Coloring Geographical Threshold Graphs

Milan Bradonjić [*] Tobias Müller [†] Allon G. Percus[‡]

Abstract

We propose a coloring algorithm for sparse random graphs generated by the geographical threshold graph (GTG) model, a generalization of random geometric graphs (RGG). In a GTG, nodes are distributed in a Euclidean space, and edges are assigned according to a threshold function involving the distance between nodes as well as randomly chosen node weights. The motivation for analyzing this model is that many real networks (e.g., wireless networks, the Internet, etc.) need to be studied by using a "richer" stochastic model (which in this case includes both a distance between nodes and weights on the nodes). Here, we analyze the GTG coloring algorithm together with the graph's clique number, showing formally that in spite of the differences in structure between GTG and RGG, the asymptotic behavior of the chromatic number is identical: $\chi = \frac{\ln n}{\ln \ln n}(1 + o(1))$. Finally, we consider the leading corrections to this expression, again using the coloring algorithm and clique number to provide bounds on the chromatic number. We show that the gap between the lower and upper bound is within $C \ln n/(\ln \ln n)^2$, and specify the constant C.

1 Introduction

Numerous approaches have been proposed in recent years to study the structure of large real-world technological and social networks, and to optimize processes on these networks. A particularly fertile approach has been to consider the network as an instance of an ensemble, arising from a suitable random generative model. One straightforward example is the random geometric graph (RGG) model, where nodes are placed at random in a Euclidean space and edges are placed between any two nodes within a threshold distance. This has the advantage of describing many aspects of systems such as sensor networks, while avoiding unnecessary detail. Even though geometric correlations in RGGs complicate the probabilistic analysis of the model, recent work has clarified many of its structural properties including threshold behavior [10, 5, 6], random walk behavior [1] and chromatic number [8, 9, 10].

RGGs fail, however, to capture heterogeneity in the network. Geographical threshold graphs (GTG) aim at generalizing RGGs, providing this heterogeneity via a richer stochastic model that nevertheless preserves much of the simplicity of the RGG model. GTGs assign to nodes both a location and a weight, which may represent a quantity such as transmission power in a wireless network or influence in a social network. Edges are placed between two nodes if a symmetric function of their weights and the distance between them exceeds a certain threshold [4].

Recent work has analyzed structural properties of GTGs, such as connectivity, clustering coefficient, degree distribution, diameter, existence and absence of the giant component [2, 3]. These properties are not merely of theoretical importance, but also play an important role in applications. In communication networks, connectivity implies the ability to reach all parts of the network. In packet routing, the diameter gives the minimal number of hops needed for transmission between two arbitrary nodes. And in the case of epidemics, the existence or absence of the giant component controls whether the epidemic spreads or is contained.

When considering wireless networks, a natural quantity to study is the chromatic number. This is the minimum number of colors needed to color vertices, such that no two adjacent vertices in the graph receive the same color. Treating the colors as the different radio channels or frequencies, the chromatic number gives the minimal number of channels needed so that neighboring radios do not interfere with each other. In

---

[*] Mathematical Modeling and Analysis Group, and Center for Nonlinear Studies, Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, `milan@lanl.gov`.

[†] School of Mathematics, Tel Aviv University. Tel Aviv 69978, Israel, `tobias@post.tau.ac.il`. Research partially supported through an ERC advanced grant.

[‡] School of Mathematical Sciences, Claremont Graduate University, Claremont, CA 91711, `allon.percus@cgu.edu`.

this paper we study the asymptotic behavior of the chromatic number for GTGs with constant mean degree. We propose a greedy coloring algorithm, and analyze the behavior of this algorithm along with the graph's clique number. This leads to upper and lower bounds on the chromatic number.

The paper is organized as follows. Section 2 defines the GTG model. Section 3 presents our main asymptotic result, based on our analysis of the coloring algorithm. We show that for graphs G of constant mean degree, both the clique number $\omega(G)$ and chromatic number $\chi(G)$ are with high probability given by $\frac{\ln n}{\ln\ln n}(1+o(1))$. Section 4 analyzes the gap between lower and upper bounds on the chromatic number, given respectively by the clique number and the greedy coloring algorithm. We show that this gap is within $C\ln n/(\ln\ln n)^2$, and specify the constant C. Finally, Section 5 concludes with open questions regarding the chromatic number for sparser and denser GTGs.

