DYNAMIC MODELS OF A ROTARY DOUBLE INVERTED PENDULUM SYSTEM

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ABSTRACT

This paper describes the dynamic models of a double rotary inverted pendulum, which has been developed for the laboratory experiments. Two different-length rigid pendulums are connected to a horizontallly rotating disc which is attached directly to a DC motor. The derivation of the dynamical equations and the linearized model are described. Finally, the time responses of open-loop system are shown and compared with the experimental data to verify the model validity.

1. INTRODUCTION

Inverted pendulum is one of widely used apparatus in control laboratories to demonstrate the modern control theory applications [1]. The conventional inverted pendulum has the structure of the cart-type system, which has the limitation of the cart length. For this reason, the rotary inverted pendulum was introduced to compensate this restriction [2]. In this work, the laboratory-scale system has been developed as part of the senior project [3] and subsequently improved to be the rotary double pendulum [4]. It is controlled in real-time by a low-end computer using RT-Linux as the operating system.

The objective of this paper is to obtain the dynamics of this system. Full derivation of the mathematical equations is presented based on the Lagrange equation. The structure of this paper is organized as follows. First, the modelling of the rotary double inverted pendulum will be described in section 2-4. It includes the dynamical equations and the linearized model obtained from the linearization at the equilibrium points. Section 5 shows the simulation results of the open-loop responses compared with the experimental data, and finally, conclusions will be discussed in section 6.

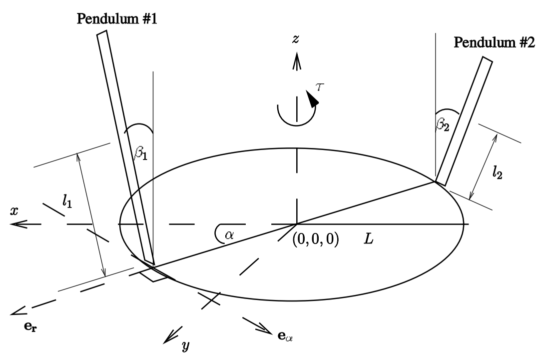
2. ROTARY DOUBLE INVERTED PENDULUMS

Consider the rotary double inverted pendulum system in Fig. 1. Two pendulums are connected to a rotating disc which is driven by a DC motor. The control objective is to balance the first pendulum in the upright position and to keep the other pendulum to be at rest in hanging position. The system variable are described as follows.

τ : The external torque applied to the disc (N·m)
α : The angular displacement of the rotating disc (rad)
β_1 : The 1st pendulum angle with respect to the vertical axis. (rad)
β_2 : The 2nd pendulum angle with respect to the vertical axis. (rad)

The system parameters are described in Table 1. In next section, The nonlinear dynamic equations can be derived by using the Lagrange equations. Then, we

(a) Physical System



(b) Schematic Drawing

Fig. 1. Rotary Inverted Pendulum System

Table 1. Parameters of rotary inverted pendulum system

Parameter	Notation	Value	Unit
Inertia of the rotating disc	J_0	0.06	kg-m^2
Inertia of the 1st pendulum	J_1	0.008	kg-m^2
Inertia of the 2nd pendulum	J_2	0.002	kg-m^2
Viscous coef. of rotating disc	c_0	0.004	N-m·s
Viscous coef. of the 1st pendulum	c_1	0.0031	N-m·s
Viscous coef. of the 2nd pendulum	c_2	0.00088	N-m·s
Mass of the 1st pendulum	m_1	0.25	kg
Mass of the 2nd pendulum	m_2	0.13	kg
The displacement from the joint to the c.m. of the 1st pendulum	l_1	0.24	m
The displacement from the joint to the c.m. of the 2nd pendulum	l_2	0.13	m
The radius of the rotating disc	L	0.172	m
The gravity constant	g	9.8	m/s
Torque constant	K_m	0.005	N.m/A
Back emf. constant	K_b	0.001	Volt/rad
Resistance in motor circuit	R	2	Ω

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linearize the nonlinear equations to obtain the linear model which will be used in the controller design.

