Wireless Ad Hoc Networks with Tunable Topology

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***Abstract*— We study the set of connectivity and topological properties achievable by *heterogeneous* Wireless Ad Hoc Networks (WANETs). Instead of using the well known Random Geometric Graph (RGG) model, which is more suitable for homogeneous WANETs, we use Geographical Threshold Graphs (GTGs) to model WANETs with tunable topology. The GTG model allows nodes to have different capabilities, e.g., power and bandwidth resources available to nodes could be chosen from a distribution, and it naturally incorporates the path-loss function characteristic of wireless transmission. The relationships between the topological properties and the parameters that characterize the resource pool of the nodes are investigated. For example, we show that by varying the parameters of the GTGs, one can obtain networks with different diameters. Similarly, we show that (i) one can design WANETs with different desired degree distributions, by choosing the nodes from a matching resource distribution, and (ii) one can compute the degree distribution, when the resource parameters of the nodes are fixed. As to be expected, networks with extensive connectivity properties, e.g., possessing heavy-tailed degree distributions, are shown to require significant power resources. However, the model provides flexibility to the designer to tune the network properties and estimate the related costs.**

I. INTRODUCTION

The wireless ad hoc networks (WANET) are complex technological systems that emerged in recent years. They are made up of a group of mobile units equipped with radio transceivers that wish to communicate with each other over wireless channels. The nodes in a WANET relay information for one another in a multi-hop fashion, as a centralized authority is lacking. Thus, nodes in a WANET *self-organize* to form a decentralized communication network that does not rely on any fixed infrastructures.

In general, the network is dynamic with nodes joining and leaving at any time. However, for any fixed instant in time, the wireless network can be modeled as a graph. The number of neighbors that a node can establish wireless links to is known as the node degree. There are several fundamental graph theoretic properties that are critical to network performance: first, connectivity must be ensured for the nodes to communicate among one another; second, the diameter of the graph is important, since it is an upper bound on the hop-count or network latency; third, the degree distribution is essential to understand as the degree distribution has implications on different network performance metrics.

Mathematical modeling of wireless ad hoc networks has attracted much attention recently. A widely adopted model is the random geometric graph (RGG), where nodes are uniformly distributed in a d-dimensional space and the coverage area of each node is a ball with the same radius [1], [2], [3], [4], [5], [6]. The graph is constructed by linking pairs of nodes that are located in each other's coverage area. The circular coverage assumption is quite realistic for open and flat terrain, but it is highly questionable in urban environments in the presence of countless objects such as buildings, trees and walls. A closely related model to RGG is known as the Poisson Boolean model, where the nodes are distributed according to a Poisson point process and each node is given identical circular coverage areas [7], [8]. In addition, other models with an irregular coverage area have been developed [9], [10], [11], [12].

A fundamental property of a wireless ad hoc network is the diameter, which corresponds to the maximum number of hops between any pair of nodes. Thus, generating a graph with a small diameter ensures that the network has low latency. Previous models of wireless ad hoc networks, such as the random geometric graph, typically have a large diameter that scales as $O(\sqrt{n/\ln n})$, where n is the number of nodes in the network. Recently, Helmy [13] and Dixit et al. [14] proposed the construction of wireless ad hoc networks with small average path length by adopting the small world paradigm [15]. Despite these advances, the construction of wireless ad hoc networks with provably small diameter remains an open question.

Another important technique examined in the wireless ad hoc network community is topology control. The goal of this technique is to control the graph topology of in order to ensure network connectivity, reduce energy consumption or minimize radio interference. A great review paper on this topic is found in [16].

Results: Our contributions in this paper are three-fold: first, we develop a generalized model for wireless ad hoc networks with a *tunable* topology; second, we analyze the diameter of the graph generated by our model and derive the conditions for creating graphs with provably small diameters; third, we address the problem of topology control: given a desired degree distribution, we derive a set of conditions that enable us to analytically calculate the required node weight distribution that will generate a graph with the desired degree distribution in our model. Since the weight is used to model a node's resources, our finding can be used to solve the resource allocation problem in generating a graph topology with a desired degree distribution. All of our analytical results are verified by large-scale simulations.

II. THE 2D-GTG MODEL

We first define our model based on the geographical threshold graph (GTG) [17]: n nodes are placed uniformly and independently into the unit disc in $\mathbb{R}^2$; now, two nodes with weights w and w' at a Euclidean distance r are connected by a link if and only if

$$G(w, w')h(r) \geq \theta \tag{1}$$

where we have $\theta \geq 0$ and the function $h(r)$ is assumed to be a decreasing function in r. A non-negative weight w_i, taken randomly and independently from a probability distribution $f(w_i) : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, is assigned to each node v_i, for $i = \{1, 2, \ldots, n\}$. Let the cumulative density function of $f(w)$ be $F(w) = \int_0^w f(u)du$. We will refer to this model as the 2D-GTG model. Note that the random geometric graph model can be obtained from our general model by setting the function $G(w, w')$ to a constant, thus ignoring all node weights.

The motivation for developing this graph model is as follows: in a real-world environment where a wireless ad hoc network is to be deployed, there are many complications that will render the previously discussed RGG model highly unrealistic. For example, there are buildings, trees and walls that obstruct wireless signals. Thus, the circular coverage assumption of the random geometric graph model is highly unrealistic.

Moreover, the wireless nodes can be highly heterogeneous in different characteristics, such as in transmission power, expected lifetime, location and bandwidth. Thus, we assign a quantitative measure w (i.e. node weight) to each node in the network. One can consider the node weight as a quantitative measure of the characteristic of the wireless node, which represents the node's *desirability* or its ability to make or attract links.

