$f_{R_{min}}(r) = \begin{cases} 2\pi nr & \text{when } 0 \leq r \leq \frac{1}{2\sqrt{n}} \\ 4nr \left[\frac{\pi}{2} - 2 \arccos \left(\frac{1}{2\sqrt{nr}} \right) \right] & \text{when } \frac{1}{2\sqrt{n}} \leq r \leq \frac{1}{\sqrt{2n}} \end{cases}$$

The density function is 0 for any r not included in the definition, which also applies to the other probability density functions in this section. The average R_{min} is found to be:

$$E(R_{min}) \approx \frac{0.3826}{\sqrt{n}}$$

Next, we consider a point source appearing in a unit disc ($R_0 = \frac{1}{\sqrt{\pi}}$) uniformly. Denote by R_s the distance from the source to the center of the disc.

$$f_{R_s}(r_s) = 2\pi r_s \quad \text{when } 0 \leq r_s \leq R_0$$

Denote by R , the distance between the source and a randomly placed sensor according to the uniform distribution. Given the source is r_s away from the center, the probability density function for R , is:

$$f_R(r | r_s) = \begin{cases} 2\pi r & \text{when } 0 \leq r \leq R_0 - r_s \\ 2\theta r & \text{when } R_0 - r_s \leq r \leq R_0 + r_s \end{cases} \quad (20)$$

$$\theta = \arccos \left(\frac{r_s^2 + r^2 - R_0^2}{2r_s r} \right)$$

Therefore,

$$P(R \leq r | r_s) = \begin{cases} \pi r^2 & \text{when } 0 \leq r \leq R_0 - r_s \\ \phi R_0^2 + \theta r^2 - R_0^2 \sin \phi \cos \phi - r^2 \sin \theta \cos \theta & \text{when } R_0 - r_s \leq r \leq R_0 + r_s \end{cases}$$

$$\phi = \begin{cases} \arcsin \left(\frac{r \sin \theta}{R_0} \right) & \text{when } R_0 - r_s \leq r \leq \sqrt{R_0^2 + r_s^2} \\ \pi - \arcsin \left(\frac{r \sin \theta}{R_0} \right) & \text{when } \sqrt{R_0^2 + r_s^2} \leq r \leq R_0 + r_s \end{cases}$$

Now, consider n sensors are randomly placed in the disc in the same fashion. That the distance between the source and the closest sensor is r is to say that there is one sensor r away from the source and the other $(n - 1)$ sensors are at least r away from the source.

$$f_{R_{min}}(r | r_s) = n f_R(r | r_s) [1 - P(R \leq r | r_s)]^{n-1}$$

Noticing that R_s is itself uniformly distributed, take the expectation on R_s .

$$f_{R_{min}}(r) = E_{R_s}[f_{R_{min}}(r | r_s)]$$

The formula is numerically evaluated for different n , and the mean value is computed.

$$R_{min} \approx \frac{0.5101}{\sqrt{n}}$$

APPENDIX II

$$\sum_{i=0}^N \left| \frac{\partial S_{\Delta}}{\partial y_i} \right| \text{ IS BOUNDED}$$

Rearrange the cubic spline function for $x_{j-1} \leq x \leq x_j$.

$$S_{\Delta}(x, y_0, y_1, \dots, y_N) = \alpha_j M_{j-1} + \beta_j M_j + \gamma_j y_{j-1} + (1 - \gamma_j) y_j$$

$$\alpha_j = \left[\frac{(x_j - x)^3}{6h_j} - \frac{h_j(x_j - x)}{6} \right], \quad |\alpha_j| \leq \frac{h_j^2}{9\sqrt{3}}$$

$$\beta_j = \left[\frac{(x - x_j)^3}{6h_j} - \frac{h_j(x - x_{j-1})}{6} \right], \quad |\beta_j| \leq \frac{h_j^2}{9\sqrt{3}}$$

$$0 \leq \gamma_j = \frac{(x_j - x)}{h_j} \leq 1$$

Differentiate the spline function about y_i .

