

Investigating Quasi Error Free (QEF) Operation with Turbo Codes

Ksenija Laković, Christopher Jones and John Villasenor

UCLA Electrical Engineering 56-125B Engineering IV, Los Angeles, CA 90095

Tel: (+01) 310 825 5178, Fax: (+01) 310 825 7928

E-mail: ksenija, christop, villa@icsl.ucla.edu

Abstract: We investigate the possibility of obtaining quasi error free (QEF) performance, i.e. residual BER requirements of 10^{-10} or lower, with schemes based on a turbo code in combination with an outer block code. We study the failure modes of these schemes, and propose a method of estimating a lower bound on QEF operating point, on the basis of measurements in the low SNR range. Results indicate that moderate levels of error correcting ability in an outer code ($T \leq 10$) are not sufficient to ensure QEF operation.

Keywords: turbo code, quasi error free, error floor

1. INTRODUCTION

Parallel concatenated codes [1] have received extensive attention due to their ability to obtain good performance over relatively noisy channels (bit error rates of the order of 10^{-5} to 10^{-6} can be achieved at signal to noise ratios less than a decibel away from modulation constrained capacities). An equally important but less well studied question concerns their performance in systems with extremely stringent, *quasi error free* (QEF) residual BER requirements of 10^{-10} or lower. Such requirements arise, for example, in some satellite communications systems.

Since parallel concatenated codes are generally known to present an error floor (BER of 10^{-6} – 10^{-7} , even for relatively large E_b/N_o values), they are not able to guarantee QEF performance alone. In this paper we explore the potential solution based on turbo codes in combination with an outer block code to correct residual turbo decoding errors. We furthermore introduce a method for establishing a lower bound on the SNR required for QEF operation. We demonstrate that the bound can be initialized using measurements from the low SNR range, where reliable results can be obtained with relatively short simulation sequences.

In Section 2, we propose the method for generating our bound. Section 3 analyzes data that result from simulation of many turbo coded blocks and discusses failure mode characteristics for a rate (1,3) BPSK realization of the codes in [1] and for a realization of the rate (2,3) 8 PSK code of [2]. Section 4 concludes the paper.

2. A LOWER BOUND ON BLOCK ERROR RATE

Consider a rate (k, n) turbo encoder, and assume that n coded bits are transmitted over a Gaussian channel using M -dimensional modulation. Denote the input sequence of $k \cdot L$ information bits with $U = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_L)$, where $\mathbf{u}_i = [u_{i1}, \dots, u_{ik}]$, $i = 1, \dots, L$, and the transmit sequence with $V = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_L)$, $\mathbf{v}_i = [v_{i1}, \dots, v_{iM}]$, $i = 1, \dots, L$. As an example, the notations are presented in Figure 1, for the two schemes considered in this paper.

Denote the noise introduced during the transmission with $W = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_L)$, $\mathbf{w}_i = [w_{i1}, \dots, w_{iM}]$, where w_{iM} , $i = 1, \dots, L$, $j = 1, \dots, M$ are the realizations of statistically independent and identically distributed Gaussian random variables, with zero mean and variance σ^2 . For a particular M -dimensional, length L noise vector, the noise power introduced along each dimension is equal to:

$$P_{L,j} = \sum_{i=1}^L w_{ij}^2, \quad j = 1, \dots, M \quad (1)$$

and $\frac{P_{L,j}}{\sigma^2}$ are realizations of a random variable that has a chi-square probability density function with L degrees of freedom [3].

At the receiver, using the iterative procedure (I iterations), the sequence $X = V + W$ is decoded into \hat{U}^I . We introduce an assumption that it is possible to directly relate the number of bit errors (η_{err}^I) in the block \hat{U}^I to the noise power in the block X . More specifically, for different threshold values $T \geq 0$, we define a critical block noise power $P_{L,crit}^T$ and assume that the probability of having more than T errors is equal to the probability of having the average noise power per dimension greater than $P_{L,crit}^T$, which can be written as:

$$P(\eta_{err}^I > T) = P\left(\frac{P_L}{\sigma^2} > \frac{P_{L,crit}^T}{\sigma^2}\right) \quad (2)$$

$$= 1 - \Phi_{\chi_L^2}\left(\frac{P_{L,crit}^T}{\sigma^2}\right) \quad (3)$$

$\Phi_{\chi_L^2}\left(\frac{P_{L,crit}^T}{\sigma^2}\right)$ represents the value of chi-square cumulative distribution function (cdf) at $\frac{P_{L,crit}^T}{\sigma^2}$.