In radio communication, the received signal level deceases as the geographical distance between the receiver and the transmitter increases, which is known as the path-loss phenomenon. The path-loss model is represented by the function $h(r) = r^{-\alpha}$, where α can be between 2 and 6 depending on the medium of transmission [18]. Toward the goal of realistically modeling the interplay of these factors, we arrive at the geographical threshold model as specified in Eq. (1). We want to emphasize that we the functions $G(w, w')$ and $h(r)$ are decoupled, with the function $G(w, w')$ characterizing the interaction strength between the pair of nodes and the function $h(r)$ modeling the path-loss phenomenon.

Since our model is very general, we now provide some specific examples where our model is equipped to handle. Consider the case where nodes have heterogeneous power: each node has a transceiver with transmission power and receiver amplifier power both equal to w, which is distributed according to $f(w)$. We obtain the following condition:

$$\frac{ww'}{r^{\alpha}} \geq \theta \tag{2}$$

In Fig. 1(a), node A transmits a signal to node B with power w and the signal is attenuated by the factor $r^{-\alpha}$; upon receiving the signal, node B amplifies the signal by a factor of w'; thus a directional link from node A to node B is established if the amplified signal is greater than the threshold θ. Similarly, node B transmits a signal to node A with power w' and node A amplifies the received signal by a factor of w. Now, a bi-directional link between node A and node B is established.

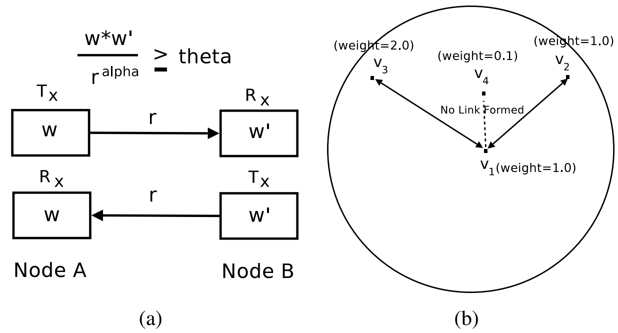

Figure 1. (a) A simple illustration of link formation between two nodes. (b) Irregular coverage: v_1 is not connected to v_4 even though v_4 is very close in distance.

In Fig. 1(b), we provide a simple illustration of how links are formed according to Eq. (2). We have four nodes v_1, v_2, v_3, v_4 with the weights $w_1 = 1$, $w_2 = 1$, $w_3 = 2$, $w_4 = 1/10$. We will only illustrate how node v_1 makes links. The Euclidean distance between node v_1 and all other points are as follows: $||v_1, v_2|| = 3\sqrt{2}/10$, $||v_1, v_3|| = 0.5$, and $||v_1, v_4|| = 0.2$. Let $\theta = 12.75$, and $\alpha = 3$, it follows that v_1 is connected to v_2 and v_3 but not to v_4, even though v_1 is much closer to v_4 than to the other two nodes, thus the coverage area of a node is irregular.

Now, consider another case where the network designer is concerned about node failures. Thus, another scenario would be to prefer making links to nodes with longer uptimes. Modeling a node's uptime by the node weight w, we arrive at the following condition:

$$\frac{w + w'}{r^{\alpha}} \geq \theta \tag{3}$$

In the graph generated in this manner, a node with longer uptime will have an enhanced ability to initiate connections as well as to receive connections. As shown, our model is very general and can be used to establishing links based on any quantifiable node characteristic that is of concern to the network designer.

III. DIAMETER AND CONNECTIVITY

In the design of wireless ad hoc networks, it is desirable to achieve low latency in the graph (i.e. the hop-count between any pair of nodes in the network is small). In other words, a graph with a small diameter is desired. In this section, we consider two fundamental problems on graphs: diameter and connectivity. We first emphasize that the threshold θ is in general a function of the network size

n, that is $\theta(n)$. We give the conditions on the threshold function $\theta(n)$ such that the graph has a desired diameter in general. Furthermore, we derive the conditions on $\theta(n)$, in terms of the cumulative distribution function on weights $F(w)$, such that $diam$ belongs to the classes $diam = O(1)$, $diam = O(\ln^q n)$ and $diam = O(\sqrt{n/\ln n})$, respectively. These classes correspond to an ultra-low, low and high latency network, respectively. Particularly, for these three classes, we give the exact expressions on $\theta(n)$ in the case of the exponentially distributed weights. We then examine the connectivity properties of the graph.

A. Diameter

In this subsection, we give the upper bound on the diameter in our unit-area disk 2D-GTG model. We have n nodes placed uniformly randomly and independently into the disk. Let u and v be two arbitrary nodes. Let us construct the sequence of adjacent squares $S_1, S_2, \ldots, S_{O(1/x)}$, of the size $x \times x$, linking u and v, such that u and v are the centers of the first and last squares, respectively[1] (see Fig. 2). The geometric distance between any two nodes is $r \leq 2/\sqrt{\pi}$. Thus, there are $O(1/x)$ squares on the straight path $u-v$ in total.

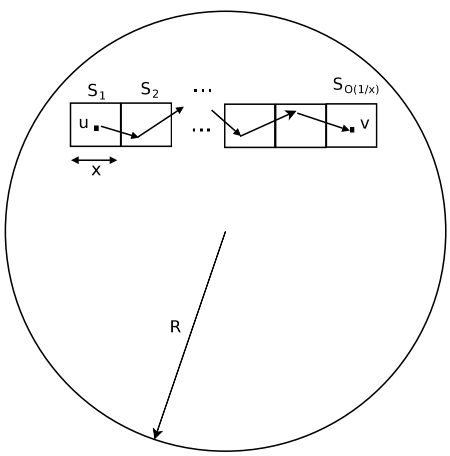

Figure 2. Illustration of our diameter proof technique: a sequence of adjacent squares of size $x \times x$ link an arbitrary pair of nodes u and v in a unit-area disc.