3. NONLINEAR DYNAMIC MODEL

The mathematical equations can be derived using the Lagrangian $\mathcal{L} = K - U$ where K is the kinetic energy and U is the potential energy. The Lagrange equation is as follows.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial W}{\partial \dot{q}_i} = F_i \tag{1}$$

where F_i, q_i, W are the generalized forces, the generalized coordinates and the loss energy, respectively.

Next, we will calculate the kinetic energy, the potential energy, and the loss energy. The system parameters used in the following are described in Table 1.

Kinetic Energy

The total kinetic energy of the system is

$$K = \frac{1}{2}J_0\dot{\alpha}^2 + \frac{1}{2}J_1\dot{\beta}_1^2 + \frac{1}{2}J_2\dot{\beta}_2^2 + \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \tag{2}$$

where v_1, v_2 are the velocities of the center of mass of the 1st pendulum and the 2nd pendulum, respectively.

In order to calculate the velocity of each pendulum, we need to find the cartesian coordinate of the center of mass by considering Fig. 2. Let $\{\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\}$ be

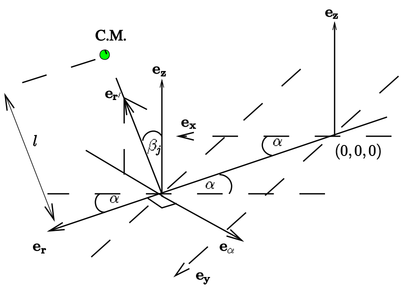

Fig. 2. Vector representatives in the cartesian and polar coordinate

the set of unit vectors of the cartesian coordinate and $\{\mathbf{e_r}, \mathbf{e}_\alpha, \mathbf{e_z}\}$ be the set of unit vectors in the angular coordinate. $\mathbf{e_{r'}}$ is the unit vector in the direction along the pendulum rod. Therefore,

$$\mathbf{e_{r'}} = \mathbf{e_z}\cos\beta_j - \mathbf{e}_\alpha \sin\beta_j \tag{3}$$

We can find the coordinate vector of the C.M. of the j^{th} pendulum as follow.

$$\overrightarrow{\text{C.M.}}_{\{r\alpha z\}} = l_j\mathbf{e_{r'}} + \mathbf{e_r}L \tag{4}$$

$$= \mathbf{e_z}l_j\cos\beta_j - \mathbf{e}_\alpha l_j \sin\beta_j + L\mathbf{e_r} \tag{5}$$

We can transform (5) into the cartesian coordinate by using

$$\mathbf{e}_\alpha = -\mathbf{e_x}\sin\alpha + \mathbf{e_y}\cos\alpha \tag{6}$$

$$\mathbf{e_r} = \mathbf{e_x}\cos\alpha + \mathbf{e_y}\sin\alpha \tag{7}$$

Thus, the cartesian coordinate of the C.M of the j^{th} pendulum is

$$\overrightarrow{\text{C.M.}}_{\{xyz\}} = (l_j\sin\beta_j\sin\alpha + L\cos\alpha)\mathbf{e_x} + (L\sin\alpha - l_j\sin\beta_j\cos\alpha)\mathbf{e_y} + (l_j\cos\beta_j)\mathbf{e_z} \tag{8}$$

Hence, the velocities of the j^{th} pendulum in xyz axis are

$$v_j^x = l_j\sin\beta_j\cos\alpha\dot{\alpha} + \sin\alpha(l_j\cos\beta_j\dot{\beta}_j - L\dot{\alpha}) \tag{9}$$

$$v_j^y = l_j\sin\beta_j\sin\alpha\dot{\alpha} - \cos\alpha(l_j\cos\beta_j\dot{\beta}_j - L\dot{\alpha}) \tag{10}$$

$$v_j^z = -l_j\sin\beta_j\dot{\beta}_j \tag{11}$$

Therefore,

$$\frac{1}{2}m_jv_j^2 = \frac{1}{2}m_j(l_j\sin\beta_j\dot{\alpha})^2 + \frac{1}{2}m_j(L\dot{\alpha})^2 + \frac{1}{2}m_j(l_j\dot{\beta}_j)^2 - ml_jL\cos\beta_j\dot{\beta}_j\dot{\alpha} \tag{12}$$