2 Geographical Threshold Graph Model

Given random points $X_1, X_2, \ldots \in [0,1]^2$ that are i.i.d., uniformly at random, and i.i.d. nonnegative weights $W_1, W_2, \ldots$, we construct a random geographical threshold graph G_n as follows. Let $N \overset{d}{=} \mathrm{Po}(n)$ be the number of nodes, independent of the X_i and W_i. Let θ_n be a given threshold parameter that depends on n. Then G_n has vertex set $V(G_n) = \{1, \ldots, N\}$, and for $i, j \in V(G_n)$, G_n has edge $ij \in E(G_n)$ iff

$$\frac{W_i + W_j}{\|X_i - X_j\|^2} \geq \theta_n. \tag{1}$$

For technical convenience we identify opposite edges of $[0,1]^2$, making it into a torus.

We will specifically analyze the regime of constant expected degree. If $\mathbf{E}(W_i)$ is a constant, then this occurs when the threshold parameter is linear in the expected number of nodes, $\theta_n = \Theta(n)$. For simplicity we take $\theta_n = n$, since if $\theta_n = cn$ for some constant $c > 0$, the weights can always be rescaled to $W_i := W_i/c$.

3 Asymptotic Results

If G is a graph then $\omega(G)$ denotes its clique number and $\chi(G)$ its chromatic number. We will show formally that the clique number and chromatic number of the geographical threshold graph are essentially the same as those for a random geometric graph with the same (constant) average degree.

Note that, since coloring a clique of size $\omega(G)$ requires $\omega(G)$ different colors, $\omega(G) \leq \chi(G)$.

Theorem 3.1 *Suppose that weights are distributed such that* $\boldsymbol{Pr}(W_i > x) = O(x^{-\gamma})$ *for some* $\gamma > 1$*. Then*

$$\frac{\omega(G_n)}{\ln n/\ln\ln n} \to 1 \quad \textit{in probability,}$$

and

$$\frac{\chi(G_n)}{\ln n/\ln\ln n} \to 1 \quad \textit{in probability,}$$

as $n \to \infty$.

The rest of this section is devoted to proving the theorem.

3.1 Lower bound.

Let $\hat{w} \in \mathbf{R}$ be such that $\mathbf{Pr}(W_i > \hat{w}) \geq 1/2$. Then the probability that G_n contains fewer than $n/3$ vertices with weight more than $\hat{w}$ is exponentially small. In fact, this probability is bounded above by the probability that a $\mathrm{Po}(n/2)$-variable is less than $n/3$, which is $\exp[-\Omega(n)]$, as can be seen by the Chernoff bound. Let G'_n be the subgraph of G_n induced by $n/3$ of the points with weights $\hat{w}$ at least. Note that if $i, j \in V(G'_n)$ and $\|X_i - X_j\|^2 < 2\hat{w}/n$ then certainly $ij \in E(G'_n)$. Thus G'_n (and hence also G_n) contains the ordinary random geometric graph $G(n/3, \sqrt{2\hat{w}/n})$ as a subgraph, with the probability $1 - \exp[-\Omega(n)]$. By Lemma 5.3 in [7],

$$\mathbf{Pr}(\omega(G_n) < (1-\varepsilon)\ln n/\ln\ln n) = o(1).$$

3.2 Upper bound.

Let us define a "level" L_k as follows:

$$\begin{array}{l} L_{-1} := \{i \leq N : W_i < 1\}, \\ L_k := \{i \leq N : 4^k \leq W_i < 4^{k+1}\} \quad \text{for } k \geq 0. \end{array}$$

Note that the set $\{X_i : i \in L_k\}$ of the points of the Poisson process corresponding to level k is in fact a Poisson process itself with intensity $n \cdot (F(4^{k+1}) - F(4^k))$ (here F denotes the cdf of W_1) on the unit square and intensity 0 elsewhere. Moreover, these Poisson processes corresponding to the levels $L_{-1}, L_0, L_1, \ldots$ are independent.