Let V_i be the number of nodes that lie within the square S_i, for $i = 1, 2, \ldots, O(1/x)$. We have $\mathrm{E}[V_i] = nx^2$. We further note that even if the nodes are placed according to the Poisson Point process, the result follows. Using Chernoff bound, the following is satisfied:

$$\Pr[V_i \leq (1-\delta)\mathrm{E}[V_i]] \leq e^{-\mathrm{E}[V_i]\delta^2/2}. \tag{4}$$

Taking $\delta = 1/2$, we get $\Pr[V_i \leq nx^2/2] \leq e^{-nx^2/8}$, i.e. in each square S_i, there are at least $nx^2/2$ nodes whp[2].

Let M_i be the event that in a square S_i, there is at least one node with weight $w \geq s_n$. We will specify s_n later. Now, we derive the lower bound on the probability $\Pr[M_i]$.

[1]The centers of the squares lie on the straight line $u - v$.

[2]We say that an event A happens with high probability if $\lim_{n\to+\infty} \Pr[A] = 1$

This probability is grater than probability conditioned on the event that there are at least $nx^2/2$ nodes in S_i, i.e.

$$\begin{aligned}\Pr[M_i] &\geq \Pr[M_i|V_i \geq nx^2/2]\Pr[V_i \geq nx^2/2] \\ &\geq (1-\Pr[W \leq s_n]^{nx^2/2})(1-e^{-nx^2/8}) \\ &= (1-F(s_n)^{nx^2/2})(1-e^{-nx^2/8}).\end{aligned} \tag{5}$$

We now explain how we choose s_n such that any two neighboring squares S_j and S_{j+1} are connected by an edge (i.e. there are two connected nodes $a \in S_j$ and $b \in S_{j+1}$). Let weights of a and b be w and w', respectively. We showed that in any square S_i there is at least one node with weight $\geq s_n$, whp. We want that the connectivity relation for nodes a and b is satisfied, i.e. $G(w,w')/r^2 \geq \theta(n)$. Maximal distance $r = ||a,b||$ between a pair of nodes is $r \leq x\sqrt{5}$. Conditioned on the events that weights w, w' are grater then s_n we have the following relation for the connectivity of nodes a and b

$$\Pr[a \sim b | w, w' \geq s_n] \geq \Pr[G(s_n, s_n)/r^\alpha \geq \theta(n)] \tag{6}$$

For the general additive and multiplicative models, we get s_n by the following relations:

1. For the general additive case $(g(w)+g(w'))/r^\alpha \geq \theta(n)$ we take s_n to be

$$\begin{aligned} g(s_n) &= \Theta(x^\alpha \theta(n)), \text{ i.e.} \\ s_n &= \Theta\big(g^{-1}(x^\alpha \theta(n))\big). \end{aligned} \tag{7}$$

2. For the general multiplicative case $g(w)g(w')/r^\alpha \geq \theta(n)$ we take s_n to be

$$\begin{aligned} g(s_n) &= \Theta(x^{\alpha/2}\theta(n)^{1/2}), \text{ i.e.} \\ s_n &= \Theta\big(g^{-1}(x^{\alpha/2}\theta(n)^{1/2})\big). \end{aligned} \tag{8}$$

In the simple additive model where $g(w) = w$, we get $s_n = \Theta(x^\alpha \theta(n))$. Analogously, for the simple multiplicative model, $s_n = \Theta(x^{\alpha/2}\theta(n)^{1/2})$.

If an arbitrary pair of nodes (u, v) is connected by a path of nodes belonging to the squares $S_1, S_2, \ldots, S_{O(1/x)}$, the following relation on $diam$ is satisfied:

$$\begin{aligned}\Pr[diam &= O(1/x)] \geq \Pr[\cap_{i=1}^{O(1/x)} M_i] \\ &= \Big((1-e^{-nx^2/8})(1-F(s_n)^{nx^2/2}\Big)^{O(1/x)},\end{aligned}$$

since the nodes, as well as weights, are distributed independently. Now, the lemma on the diameter follows:

Lemma 1: Let the cumulative weight distribution function be $F(w)$ in our 2D-GTG model. Let the sequence s_n and x be such that

$$\lim_{n\to\infty} \Big(1 - F(s_n)^{nx^2/2}\Big)^{1/x} = 1. \tag{9}$$

Then, whp $diam = O(1/x)$. The sequence s_n is given by Eq. (7) and (8) for the general additive and multiplicative cases, respectively.

Proof: Proof follows from the previous discussion. ■

B. Some Classes of Diameter

We now analyze conditions on $\theta(n)$ such that $diam = O(1)$, $diam = O(\ln^q n)$ and $diam = O(\sqrt{n/\ln n})$. W.l.o.g. we state the results for the simple additive and multiplicative models, i.e. with $g(w) = w$. The results for the general additive and multiplicative models are analogous by using the inverse function g^{-1}. Finally, we work out the case when the weight distribution is exponential, $f(w) = e^{-w}, w \geq 0$, (i.e. $F(w) = 1-e^{-w}, w \geq 0$) and derive the upper bound on the threshold function $\theta(n)$ in this particular case. For some other weight distribution, the analysis would be similar.

1) Ultra-low Latency: $diam = O(1)$: For the diameter to be a constant, let $x < 1$ be a constant. Invoking Lemma 1, it follows that $diam = O(1)$ whp if and only if $1 - F(s_n)^{nx^2/2} \to 1$, i.e. if and only if $F(s_n)^n \to 0$. The condition on the size of $diam$ is given by the following relations.

1. Additive model:
   if $F(\theta(n))^n \to 0$, then $diam = O(1)$ whp.
2. Multiplicative model:
   if $F(\theta(n)^{1/2})^n \to 0$, then $diam = O(1)$ whp.

These relations give us the bounds on $\theta(n)$, and we can derive $\theta(n)$ such that $diam = O(1)$ whp.