Substitute (12) into (2), we obtain the total kinetic energy.

$$K = \frac{1}{2}J_0\dot{\alpha}^2 + \frac{1}{2}J_1\dot{\beta}_1^2 + \frac{1}{2}J_2\dot{\beta}_2^2 + \frac{1}{2}m_1(l_1\sin\beta_1\dot{\alpha})^2 + \frac{1}{2}m_1(L\dot{\alpha})^2 + \frac{1}{2}m_1(l_1\dot{\beta}_1)^2 - ml_1L\cos\beta_1\dot{\beta}_1\dot{\alpha} + \frac{1}{2}m_2(l_2\sin\beta_2\dot{\alpha})^2 + \frac{1}{2}m_2(L\dot{\alpha})^2 + \frac{1}{2}m_2(l_2\dot{\beta}_2)^2 - m_2l_2L\cos\beta_2\dot{\beta}_2\dot{\alpha} \tag{13}$$

Potential Energy

The potential energy of two pendulums is

$$U = m_1gl_1\cos\beta_1 + m_2gl_2\cos\beta_2 \tag{14}$$

Loss energy

The energy lost due to the frictional force is

$$W = \frac{1}{2}c_0\dot{\alpha}^2 + \frac{1}{2}c_1\dot{\beta}_1^2 + \frac{1}{2}c_2\dot{\beta}_1^2 \tag{15}$$

From (13) and (14), we can write the Lagrange function $\mathcal{L} = K - U$ as follows.

$$\mathcal{L} = \frac{1}{2}J_0\dot{\alpha}^2 + \frac{1}{2}J_1\dot{\beta}_1^2 + \frac{1}{2}J_2\dot{\beta}_2^2 + \frac{1}{2}m_1(l_1\sin\beta_1\dot{\alpha})^2 + \frac{1}{2}m_1(L\dot{\alpha})^2 + \frac{1}{2}m_1(l_1\dot{\beta}_1)^2 - ml_1L\cos\beta_1\dot{\beta}_1\dot{\alpha} + \frac{1}{2}m_2(l_2\sin\beta_2\dot{\alpha})^2 + \frac{1}{2}m_2(L\dot{\alpha})^2 + \frac{1}{2}m_2(l_2\dot{\beta}_2)^2 - m_2l_2L\cos\beta_2\dot{\beta}_2\dot{\alpha} - m_1gl_1\cos\beta_1 - m_2gl_2\cos\beta_2 \tag{16}$$

The set of generalized coordinates of this system is $\{\alpha, \beta_1, \beta_2\}$ and the corresponding generalized forces is $\{\tau, 0, 0\}$.

From (1) and (16), the dynamic equations of this system are as follow.

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta}_1 \\ \ddot{\beta}_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{17}$$

where

$$P_{11} = J_0 + m_1 l_1^2 \sin^2 \beta_1 + m_1 L^2 + m_2 l_2^2 \sin^2 \beta_2 + m_2 L^2$$
$$P_{12} = -m_1 l_1 L \cos \beta_1$$
$$P_{13} = -m_2 l_2 L \cos \beta_2$$
$$P_{21} = -m_1 l_1 L \cos \beta_1$$
$$P_{22} = J_1 + m_1 l_1^2 \quad (18)$$
$$P_{23} = 0$$
$$P_{31} = -m_2 l_2 L \cos \beta_2$$
$$P_{32} = 0$$
$$P_{33} = J_2 + m_2 l_2^2$$