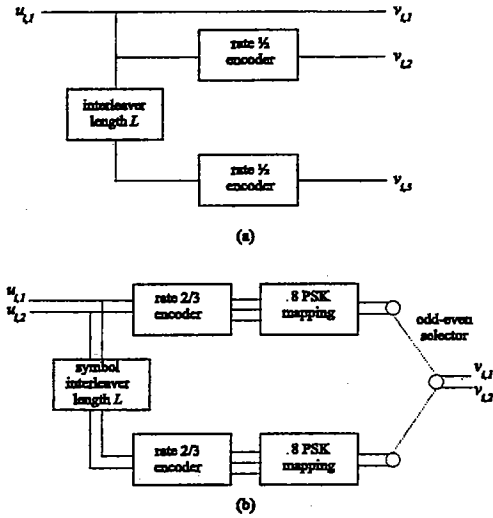


Figure 1: (a) Rate (1,3) turbo code, with binary modulation (TBPSK). Coded bits are sent independently, $M = 3$. (b) Rate (2,3) turbo-trellis code (TTCM). Coded bits are mapped to two-dimensional symbols, using 8PSK, $M = 2$.

It can be easily noticed that the estimation of $P_{L,crit}^T$ is equivalent to the estimation of $\sigma_{L,crit}^T$ since:

$$\sigma_{L,j}^2 = \frac{\sum_{i=1}^L w_{i,j}^2}{L} = \frac{P_{L,j}}{L}. \quad (4)$$

Furthermore, E_b/N_o and σ^2 are related by:

$$E_b/N_o = 10 \log_{10} \frac{E\{\mathbf{v}_i^2\}}{2 \cdot \sigma^2 \cdot k} \quad (5)$$

where $E\{\mathbf{v}_i^2\}$ represents the average power per transmitted symbol ($E\{\mathbf{v}_i^2\} = E\{\sum_{j=1}^M v_{i,j}^2\}$), and k is the number of information bits per symbol \mathbf{v}_i . Therefore, considering the relations (3), (4), and (5), the chi-square bound on the probability of block failure can be expressed in terms of $E_b/N_{o,L,crit}^T$, as :

$$P(\eta_{err}^I > T) = 1 - \Phi_{\chi_L^2} \left(L \cdot \frac{\sigma_{L,crit}^T}{\sigma^2} \right) \quad (6)$$

$$= 1 - \Phi_{\chi_L^2} \left(L \cdot 10^{(E_b/N_o - E_b/N_{o,L,crit}^T)/10} \right) \quad (7)$$

The estimation of $E_b/N_{o,L,crit}^T$ is described in the following section.

3. DATA ANALYSIS

3.1. Turbo-Trellis Coding Scheme (TTCM)

In this section we consider the TTCM code, with eight-state constituent encoders, and $\{11,02,04\}$ generator polynomials, as in [2]. We perform the estimation of $E_b/N_{o,L,crit}^T$, compute the bound on the probability of block failure (7), and compare it to simulation results. We use the blocks of length $L = 10000$, and measure the number of decoding errors after 8 decoding iterations. The block error data were gathered by running blocks to target a desired average signal to noise ratio $E[E_b/N_o]$. The actual signal to noise ratio of the block is measured as:

$$E_b/N_{o,L} = 10 \log_{10} \frac{\sum_{j=1}^M \sum_{i=1}^L v_{i,j}^2}{2 \cdot k \cdot \sum_{j=1}^M \sum_{i=1}^L (x_{i,j} - v_{i,j})^2} \quad (8)$$

where $v_{i,j}$ and $x_{i,j}$ represent the sent and received symbols, and k the number of information bits per symbol, as in Section 2.