Exponential weight distribution

1. For an exponential weight distribution in the additive model, it follows that $F(\theta(n))^n = (1 - e^{-\theta(n)})^n = (1-e^{-\theta(n)})^{e^{\theta(n)}(n/e^{\theta(n)})} \to e^{-n/e^{\theta(n)}}$. The last equation tends to 0 if and only if $n/e^{\theta(n)} \to \infty$. That is, $diam = O(1)$ if
$$\theta(n) = o(\ln n). \tag{10}$$
2. For the exponential weight distribution in the multiplicative model, it similarly follows that $diam = O(1)$
$$\theta(n) = o(\ln^2 n). \tag{11}$$

2) Low Latency: $diam = O(\ln^q n)$: Let us choose $x = 1/\ln^q n$. Invoking Lemma 1, we obtain:

$$(1 - F(s_n)^{nx^2/2})^{1/x} = \left(1 - F(s_n)^{\frac{n}{2\ln^{2q} n}}\right)^{\ln^q n} \tag{12}$$

For $s_n \to 0$, the last expression tends to 1, if and only if

$$F(s_n)^{\frac{n}{2\ln^{2q} n}} \ln^q n \to 0, \tag{13}$$

by using $\lim_{t\to+\infty}(1 - 1/t)^t = 1/e$. The condition on the size of $diam$ is given by the following relations.

1. Additive model:
   if $F((\ln^q n)^\alpha \theta(n))^{\frac{n}{2\ln^{2q} n}} \ln^q n \to 0$, then $diam = O(\ln^q n)$ whp.
2. Multiplicative model:
   if $F((\ln^q n)^{\alpha/2} \theta(n)^{1/2})^{\frac{n}{2\ln^{2q} n}} \ln^q n \to 0$, then $diam = O(\ln^q n)$ whp.

Exponential Weight Distribution The following is to be satisfied:

$$\begin{aligned} F(s_n)^{\frac{n}{2\ln^{2q} n}} \ln^q n &= \ln^q n(1 - e^{-s_n})^{\frac{n}{2\ln^{2q} n}} \\ &\to s_n^{n/(2\ln^{2q} n)} \ln^q n \to 0, \end{aligned} \tag{14}$$

which is equivalent to $s_n = o\Big((\ln n)^{-\frac{2q}{n}\ln^{2q} n)}\Big)$.

1. For the additive case, the diameter is $diam = O(\ln^q n)$ if
$$\theta(n) = o\Big((\ln n)^{q(\alpha - \frac{2}{n}\ln^{2q} n)}\Big). \tag{15}$$
2. For the multiplicative case, the diameter is $diam = O(\ln^q n)$ if
$$\theta(n) = o\Big((\ln n)^{q(\alpha - \frac{4}{n}\ln^{2q} n)}\Big). \tag{16}$$

3) High Latency: $diam = O(\sqrt{\frac{n}{\ln n}})$: Let us choose $x = \sqrt{\ln n/n}$. Invoking Lemma 1, we get:

$$(1 - F(s_n)^{nx^2/2})^{1/x} = (1 - F(s_n)^{\ln n})^{\sqrt{\frac{n}{\ln n}}} \tag{17}$$

It can be shown that the last expression tends to 1 if and only if $\sqrt{n/\ln n} F(s_n)^{\ln n} \to 0$, by using $\lim_{t\to+\infty}(1 - 1/t)^t = 1/e$. The condition on the size of $diam$ is given by the following relations.

1. Additive model:
   if $\sqrt{n/\ln n} F((\ln n/n)^{\alpha/2}\theta(n))^{\ln n} \to 0$, then $diam = O(\sqrt{n/\ln n})$ whp.
2. Multiplicative model:
   if $\sqrt{n/\ln n} F((\ln n/n)^{\alpha/2}\theta(n)^{1/2})^{\ln n} \to 0$, then $diam = O(\sqrt{n/\ln n})$ whp.

Exponential weight distribution The following is to be satisfied

$$\begin{aligned} \sqrt{n/\ln n} F(s_n)^{\ln n} &= \sqrt{n/\ln n}(1 - e^{-s_n})^{\ln n} \\ &\to \sqrt{n/\ln n} {s_n}^{\ln n} \to 0, \end{aligned} \tag{18}$$

which is equivalent to

$$s_n = o\Big((\ln n/n)^{1/(2\ln n)}\Big). \tag{19}$$

1. For the additive case, the diameter is $diam = O(\sqrt{n/\ln n})$ if
$$\theta(n) = o\Big((n/\ln n)^{\alpha/2 - 1/(2\ln n)}\Big). \tag{20}$$
2. For the multiplicative case, diameter is $diam = O(\sqrt{n/\ln n})$ if
$$\theta(n) = o\Big((n/\ln n)^{\alpha/2 - 1/\ln n}\Big). \tag{21}$$

C. Connectivity

Definition 2: (Random Geometric Graph)[1] Let $\mathcal{G}_{n,r(n)}$ be a graph formed when n nodes are placed uniformly and independently onto the unit disc in $\mathbb{R}^2$. Two vertices are connected if and only if they are within distance $r(n)$, under the Euclidean norm L_2.

Theorem 3: [2] A graph $\mathcal{G}_{n,r(n)}$ with $\pi r^2(n) = \frac{\log n + c(n)}{n}$ is connected with probability one iff $c(n) \to +\infty$.