$$\begin{aligned} p_1 &= m_1 l_1^2 \dot{\beta}_1 \dot{\alpha} \sin(2\beta_1) + m_1 l_1 L \dot{\beta}_1^2 \sin \beta_1 + c_0 \dot{\alpha} \\ &\quad + m_2 l_2^2 \dot{\beta}_2 \dot{\alpha} \sin(2\beta_2) + m_2 l_2 L \dot{\beta}_2^2 \sin \beta_2 \\ p_2 &= -m_1 l_1^2 \dot{\alpha}^2 \sin \beta_1 \cos \beta_1 - m_1 g l_1 \sin \beta_1 + c_1 \dot{\beta}_1 \\ p_3 &= -m_2 l_2^2 \dot{\alpha}^2 \sin \beta_2 \cos \beta_2 - m_2 g l_2 \sin \beta_2 + c_2 \dot{\beta}_2 \end{aligned} \quad (19)$$

From (17), the control input is the torque applied to the hub of the rotating disc. In practice, we use a DC motor to drive the rotating disc, so the control input is changed to be the voltage.

In case of neglecting the effect of inductor in the motor circuit, the relationship between the torque, τ, and the voltage, V, is given by

$$\tau = \frac{K_m V}{R} - \frac{K_m K_b \dot{\alpha}}{R} \quad (20)$$

where K_m, K_b, and R are the motor parameters shown in Table 1. Substitute (20) into (17), we obtain the rotary inverted pendulum plus the motor dynamics as

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta}_1 \\ \ddot{\beta}_2 \end{bmatrix} + \begin{bmatrix} p'_1 \\ p'_2 \\ p'_3 \end{bmatrix} = \frac{K_m}{R} \begin{bmatrix} V \\ 0 \end{bmatrix} \quad (21)$$

where

$$\begin{aligned} p'_1 &= m_1 l_1^2 \dot{\beta}_1 \dot{\alpha} \sin(2\beta_1) + m_1 l_1 L \dot{\beta}_1^2 \sin \beta_1 \\ &\quad + m_2 l_2^2 \dot{\beta}_2 \dot{\alpha} \sin(2\beta_2) + m_2 l_2 L \dot{\beta}_2^2 \sin \beta_2 \\ &\quad + \left(c_0 + \frac{K_m K_b}{R}\right) \dot{\alpha} \quad (22) \\ p'_2 &= -m_1 l_1^2 \dot{\alpha}^2 \sin \beta_1 \cos \beta_1 - m_1 g l_1 \sin \beta_1 + c_1 \dot{\beta}_1 \\ p'_3 &= -m_2 l_2^2 \dot{\alpha}^2 \sin \beta_2 \cos \beta_2 - m_2 g l_2 \sin \beta_2 + c_2 \dot{\beta}_2 \end{aligned}$$

4. LINEARIZED DYNAMIC MODEL

Since the control objective is to keep two pendulums vertically, the operating points are $\beta_1^* = \beta_2^* = 0°$. From the Taylor series, at the small value of x,

$$x^2 \approx 0\ ,\ \ \sin x \approx x\ ,\ \ \sin^2 x \approx 0.$$

Hence, we make the approximation of the nonlinear equations in (21) around the equilibrium points, which are $\beta_1^* = \beta_2^* = 0$.

$$(J_0 + m_1 L^2 + m_2 L^2)\ddot{\alpha} - m_1 l_1 L \ddot{\beta}_1 - m_2 l_2 L \ddot{\beta}_2 + \dot{\alpha}\left(c_0 + \frac{K_m K_b}{R}\right) = \frac{K_m}{R}\tau \quad (23)$$

$$-m_1 l_1 L \ddot{\alpha} + (J_1 + m_1 l_1^2)\ddot{\beta}_1 - m_1 g l_1 \beta_1 + c_1 \dot{\beta}_1 = 0 \quad (24)$$

$$-m_2 l_2 L \ddot{\alpha} + (J_2 + m_2 l_2^2)\ddot{\beta}_2 - m_2 g l_2 \beta_2 + c_2 \dot{\beta}_2 = 0 \quad (25)$$