For $x \in [0,1]^2$ let us denote

$$M_x := \sum_{k=-1}^{\infty} |\{i \in L_k : \|X_i - x\| \leq 6\sqrt{2} \cdot 2^k/\sqrt{n}\}|,$$

and let us set

$$M := \max_{x \in [0,1]^2} M_x.$$

Very roughly, M represents the greatest number of neighbors that a sufficiently high-weighted node can have.

Then we have the following:

Lemma 3.2 *The chromatic number satisfies* $\chi(G_n) \leq M$.

Proof Let us order the vertices by nondecreasing weight and greedily color them. That is, we first color the vertex with smallest weight, then the vertex with second smallest weight and so on; and when we choose a color for a vertex we always pick the smallest available color (i.e., the smallest color that does not occur among the neighbors of the vertex that have already been colored). We claim that in this way we will never need more than M colors.

For ease of notation let us assume (w.l.o.g.) that $W_1 \leq W_2 \leq \ldots \leq W_N$. Let $N_<(i)$ represent all neighbors of node i with lower weight than i:

$$N_<(i) := \{j < i : ij \in E(G_n)\}.$$

Note that if $i \in L_k$ and $j \in N_<(i)$ then $\|X_i - X_j\| \leq 2^{k+3/2}/\sqrt{n}$. For $1 \leq i \leq N$ let $c(i)$ denote the color that the algorithm has assigned to vertex i.

Now let i be an arbitrary vertex, and let k_0 denote the level of i. For each of the colors $1, \ldots, c(i) - 1$ there is a $j \in N_<(i)$ with $c(j)$ equal to that color. Let $c_1 < c(i)$ be the largest color for which there is no $j \in L_{k_0} \cap N_<(i)$ with $c(j) = c_1$. It is possible that no such c_1 exists, in which case

$$\begin{array}{rcl} c(i) & \leq & |L_{k_0} \cap N_<(i)| + 1 \\ & \leq & |\{j \in L_{k_0} : \|X_j - X_i\| \leq 2^{k_0+3/2}/\sqrt{n}\}| \\ & \leq & M_{X_i} \\ & \leq & M. \end{array}$$

If c_1 exists, then let us pick a $j_1 \in N_<(i) \setminus L_{k_0}$ with $c(j_1) = c_1$, and for notational convenience, define $j_0 := i$. Let k_1 denote the level of j_1. All colors $1, \ldots, c_1 - 1$ must occur in $N_<(j_1)$. Let $c_2 < c_1$ be the largest color for which there is no $j \in L_{k_1} \cap N_<(j_1)$ with $c(j) = c_2$. It is possible that no such c_2 exists, in which case

$$\begin{array}{rcl} c(i) & \leq & |L_{k_0} \cap N_<(j_0)| + |L_{k_1} \cap N_<(j_1)| + 2 \\ & \leq & |\{j \in L_{k_0} : \|X_j - X_{j_0}\| \leq 2^{k_0+3/2}/\sqrt{n}\}| + |\{j \in L_{k_1} : \|X_j - X_{j_1}\| \leq 2^{k_1+3/2}/\sqrt{n}\}| \\ & \leq & |\{j \in L_{k_0} : \|X_j - X_{j_1}\| \leq 2 \cdot 2^{k_0+3/2}/\sqrt{n}\}| + |\{j \in L_{k_1} : \|X_j - X_{j_1}\| \leq 2^{k_1+3/2}/\sqrt{n}\}| \\ & \leq & M_{X_{j_1}} \\ & \leq & M. \end{array}$$

Here, the first line follows from the fact that each color $\leq c(j_0)$ must either occur as the color of j_0 or j_1, or of a neighbor of j_0 of level k_0 or of a neighbor of j_1 of level k_1. The second line uses the fact that

$\|X_j - X_{j_0}\| \leq 2^{k_0+3/2}/\sqrt{n}$ if $j \in N_<(j_0) \cap L_{k_0}$ and that $\|X_j - X_{j_1}\| \leq 2^{k_1+3/2}/\sqrt{n}$ if $j \in N_<(j_1) \cap L_{k_1}$. The third line uses the triangle inequality: $\|X_j - X_{j_1}\| \leq \|X_j - X_{j_0}\| + \|X_{j_0} - X_{j_1}\|$.