Furthermore, the second interesting effect appears for the same case $diam = O(\sqrt{n/\ln n})$. The experimental value for the critical $\theta(n)$ (the critical threshold when the graph is globally connected) matches with Eq. (20), which is the

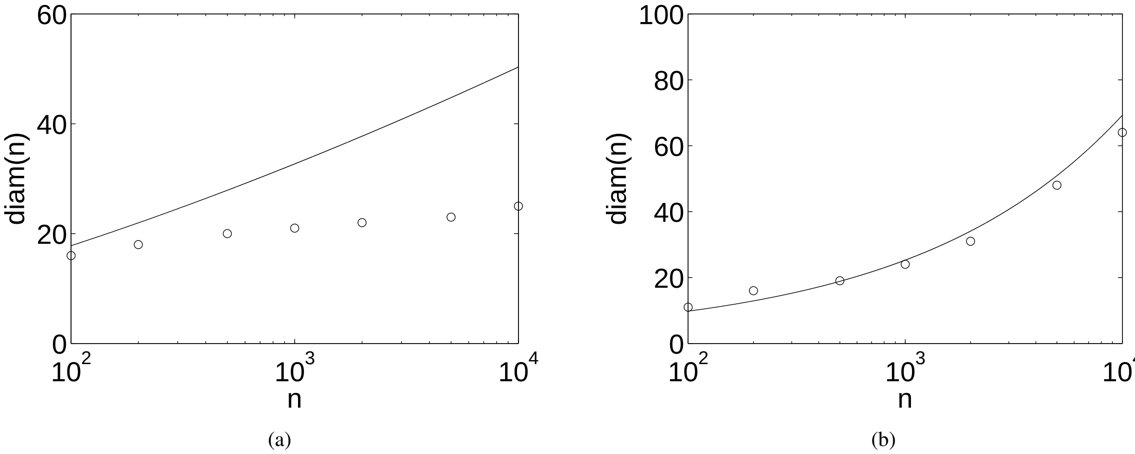

Figure 3. Simulations are done for the additive case with the path-loss exponent $\alpha = 3$; exponentially distributed weights with mean 1 are used; the network sizes simulated are: $n = \{100, 200, 500, 1000, 2000, 10000\}$; the threshold values $\theta(n)$ for the two cases are obtained by invoking Eq. (15) and (20), respectively. (a) For the case of $diam = O(\ln^q n)$, with $q = 1.5$, the analytical solid curve is the upper bound on $diam(n)$. Thus, our simulation results match perfectly with theoretical predictions, since the simulation points all lie below the analytical curve. (b) For the case of $diam = O(\sqrt{n/\ln n})$, the solid curve plots the upper bound on $diam(n)$, and this bound exactly matches with the experimental values.

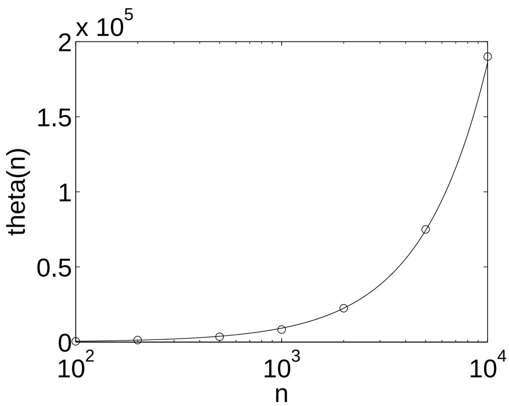

Figure 4. Critical $\theta(n)$ for global connectivity: for each network size n, we found through simulation the critical θ above which the graph is no longer globally connected. The simulation results agree well with the theoretical bound from Eq. (20).

upper bound for the threshold, needed for diameter to be $O(\sqrt{n/\ln n})$ (see Fig. 4).

We proceed with the sufficient conditions on the network reliability for the additive model.

Proposition 4: Let s_n be such that $nF(s_n) \to 0$. Let the threshold $\theta(n)$ satisfy $\frac{G(s_n,s_n)}{\theta(n)} \geq (\frac{\log n + c(n)}{\pi n})^{\alpha/2}$ for $c(n) \to +\infty$. Then the graph in our 2D-GTG model is asymptotically connected with probability one.

The idea of the proof is the following. We consider the event where every node has at least weight s_n (the sequence $s_n \to 0$ is specified by $nF(s_n) \to 0$). We use the result from [2], when the graph is globally connected, that gives us the bound on the asymptotic behavior of the critical connectivity threshold function $\theta(n)$. Later, our results are verified by simulations.

Proof: Let us consider the sequence $s_n \to 0$ as $n \to \infty$, such that $nF(s_n) \to 0$. The sequence s_n always exists since $\lim_{t\to 0} F(t) = 0$. For $i = 1, 2, \ldots, n$ let A_i be the event such that a weight of node i is greater than or equal to s_n, i.e. $A_i = \{W_i \geq s_n\}$. A_i's are independent events and $\Pr[A_i] = 1 - F(s_n)$. Let us denote the intersection $A = A_1 A_2 \ldots A_n$. Let $\mathcal{C}$ be the event that the graph is connected. The probability of the connectedness $\Pr[\mathcal{C}]$ satisfies:

$$\begin{aligned} 1 &\geq \Pr[\mathcal{C}] \geq \Pr[\mathcal{C}|A]\Pr[A] = \Pr[\mathcal{C}|A]\Pi_{i=1}^{n}\Pr[A_i] \\ &= \Pr[\mathcal{C}|A](1 - F(s_n))^n \to \Pr[\mathcal{C}|A]. \end{aligned}$$

The last line follows from the fact that $s_n \to 0$ and $nF(s_n) \to 0$, what implies $(1 - F(s_n))^n \to 1$.

In the following, for two connected nodes v_i and v_j we use the notation $v_i \sim v_j$. For any two different vertices v_i, v_j, conditioned on the events A_i, A_j, the following is satisfied

$$\begin{aligned} \Pr[v_i \sim v_j | A_i, A_j] &\geq \Pr[\frac{G(s_n, s_n)}{r^\alpha} \geq \theta(n)] \\ &= \Pr[r \leq (\frac{G(s_n, s_n)}{\theta(n)})^{1/\alpha}]. \quad (22) \end{aligned}$$

Let us now consider the random geometric random graph $\mathcal{G}_{n,r(n)}$, with $r(n) = \sqrt{(\ln n + c(n))/(\pi n)}$. Conditioning on the events A_i's, from Eq. 22 it follows that $\mathcal{G}_{n,r(n)}$ is embedded into our model, since

$$G(s_n, s_n)/\theta(n) \geq \Big((\ln n + c(n))/(\pi n)\Big)^{\alpha/2}. \quad (23)$$