For each block $E_b/N_{o,L}$ is computed using this relation (8), and the number of errors (η_{err}^I) is recorded. $E_b/N_{o,L}$ values are then quantized, and the curves are plotted against the percentage of blocks at a block $E_b/N_{o,L}$ with $\eta_{err}^I > T$, as depicted in Figure 2. $E_b/N_{o,L,crit}^T$ is estimated from this plot, as the value of $E_b/N_{o,L}$ in which the percentage of blocks with $\eta_{err}^I > T$ is equal to 0.5. More specifically, the SNR threshold used to separate 'bad' blocks from 'good' blocks lies at the midpoint of the block error probability range and is used to initialize the chi-square bound. Hence, the performance predicted by the analytic bound (7) assumes *ideal* behaviour in the code, where blocks having actual signal to noise ratios less than some critical value are assumed to decode with more than T errors and blocks with SNRs above this critical value are assumed to decode with less than T errors.

Simulated results, however, demonstrate deviations from this model. In order to quantify the level of deviation we introduce two measures. One is *transition zone width*, defined as the difference of the values of $E_b/N_{o,L}$ in which the percentage of erroneous blocks (blocks with $\eta_{err}^I > T$) is equal to 0.01 and 0.99 respectively:

$$w_L^T = E_b/N_{o,L}^{T,0.01} - E_b/N_{o,L}^{T,0.99}$$

Another is *relative symmetry misadjustment*, defined as:

$$\delta_L^T = \frac{\left| 0.5(E_b/N_{o,L}^{T,0.99} + E_b/N_{o,L}^{T,0.01}) - E_b/N_{o,L,crit}^T \right|}{w_L^T}$$

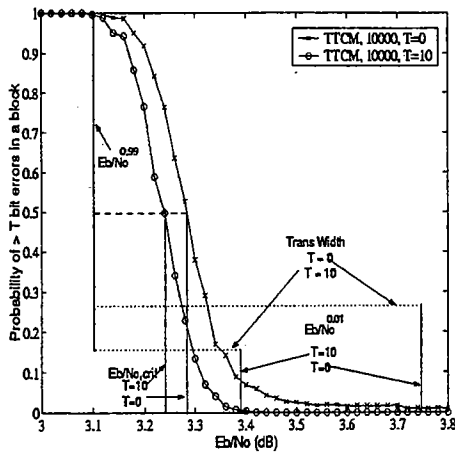


Figure 2: Illustration of block error symmetry properties and definitions for TTCM, $L = 10000$.

T	$E_b/N_o^{T,0.99}$	$E_b/N_o^{T,crit}$	$E_b/N_o^{T,0.01}$	w_L^T	δ_L^T
0	3.135	3.280	3.750	0.615	0.264
2	3.120	3.265	3.495	0.375	0.113
4	3.110	3.255	3.425	0.315	0.040
6	3.105	3.235	3.390	0.285	0.044
10	3.100	3.225	3.365	0.265	0.028

Figure 3: Symmetry progression for TTCM, $L = 10000$.

We present these parameters for $T \in \{0, 10\}$ in Figure 3. We notice a significant reduction in the misadjustment with increases in T , and thus anticipate closer adherence to the chi-square bound. This is verified in Figure 4, which provides the comparison between the simulation results, and results estimated with relation (7), using the values $E_b/N_o^{T,crit}$ from Figure 3. As T moves from 0 to 10 the block error floor is reduced by two orders of magnitude. More specifically, we note a shift from 10^{-2} for $T = 0$ to 10^{-4} for $T = 10$. We also add that this relatively modest level of error correction clearly does not enable the code to approach QEF levels.