Now suppose that $j_1 > \ldots > j_m$ and $k_1 > \ldots > k_m$ have been defined in such a way that, for $p = 0, \ldots, m$, we have $j_{p+1} \in L_{k_p} \cap N_<(j_p)$ and $c(j_{p+1}) < c(j_p)$ is the largest color that does not occur in $\{c(j) : j \in N_<(j_p) \cap L_{k_p}\}$. Let c_{m+1} be the largest color such that there is no $j \in N_<(j_m) \cap L_{k_m}$ with $c(j) = c_{m+1}$. If no such c_{m+1} exists, then

$$\begin{array}{rcl} c(i) & \leq & \sum_{p=0}^{m} |L_{k_p} \cap N_<(j_p)| + m + 1 \\ & \leq & \sum_{p=0}^{m} |\{j \in L_{k_p} : \|X_j - X_{j_m}\| \leq 6\sqrt{2} \cdot 2^{k_p}/\sqrt{n}\}| \\ & \leq & M_{X_{j_m}} \\ & \leq & M. \end{array}$$

The first line follows because necessarily $\{1, \ldots, c_m - 1\} \subseteq \{c(j) : j \in N_<(j_m) \cap L_{k_m}\}$ and $c(j_m) = c_m$, $\{c_m + 1, \ldots, c_{m-1} - 1\} \subseteq \{c(j) : j \in N_<(j_{m-1}) \cap L_{k_{m-1}}\}$ and $c(j_{m-1}) = c_{m-1}$, and so on. The second line follows because, again by the triangle inequality:

$$\begin{array}{rcl} \|X_{j_p} - X_{j_m}\| & \leq & \displaystyle\sum_{q=p}^{m-1} \|X_{j_q} - X_{j_{q+1}}\| \\ & \leq & \displaystyle\sum_{q=p}^{m-1} \frac{2^{k_q+3/2}}{\sqrt{n}} \\ & < & 2^{k_p+5/2}/\sqrt{n}, \end{array}$$

for all $1 \leq p \leq m$. And hence, for any $j \in N_<(j_p) \cap L_{k_p}$, we have

$$\begin{array}{rcl} \|X_j - X_{j_m}\| & \leq & \|X_{j_p} - X_{j_m}\| + \|X_j - X_{j_p}\| \\ & \leq & (2^{k_p+5/2} + 2^{k_p+3/2})/\sqrt{n} \\ & < & 6\sqrt{2} \cdot 2^{k_p}/\sqrt{n}. \end{array}$$

If c_{m+1} exists then we can choose $j_{m+1} \in N_<(j_m) \setminus L_{k_m}$ such that $c(j_{m+1}) = c_{m+1}$ and set k_{m+1} equal to the level of j_{m+1}, and continue by attempting to pick a c_{m+2}. It is clear that the process of picking new c_m's cannot continue indefinitely (certainly there can be no more than N steps), so we can conclude that $c(i) \leq M$. Since the vertex i was arbitrary, the claim follows.

To finish the proof of the theorem it now suffices to prove the following lemma:

Lemma 3.3 *Suppose that weights are distributed such that* $\boldsymbol{Pr}(W_i > x) = O(x^{-\gamma})$ *for some* $\gamma > 1$. *Then*

$$\limsup_{n\to\infty} \frac{M}{\ln n / \ln\ln n} \leq 1 \qquad a.s.$$

Proof Let us set

$$M'_x := \sum_{k=-1}^{\infty} |\{i \in L_k : \|X_i - x\| \leq 12\sqrt{2} \cdot 2^k/\sqrt{n}\}|,$$

and note that if $A := \{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}) : 0 \leq a, b \leq \sqrt{n}\}$ where a and b are integers. Now, for any $x \in [0,1]^2$ there is $z \in A$ with $\|x - z\| \leq \sqrt{2}/\sqrt{n}$. So if $\|X_i - x\| \leq 6\sqrt{2} \cdot 2^k/\sqrt{n}$ then $\|X_i - z\| \leq \sqrt{2}/\sqrt{n} + 6\sqrt{2} \cdot 2^k/\sqrt{n} < 12\sqrt{2} \cdot 2^k/\sqrt{n}$ for all $k \geq -1$, and thus