The asymptotic connectedness, with probability one, of $\mathcal{G}_{n,r(n)}$, implies $\Pr[\mathcal{C}|A] = 1$, which further implies that the graph in our 2D-GTG model, is asymptotically connected with probability one, i.e. $\Pr[\mathcal{C}] = 1$. ■

We now analyze the general additive and general multiplicative cases. The lower bound on the critical connectivity threshold function $\theta(n)$ (when the graph is globally connected) is given by the following conditions:

1. For the general additive case $(g(w)+g(w'))/r^\alpha \geq \theta(n)$, we have:

$$\theta(n) \leq \frac{2}{g(s_n)}\Big(\frac{\pi n}{\ln n + c(n)}\Big)^{\alpha/2}. \quad (24)$$

2. For the general multiplicative case $g(w)g(w')/r^\alpha \geq \theta(n)$, we have:

$$\theta(n) \leq \frac{1}{g^2(s_n)}\Big(\frac{\pi n}{\ln n + c(n)}\Big)^{\alpha/2}. \quad (25)$$

IV. Topology Control

A. *Motivation*

With a general model based on geographical threshold graph in hand, we are now ready to investigate the fundamental properties of this model. Toward the goal of better harnessing the topology of the wireless network, we pose the following two questions: first, for a given desired topology (e.g. with a given degree distribution), how should different resources be allocated, i.e. what should the form for $f(w)$ and $G(w, w')$? A related question would be: given constraints in resources (i.e. the functions $f(w)$ and $G(w, w')$ are given), what is the resulting graph topology and its properties such as degree distribution?

The formulation of these two questions is related to the problem of *topology control* in the design of wireless ad hoc networks. Previous works on topology control focus on developing techniques to control the topology of the wireless connection graph to achieve a network-wide goal such as connectivity, reduction of energy consumption and minimizing radio interference. We refer the readers to the review paper by Santi [16] and the references therein for an overview. The degree distribution encodes many properties of a graph. Given a desired degree distribution, we calculate the required node weight distribution that will generate the desired graph with the given degree distribution. We now first analyze our model.

B. *Mathematical Analysis of the Model*

First, we modify the model as defined in Sec. II. In this section, nodes are uniformly and independently distributed with density ρ over the entire d-dimensional Euclidean space, with each node denoted by the coordinate $(x_1, x_2, ..., x_d)$. For a node v with an arbitrary weight w, we compute $k(w)$, which is the degree of the node as a function of its weight w. From the monotonicity of the function $h(r)$ it follows that the inverse h^{-1} exists. Let the value r_0 be given by:

$$r_0 = h^{-1}(\theta/G(w, w')). \quad (26)$$

Now, every node v' with the weight w' which lies within the ball of the radius r_0 (i.e. $\mathrm{B}^d(v, r_0)$) is connected to the vertex v. As a function of the weight w, the degree of node v is calculated as follows:

$$\begin{aligned} k(w) &\approx \int_{w'} f(w')[\text{No. of nodes in } \mathrm{B}^d(v, r_0)]dw', \\ &= \int_{w'} f(w')\rho Vol(\mathrm{B}^d(v, r_0))dw', \end{aligned} \quad (27)$$

where $Vol(\mathrm{B}^d(v, r)) = \pi^{\frac{d}{2}}\frac{r^d}{\Gamma(\frac{d}{2}+1)}$ is the volume of the ball in the d-dim space, and ρ is the average density of the nodes. We now obtain:

$$k(w) = \pi^{\frac{d}{2}}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\int_{w'} f(w')\Big(h^{-1}(\frac{\theta}{G(w, w')})\Big)^d dw'. \quad (28)$$

It is clear that in the general case Eq. (28) is not solvable.

For the path-loss function $h(r) = r^{-\alpha}$, $2 \leq \alpha \leq 6$, the inverse of h is given by $h^{-1}(t) = t^{-1/\alpha}$. Eq. (28) now simplifies to:

$$k(w) = C_0 \int_{w'} f(w')\Big(\frac{G(w, w')}{\theta}\Big)^{\frac{d}{\alpha}} dw' \quad (29)$$

where $C_0 = \pi^{\frac{d}{2}}\frac{\rho}{\Gamma(\frac{d}{2}+1)}$. Since Leibnitz's criterion is satisfied, we have:

$$\begin{aligned} \frac{dk}{dw} &= C_0 \int_{w'} f(w')\frac{\partial}{\partial w}\Big(\frac{G(w, w')}{\theta}\Big)^{\frac{d}{\alpha}} dw' \\ &= C_0 \frac{d}{\alpha}\frac{1}{\theta^{\frac{d}{\alpha}}}\int_{w'} f(w')G(w, w')^{\frac{d}{\alpha}-1}\frac{\partial G(w, w')}{\partial w}dw' \\ &= C_1 \int_{w'} f(w')G(w, w')^{\frac{d}{\alpha}-1}\frac{\partial G(w, w')}{\partial w}dw', \end{aligned} \quad (30)$$

where $C_1 = C_0\frac{d}{\alpha}\theta^{-\frac{d}{\alpha}} = \pi^{\frac{d}{2}}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\frac{d}{\alpha}\theta^{-\frac{d}{\alpha}}$. Furthermore, we will define $\nu = \frac{d}{\alpha}$.

The node degree probability density function $p_d(k)$ can be obtained by the change of variable technique:

$$p_d(k) = f(w)\left|dw/dk\right|. \quad (31)$$

Thus, given the node weight distribution $f(w)$ and the derivative of the relation between node degree and node weight dk/dw, we can find the degree distribution $p_d(k)$. Similarly, in order to compute $f(w)$, we need to determine dk/dw as well as to know the degree distribution $p_d(k)$. In the following section we examine the following cases: $G(w, w')$ is multiplicatively separable and $G(w, w')$ is additively separable.