Let $x = \begin{bmatrix} \alpha & \beta_1 & \beta_2 & \dot{\alpha} & \dot{\beta}_1 & \dot{\beta}_2 \end{bmatrix}^T$ be a state variable, and $u = V$ be the input, we can arrange (23)-(25) into the form,

$$\begin{aligned} E\dot{x} &= Fx + Gu \\ \dot{x} &= E^{-1}Fx + E^{-1}Gu =: Ax + Bu \end{aligned} \quad (26)$$

where,

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_0 + m_1 L^2 + m_2 L^2 & -m_1 l_1 L & -m_2 l_2 L \\ 0 & 0 & 0 & -m_1 l_1 L & J_1 + m_1 l_1^2 & 0 \\ 0 & 0 & 0 & -m_2 l_2 L & 0 & J_2 + m_2 l_2^2 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\left(c_0 + \frac{K_m K_b}{R}\right) & 0 & 0 \\ 0 & m_1 g l_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & m_2 g l_2 & 0 & 0 & -c_2 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 & \frac{K_m}{R} & 0 & 0 \end{bmatrix}^T$$

Next we will find the linearized model using the equilibrium points where $\beta_1^* = \beta_2^* = 180°$ to compare the dynamics with the nonlinear model. Similarly, we expand the Taylor series at $x = \pi$

$$\sin x \approx \pi - x\ ,\ \ \sin^2 x \approx 0\ ,\ \ \cos x \approx -1$$

The approximated dynamical equations become

$$(J_0 + m_1 L^2 + m_2 L^2)\ddot{\alpha} + m_1 l_1 L \ddot{\beta}_1 + m_2 l_2 L \ddot{\beta}_2 + \dot{\alpha}\left(c_0 + \frac{K_m K_b}{R}\right) = \frac{K_m}{R}\tau \quad (27)$$

$$m_1 l_1 L \ddot{\alpha} + (J_1 + m_1 l_1^2)\ddot{\beta}_1 + m_1 g l_1 (\beta_1 - \pi) + c_1 \dot{\beta}_1 = 0 \quad (28)$$

$$m_2 l_2 L \ddot{\alpha} + (J_2 + m_2 l_2^2)\ddot{\beta}_2 + m_2 g l_2 (\beta_2 - \pi) + c_2 \dot{\beta}_2 = 0 \quad (29)$$

Let $x = \begin{bmatrix} \alpha & \beta_1 - \pi & \beta_2 - \pi & \dot{\alpha} & \dot{\beta}_1 & \dot{\beta}_2 \end{bmatrix}^T$ be a state variable, and $u = V$ be the input, we can write (27)-(29) into the form,

$$\begin{aligned} E_1\dot{x} &= F_1 x + G_1 u \\ \dot{x} &= E_1^{-1}F_1 x + E_1^{-1}G_1 u =: A_1 x + B_1 u \end{aligned} \quad (30)$$

where,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_0 + m_1 L^2 + m_2 L^2 & m_1 l_1 L & m_2 l_2 L \\ 0 & 0 & 0 & m_1 l_1 L & J_1 + m_1 l_1^2 & 0 \\ 0 & 0 & 0 & m_2 l_2 L & 0 & J_2 + m_2 l_2^2 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\left(c_0 + \frac{K_m K_b}{R}\right) & 0 & 0 \\ 0 & -m_1 g l_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & -m_2 g l_2 & 0 & 0 & -c_2 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{K_m}{R} & 0 & 0 \end{bmatrix}^T$$

5. SIMULATION RESULTS

In this section, we examine the zero-input responses of the pendulum angle near the operating points $\beta_1^* = \beta_2^* = 0^\circ$ where $\beta_1(0) = 68^\circ$ and $\beta_2(0) = 70^\circ$ in Fig. 3. The responses of the nonlinear model and the experimental data are generally the same by considering the magnitude and the oscillation period of the responses. However, the experimental responses converge more rapidly than those of the simulations due to the friction effect at the low velocities which may be higher in the real system. Likewise, the zero-input responses near the equilibrium points $\beta_1^* = \beta_2^* = 180^\circ$ where $\beta_1(0) = 175^\circ$ and $\beta_2(0) = 178^\circ$ are shown in Fig. 4. In this case, we use the linear model in (30). We can see that the longer pendulum angle (β_1) responses are almost the same but not in the case of the shorter one (β_2). This can be explained that since the initial velocities depend on the pendulum length and the initial angle, the shorter pendulum start moving with the less initial velocity. Therefore, the friction considerably had an effect on the shorter pendulum motion.