$$M \leq \max_{x \in A} M'_x. \tag{2}$$

Let $x \in \mathbf{R}^2$ be arbitrary and note that $M'_x \overset{d}{=} \sum_{k=-1}^{\infty} Z_k$, where the Z_k are independent Poisson random variables, and $\mathbf{E}(Z_k) \leq \pi(12\sqrt{2})^2 \cdot 4^{k+1} \cdot \mathbf{Pr}(W_1 \geq 4^k) = O(4^{k(1-\gamma)})$. So in particular M'_x is itself Poisson with a mean that is bounded above by some constant, μ say. Using a well known bound, see for instance the Lemma 4.4 of [7], we obtain that

$$\begin{array}{rcl} \mathbf{Pr}(M'_x > (1+\varepsilon)\ln n/\ln\ln n) & \leq & \left(\dfrac{e\mu}{(1+\varepsilon)\ln n/\ln\ln n}\right)^{(1+\varepsilon)\ln n/\ln\ln n} \\ & = & \exp\big(-(1+\varepsilon+o(1))\ln n\big). \end{array} \tag{3}$$

Hence, by Eq. (2), applying the union bound,

$$\begin{aligned}\mathbf{Pr}\Big(M > (1+\varepsilon)\ln n/\ln\ln n\Big) &\le ne^{-(1+\varepsilon+o(1))\ln n} \\ &\le n^{-\frac{\varepsilon}{2}}. \qquad (4)\end{aligned}$$

The last inequality holds for n sufficiently large. This shows that $M/(\ln n/\ln\ln n)$ is upper bounded by $1+\varepsilon$, with high probability.

Remark: It is possible to adapt a subsequence trick from [10] (page 123) to strengthen the type of convergence in Theorem 3.1 from convergence in probability to almost sure convergence.

4 Mind the Gap!

In this section we analyze the gap between lower and upper bounds on the chromatic number, given respectively by the clique number (Subsection 4.1) and the greedy coloring algorithm (Subsection 4.2). In Subsection 4.3 we show that this gap is within $C\ln n/(\ln\ln n)^2$, and specify the constant C.

4.1 Lower Bound.

Informally: we tile the space $[0,1]^2$ and inscribe a ball in each tile. Then the number of nodes within a clique in an arbitrarily chosen ball will give us a lower bound on the chromatic number of the entire geographical threshold graph within $[0,1]^2$. The number of the balls, or how we tile the space $[0,1]^2$, is a parameter that we discuss later. Formally, the argument is the following.

For some threshold weight w_0, let α be defined by $\mathbf{Pr}(W \le w_0) = \alpha$. We will appropriately choose the constant w_0 (and hence α) later. Let us define a radius $r_0 = \sqrt{w_0/(2\theta_n)}$. We consider $b = 1/(2r_0)^2$ disjoint balls with radii r_0 (see Figure 1 and call these balls B_i. For convenience, tile the square $[0,1]^2$ into $b = 1/(2r_0)^2$ sub-squares of the size $2r_0 \times 2r_0$, and within each of the squares inscribe a ball of radius r_0. The number of nodes within B_i is given by Poisson distribution $\mathrm{Po}(nr_0^2\pi)$, while the number of nodes with weights $\ge w_0$ within B_i is given by $\mathrm{Po}((1-\alpha)nr_0^2\pi)$. For convenience we let $\lambda = (1-\alpha)nr_0^2\pi$. Let us note that for $\theta_n = n$ it follows $b = 1/(4r_0^2) = \theta_n/(2w_0) = n/(2w_0)$ (this is $\Theta(n)$) and $\lambda = \frac{\pi}{2}(1-\alpha)w_0$ (this is $\Theta(1)$).