1) Multiplicatively Separable Case: Now we assume that the function $G(w, w')$ is multiplicatively separable in w, w', i.e.,

$$G(w, w') = g(w)g(w'). \quad (32)$$

Then, Eq. (29) becomes:

$$\begin{aligned} k(w) &= C_0 \int_{w'} f(w')\Big(\frac{g(w)g(w')}{\theta}\Big)^{\nu} dw' \\ &= C_0\theta^{-\nu}g^{\nu}(w)\int_{w'} f(w')g^{\nu}(w')dw' \\ &= Cg^{\nu}(w), \end{aligned} \quad (33)$$

where $C = C_0\theta^{-\nu}\int f(w')g(^{\nu}(w')dw'$. The first derivative is now given as:

$$\frac{dk}{dw} = C\nu g^{\nu-1}(w)g'(w). \quad (34)$$

2) Additively Separable Case: If $G(w,w')$ is additively separable, i.e.,

$$G(w,w') = g(w) + g(w') \tag{35}$$

and $\nu = 1$, we have:

$$\begin{aligned} k(w) &= C_1 \int_{w'} f(w')(g(w) + g(w'))dw' \\ &= C_1 g(w) + D, \end{aligned} \tag{36}$$

where $D = C_1 \int f(w')g(w')dw'$ is a constant. Also we have

$$\frac{dk}{dw} = C_1 g'(w). \tag{37}$$

3) Exact Computation of the Weight Distribution: Let the following assumptions hold:

(a_1) The function $G(w,w')$ is an increasing in both arguments; for fixed w, higher the weight w' leads to greater probability that nodes v and v' are connected.

(a_2) The function $g(w)$ is continuous and differentiable, with continuous first derivatives.

(a_3) A codomain of $g(w)$ covers the domain of $p_d(k)$, i.e. there is $c > 0$ such that $[d_{min}, d_{max}] \subseteq \{cg(w)^\nu | w \in$ *domain* $g(w)\}$.

That is, for some finite M, a bounded function $g(w) \leq M$ cannot generate a degree distribution with $d_{max} \to \infty$.

Proposition 5: Let the connection between two vertices in the network be defined by Eq. (1). Also, let $G(w,w')$ be multiplicatively separable (i.e., $G(w,w') = g(w)g(w')$). Then, if the desired degree distribution of the network is $p_d(k)$, with a finite first moment, the weight distribution $f(w)$ can be computed as:

$$f(w) = p_d(Cg(^\nu(w))\nu C g^{\nu-1}(w)g'(w) \tag{38}$$

with the domain $g^{-1}((d_{min}/C)^{1/\nu}) \leq w \leq g^{-1}((d_{max}/C)^{1/\nu})$, where $\mathcal{E} = \int p_d(\mu)\mu d\mu$, $C_1 = \pi^{d/2}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\frac{d}{\alpha}\theta^{-\frac{d}{\alpha}}$ and $C = \sqrt{C_1 \mathcal{E}}$.

Proof: See the Appendix. ■

Proposition 6: Let the connection between two vertices in the network be defined by Eq. (1). With the parameter $\nu = 1$, let $G(w,w')$ be additively separable (i.e., $G(w,w') = g(w) + g(w')$). Then, if the desired degree distribution of the network is $p_d(k)$ with a finite first moment, the weight distribution $f(w)$ can be computed as:

$$f(w) = p_d(C_1 g(w) + D)g('(w). \tag{39}$$

with the domain $g^{-1}(\frac{d_{min}-D}{C_1}) \leq w \leq g^{-1}(\frac{d_{max}-D}{C_1})$, where $\mathcal{E} = \int p_d(\mu)\mu d\mu$, $C_1 = \pi^{d/2}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\theta^{-1}$, and $D = (C_1 + \mathcal{E})/2$.

Proof: See the Appendix. ■

Simulation Results: we want to verify the previous derived results. In this example, the desired degree distribution is a power law: $\frac{1}{\gamma-1}k^{-\gamma}, \gamma = 3$. The node density is modified by varying the size of the 2-dimensional space: from $10^2 \times 10^2$ to $10^5 \times 10^5$. The threshold θ and the number of nodes n are kept fixed. Other parameters are: $d = 2$, $\alpha = 2$, hence $\nu = 1$. We examined the multiplicatively separable case where $g(w) = e^w$. For a given power law degree distribution with exponent $\gamma = 3$, we computed $f(w)$ according to Eq. (38) and obtained an exponential weight distribution: $f(w) = \frac{1}{\nu(\gamma-1)}e^{-\nu(\gamma-1)w}$. We distribute the weights over the nodes in the network. We then connect any pair of nodes that satisfy the connectivity relation Eq. (1). In Fig. 5, we plot the degree distribution of the generated network. The obtained degree distribution matches with the expected power law degree distribution with $\gamma = 3$.

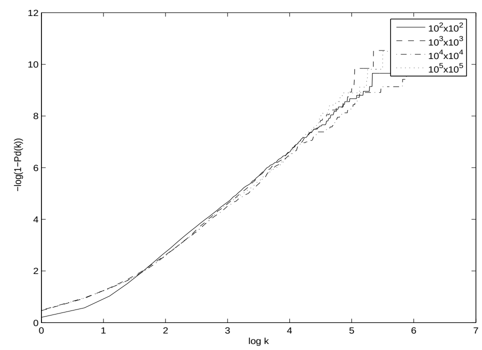

Figure 5. For each node density level, we first invoked Eq. (38) to obtain the required weight distribution $f(w)$. We then generate the graph by distributing weights according to $f(w)$. The degree distributions of the generated graphs are verified to have power law exponents of 3 as predicted.

C. Resource Allocation Case Study: Power Law Network Generation

As discussed in Sec. II, the weight of a node characterizes a node's available resources, such as power and bandwidth. Thus, computing the weight distribution $f(w)$ corresponds to finding how resources should be allocated. In this subsection, we examine the constraints in the construction of a power law network.