3.2. Binary Turbo-Coding Scheme (TBPSK)

Using the procedure described in the preceding section, we estimated $E_b/N_o^{T,crit}$ for the binary turbo-coding scheme (BPSK) of [4] with eight-state, d_2 optimized constituent codes ($d(D) = 15$, $n(D) = 13$). Simulations were run for blocks of length $L = 10000$ and 8 decoding iterations were used. Even for low levels of T we noticed levels of symmetry similar to those found with high levels of T in the TTCM scheme. This observation supports the better initial adherence to the chi-square bound and simultaneously points toward the significantly lower error floor of the TBPSK scheme. Figure 5 shows that the

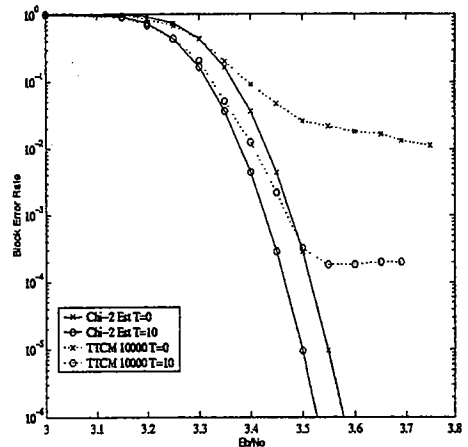


Figure 4: Length 10000 TTCM against bound for $T = 0$, $T = 10$.

$T = 0$ block error floor lies near 10^{-4} and that the corresponding bound lies less than 0.1 dB away until the error floor occurs.

With the increase of T , $E_b/N_o^{T,crit}$ values decrease at a lower rate in this case than in the case of the TTCM scheme. We can therefore not expect a reduction in the error floor beyond two orders of magnitude using T values less than 10 since the more initially asymmetric case of TTCM realized just over two orders of magnitude gain with $T = 10$. A simulation that sought to gather a reliable collection of block errors for the $T = 10$ error floor would require $10^8 \cdot L$ data symbols, which translates to 10^{12} bits per E_b/N_o with a TBPSK scheme having a block length equal to 10^4 . Therefore, values extending to the error floor are not included in Figure 5, where we show the chi-square bound for $T = 10$, and simulation results in the lower E_b/N_o range.

4. CONCLUSIONS

We have studied the improvement in error floor performance due to embedding a turbo code within an outer block code, and have proposed a method for estimating lower bounds on the probability of block failure. The procedure is based on the assumption that block failure is directly related to the actual level of noise power introduced during transmission and that an SNR threshold delineating blocks likely to have more than T bit errors from blocks likely to have less than T errors can be defined. Data analyses indicate that for low values of T (0-5) significant deviations from the ideal model exist. Increases in the correcting level T cause a shift in turbo code error performance toward the ideal model. This shift can be observed by both an increase in symmetry as shown in Figure 2 and by improved adherence to the chi-square bound (Figure 4). This effect is well

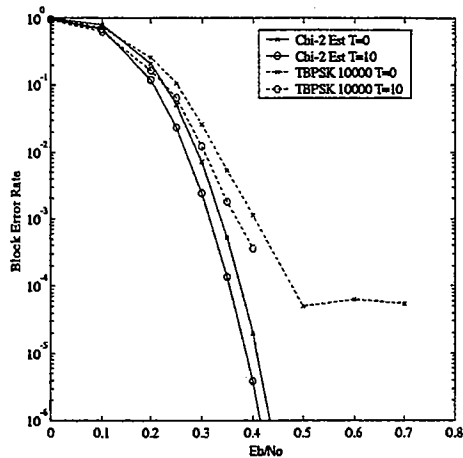


Figure 5: Length 10000 TBPSK against bound for $T = 0$, $T = 10$.

displayed in the case of the TTCM code, but is less apparent for TBPSK both due to a lower overall error floor and to a high degree of initial symmetry in the $T = 0$ case. For both schemes it is apparent that practical error correcting levels T (0-10) can reduce the error floor by at most two to three orders of magnitude. Hence, achievement of QEF operation cannot be attained through the use of an outer code alone. Another code enhancement mechanism, such as a block interleaver placed between the outer and inner codes, should be investigated.

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