In addition, we have conducted parameter estimation and designed the LQR controller based on the model derived here to stabilize the longer pendulum to be the upright position and the shorter one to the rest position [4]. The control results will be presented in the future.

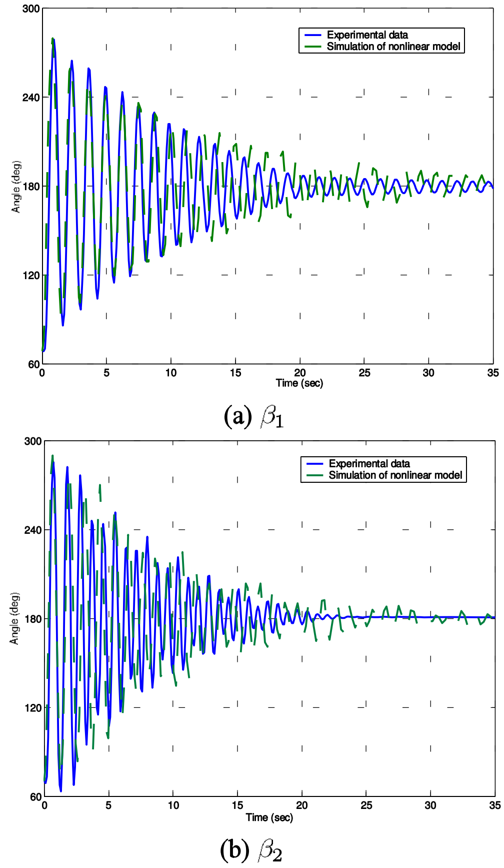

Fig. 3. The zero-input response of β_1 and β_2: $\beta_1(0) = 68^\circ$, $\beta_2(0) = 70^\circ$

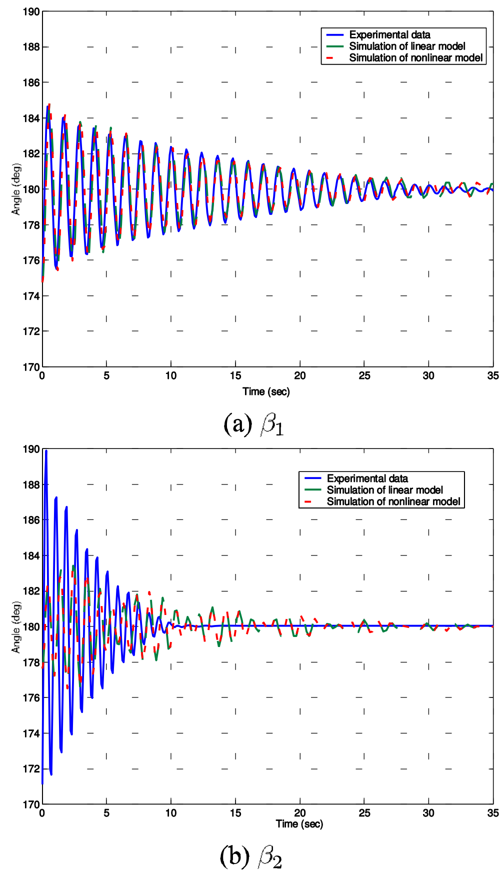

Fig. 4. The zero-input response of β_1 and β_2: $\beta_1(0) = 175^\circ$, $\beta_2(0) = 178^\circ$

6. CONCLUSIONS

The dynamical equations of the rotary double pendulum system were derived in this work. Thereafter, the linear model was obtained by the linearization at the specified operating points. The simulation results shown that the derived model in (27)-(29) give the similar responses to those of the experimental data.

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