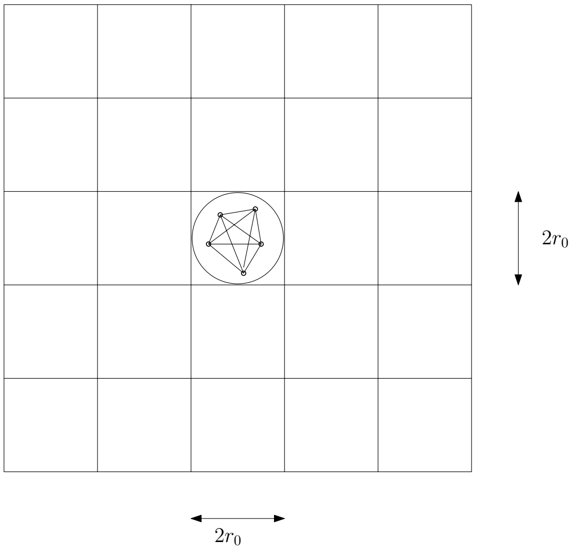

Figure 1: Tiling of the unit square, showing one ball B_i and the high-weighted nodes ($W \ge w_0$) within it.

Let us now consider only those nodes with weights $W \ge w_0$ within a given ball B_i. All such nodes form a clique, since by construction, each pair within B_i satisfies the connectivity relation Eq. (1). Let k be a positive integer to be specified later. Since the number of nodes n_i within B_i is a Poisson random variable with mean λ,

$$\mathbf{Pr}(n_i \geq k) \geq e^{-\lambda}\frac{\lambda^k}{k!}. \tag{5}$$

Denote $p := e^{-\lambda}\lambda^k/k!$, and let I_i be an indicator of the event $\{n_i \geq k\}$, so that $\mathbf{Pr}(I_i = 1) \geq p$. Let us define $J = \sum_{i=1}^{b} I_i$. We will show that for sufficiently large k, $\mathbf{Pr}(J = 0) \to 0$. First, $J = 0$ iff all I_i are 0. Second, the indicators I_i are mutually independent, since the balls B_i are mutually disjoint. Thus, $\mathbf{Pr}(\cap I_i^c) = \mathbf{Pr}(I_i^c)^b \leq (1-p)^b = \exp(\ln(1-p)b)$. We have seen that $b = \Theta(n)$. Now choose k so that $p = \ln n/n$. In that case, $\mathbf{Pr}(J > 0) \geq 1 - \exp(\ln(1-p)b) = 1 - \exp(-\Theta(\ln n)) = 1 - n^{-\Theta(1)}$. Thus, we must solve the following equation in k

$$e^{-\lambda}\frac{\lambda^k}{k!} = \frac{\ln n}{n}. \tag{6}$$

Taking the logarithm,

$$\lambda - k\ln\lambda + \ln k! = \ln n - \ln\ln n. \tag{7}$$

According to Stirling's formula $k! = \sqrt{2\pi k}\, k^k e^{-k+\beta/12k}$ for some $\beta \in (0,1)$, and applying the logarithm, $\ln k! = \frac{1}{2}\ln 2\pi k + k(\ln k - 1) + O(1/k)$. Now, Eq.(7) is equivalent to

$$k(\ln k - 1 - \ln\lambda) = \ln n - \ln\ln n - \lambda - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln k - O(1/k). \tag{8}$$

Let $\eta = 1 + \ln\lambda$, and introduce the (rescaled) variables $y = e^{-\eta}k$ and $x = e^{-\eta}(\ln n - \ln\ln n - \lambda - \frac{1}{2}\ln 2\pi - O(1/k) - \eta/2)$. Then, Eq. (8) may be recast as

$$y = \frac{x}{\ln y} - \frac{e^{-\eta}}{2}. \tag{9}$$

For given x and η, Eq. (9) has a unique solution in y. It is not hard to verify that the solution satisfies

$$y = \frac{x}{\ln x}\Big(1 + \frac{\ln\ln x}{\ln x}(1 + o(1))\Big). \tag{10}$$

Furthermore, from the definition of x,

$$x = e^{-\eta}\ln n\Big(1 - O\Big(\frac{\ln\ln n}{\ln n}\Big)\Big), \tag{11}$$

so

$$\begin{aligned} \ln x &= \ln\Big(e^{-\eta}\ln n(1 - o(1))\Big) \\ &= \ln\ln n - \eta - o(1) \end{aligned} \tag{12}$$

and

$$\begin{aligned} \ln\ln x &= \ln\Big(\ln\ln n(1 - o(1))\Big) \\ &= \ln\ln\ln n - o(1). \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned} \ln y &= \ln x - \ln\ln x + o(1) \\ &= \ln\ln n - \ln\ln\ln n - \eta - o(1) \\ &= \ln\ln n\Big(1 - \frac{\ln\ln\ln n + \eta + o(1)}{\ln\ln n}\Big). \end{aligned} \tag{14}$$