$$\frac{\partial S_{\Delta}}{\partial y_i} = \alpha_j \frac{\partial M_{j-1}}{\partial y_i} + \beta_j \frac{\partial M_j}{\partial y_i} + \gamma_j \delta_{i,j-1} + (1 - \gamma_j) \delta_{i,j}$$

$$\delta_{i,j} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{otherwise} \end{cases}$$

Take the absolute values on both sides and sum all the equations over $i = 0, 1, \dots, N$.

$$\sum_{i=0}^N \left| \frac{\partial S_{\Delta}}{\partial y_i} \right| \leq \frac{h_j^2}{9\sqrt{3}} \sum_{i=0}^N \left[\left| \frac{\partial M_{j-1}}{\partial y_i} \right| + \left| \frac{\partial M_j}{\partial y_i} \right| \right] + 1. \quad (21)$$

Now, differentiate both sides of Equation (15) about y_i .

$$\begin{bmatrix} \frac{\partial M_0}{\partial y_i} \\ \frac{\partial M_1}{\partial y_i} \\ \vdots \\ \frac{\partial M_N}{\partial y_i} \end{bmatrix} = \begin{bmatrix} 2 & \lambda_0 & \dots & 0 \\ \mu_1 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial d_0}{\partial y_i} \\ \frac{\partial d_1}{\partial y_i} \\ \vdots \\ \frac{\partial d_N}{\partial y_i} \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} \frac{\partial d_0}{\partial y_i} \\ \frac{\partial d_1}{\partial y_i} \\ \vdots \\ \frac{\partial d_N}{\partial y_i} \end{bmatrix} \quad (22)$$

Notice:

$$\frac{\partial d_j}{\partial y_i} = \frac{6\delta_{i,j+1}}{h_{j+1}(h_j + h_{j+1})} - \frac{6\delta_{i,j}}{h_j h_{j+1}} + \frac{6\delta_{i,j-1}}{h_j(h_j + h_{j+1})}$$

Take the absolute values on both sides of Equations (22), and sum over $i = 0, 1, \dots, N$. The following is obtained.

$$\sum_{i=0}^N \begin{bmatrix} \left| \frac{\partial M_0}{\partial y_i} \right| \\ \left| \frac{\partial M_1}{\partial y_i} \right| \\ \vdots \\ \left| \frac{\partial M_N}{\partial y_i} \right| \end{bmatrix} \leq \|\mathbf{B}^{-1}\| \sum_{i=0}^N \begin{bmatrix} \left| \frac{\partial d_0}{\partial y_i} \right| \\ \left| \frac{\partial d_1}{\partial y_i} \right| \\ \vdots \\ \left| \frac{\partial d_N}{\partial y_i} \right| \end{bmatrix} \leq \frac{12}{h_j h_{j+1}} \|\mathbf{B}^{-1}\|$$

in which $\|\mathbf{B}^{-1}\|$ is the row-max norm of matrix \mathbf{B}^{-1} (p. 20 [15]). For proper end conditions ($\lambda_0, \mu_N < 2$), [15] showed the following is true.

$$\|\mathbf{B}^{-1}\| \leq \max \left[(2 - \lambda_0)^{-1}, (2 - \mu_N)^{-1}, 1 \right]$$

Therefore,

$$\sum_{i=0}^N \left| \frac{\partial S_{\Delta}}{\partial y_i} \right| \leq 1 + \frac{8\sqrt{3}\eta}{9},$$

$$\eta = \max \left[(2 - \lambda_0)^{-1}, (2 - \mu_N)^{-1}, 1 \right] \left(\frac{h_j}{h_{j+1}} \right)$$

Thus, for meshes with bounded $\left(\frac{h_j}{h_{j+1}} \right)$, the quantity $\sum_{i=0}^N \left| \frac{\partial S_{\Delta}}{\partial y_i} \right|$ is bounded.

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