Claim 7: For any given distribution of the weights $f(w)$ such that the assumption (a_3) is satisfied (i.e. we have the multiplicatively separable case $(G(w,w') = g(w)g(w'))$, there exists a function $g(w)$ such that the generated degree distribution of the network is a power law $p_d(k) = (\gamma - 1)k^{-\gamma}$, with $\gamma > 1$.

Proof: See the Appendix. ■

The class of the weight distributions $f(w)$ for which the degree distribution is a power law is much wider than the class of the functions that have been analyzed in previous works[17]. For exponential and Pareto distributed $f(w)$, the following two scenarios will give us $g(w)$ such that the degree distribution is a power law.

Scenario 1 For the exponential distribution $f(w) = \lambda e^{-\lambda w}$, it follows that $g(w) \propto e^{\lambda w/s}$, for $x > 0$ and $s = \nu(\gamma - 1)$.

Scenario 2 Similarly, for the Pareto distribution $f(w) = \frac{a}{w_0}(\frac{w_0}{w})^{a+1}$ with $w > w_0$, it follows that $g(w) \propto w^{a/s}$, for $w > w_0$ and $s = \nu(\gamma - 1)$.

V. Conclusion

In this paper, we developed a generalized model for wireless ad hoc networks (WANET). Our model is based on the geographical threshold graph: the presence of a link between a pair of nodes depends on their weights offset by the path-loss function. In our model, a node's weight is a general measure that can be used to quantify a wireless node's many characteristics and resources, such as power and bandwidth. For the graph generated by our model, we further derived analytical results on the diameter of the graph, connectivity and topology control. Thus, our model allows network designers to more realistically model WANET under a wide range of criteria.

VI. Appendix

Proof: (Proposition 5) Let $G(w,w') = g(w)g(w')$ be the additively separable function. Since $g(w)$ is the monotonically continuous increasing function, it follows that there exists the inverse g^{-1}. Then we can compute

$$w = g^{-1}((k/C)^{1/\nu}). \tag{40}$$

From Eq. (31) and (40) we have:

$$\begin{aligned} f(w) &= p_d(Cg^\nu(w))|C\frac{d}{dw}g^\nu(w)| \\ &= p_d(Cg^\nu(w))\nu Cg^{\nu-1}(w)g'(w) \end{aligned} \tag{41}$$

with the domain for w

$$g^{-1}((d_{min}/C)^{1/\nu}) \leq w \leq g^{-1}((d_{max}/C)^{1/\nu}). \tag{42}$$

The only unknown in Eq. (41) is C. Now we show that $C = \sqrt{C_1\mathcal{E}}$, where $\mathcal{E} = \int p_d(\mu)\mu d\mu$ and $C_1 = \pi^{d/2}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\frac{d}{\beta}\theta^{-\frac{d}{\beta}}$. Recall that $C/C_1 = \int f(w')g(w')^\nu dw'$. Then we have

$$\begin{aligned} \frac{C^2}{C_1} &= \int_{w_{min}}^{w_{max}} p_d(Cg^\nu(w))Cg^\nu(w)dCg^\nu(w) \\ &= K^\nu \int_{d_{min}}^{d_{max}} p_d(\mu)\mu d\mu = \mathcal{E}. \end{aligned} \tag{43}$$

Than clearly $C = \sqrt{C_1\mathcal{E}}$. Now, the proof of (Proposition) follows. ∎

Proof: (Proposition 6) Let $G(w,w') = g(w)+g(w')$ be the additively separable function. From Eq. (31), it follows,

$$f(w) = p_d(C_1 g(w) + D)C_1 g'(w). \tag{44}$$

The only unknown in Eq. (44) is D. We will show that $D = (C_1+\mathcal{E})/2$, where $\mathcal{E} = \int p_d(\mu)\mu d\mu$.

$$D = C_1 \int f(w)g(w)dw \tag{45}$$

$$\begin{aligned} &= C_1 \int p_d(C_1 g(w) + D)C_1 g'(w)g(w)dw \\ &= C_1 \int p_d(\mu)d\mu(1 + \frac{\mu - D}{C_1}) \\ &= C_1 - D + \int p_d(\mu)\mu d\mu. \end{aligned} \tag{46}$$

Then clearly $D = (C_1+\mathcal{E})/2$. Now the proof follows. ∎

Proof: (Claim 7) starting from the following relation

$$f(w) = p_d(Cg^\nu(w))Cg^{\nu-1}(w)g'(w) \tag{47}$$

we want to find $g(w)$ s.t. the degree distribution is a power-law $p_d(k) = (\gamma-1)k^{-\gamma}$, where $k \geq 1$, and $\gamma > 1$. It follows

$$C^{\gamma-1}f(w)/(\gamma-1) = \psi^{-(\nu(\gamma-1)+1)}(w)\psi'(w). \tag{48}$$

Let $s = \nu(\gamma-1) > 0$ and $B = C^{\gamma-1}/(\gamma-1)$ be the known constants. That is $Bf(w) = g^{-(s+1)}(w)g'(w)$, with the exponent $s+1 > 1$. Taking the integral over the last equation we have $BF(w) = -g^{-s}(w)/s + K$, for some unknown constant K. That is $g(w) = [s(K - BF(w))]^{-1/s}$. We have a freedom for choosing the constant K. The restriction is $K > BF(w)$ for every w, that is $K \geq B$, since $\lim_{w\to+\infty} F(w) = 1$. ∎

Furthermore, the conditions (a_1)-(a_3) are satisfied: (a_1) $g(w)$ is increasing in w; (a_2) as far as a starting $f(w)$ is continuous, it implies that $g(w)$ is continuous, differentiable, with the continuous first derivative; (a_3) is satisfied by the assumption in the claim (Power Law Creation).

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