Plugging Eq. (11) and Eq. (14) into the right-hand side of Eq. (9) gives

$$\begin{aligned} y &= \frac{e^{-\eta}\ln n\Big(1 - O\Big(\frac{\ln\ln n}{\ln n}\Big)\Big)}{\ln\ln n\Big(1 - \frac{\ln\ln\ln n + \eta + o(1)}{\ln\ln n}\Big)} - \frac{e^{-\eta}}{2} \\ &= \frac{e^{-\eta}\ln n}{\ln\ln n}\Big(1 + \frac{\ln\ln\ln n + \eta + o(1)}{\ln\ln n}\Big) - \frac{e^{-\eta}}{2}, \end{aligned}$$

so

$$\begin{aligned} k &= e^{\eta} y \\ &= \frac{\ln n}{\ln\ln n}\Big(1 + \frac{\ln\ln\ln n + \eta + o(1)}{\ln\ln n}\Big). \end{aligned}$$

We know that there is a clique of size at least k within some ball B_i, with probability $\geq 1 - n^{-\Theta(1)}$. Since $k \leq \omega(G_n) \leq \chi(G_n)$, it follows that

$$\frac{\ln n}{\ln\ln n}\Big(1 + \frac{\ln\ln\ln n + \eta + o(1)}{\ln\ln n}\Big) \leq \chi(G_n).$$

4.2 Upper Bound.

In this subsection we derive an upper bound on the chromatic number, given by the greedy coloring algorithm in Section 3. Let us consider the inequality (3).

$$\begin{aligned} \mathbf{Pr}\Big(M'_x > (1+\varepsilon)\ln n/\ln\ln n\Big) &\leq \Big(\frac{e\mu}{(1+\varepsilon)\ln n/\ln\ln n}\Big)^{(1+\varepsilon)\ln n/\ln\ln n} \qquad (15) \\ &= \exp\Big\{\Big(B - \ln\big((1+\varepsilon)\frac{\ln n}{\ln\ln n}\big)\Big)(1+\varepsilon)\frac{\ln n}{\ln\ln n}\Big\} \\ &= \exp\Big\{\ln n\Big(\frac{B(1+\varepsilon)}{\ln\ln n} - \big(\ln(1+\varepsilon) + \ln\ln n - \ln\ln\ln n\big)\frac{(1+\varepsilon)}{\ln\ln n}\Big)\Big\} \\ &= \exp\Big\{\ln n\Big(\frac{B(1+\varepsilon)}{\ln\ln n} - \frac{(1+\varepsilon)\ln(1+\varepsilon)}{\ln\ln n} - (1+\varepsilon) + (1+\varepsilon)\frac{\ln\ln\ln n}{\ln\ln n}\Big)\Big\} \\ &= \exp\Big\{\ln n\Big(\frac{B}{\ln\ln n} + \frac{B\varepsilon}{\ln\ln n} - \frac{\varepsilon + O(\varepsilon^2)}{\ln\ln n} - 1 - \varepsilon + \frac{\ln\ln\ln n}{\ln\ln n} + \varepsilon\frac{\ln\ln\ln n}{\ln\ln n}\Big)\Big\}, \end{aligned}$$

where $B = \ln(\mu e)$. Let us choose ε to be

$$\varepsilon = \frac{\ln\ln\ln n + s}{\ln\ln n}, \tag{16}$$

then it follows that

$$\begin{aligned} \mathbf{Pr}(M'_x > (1+\varepsilon)\ln n/\ln\ln n) &\leq \exp\Big\{\ln n\Big(-1 + \frac{B-s}{\ln\ln n} + \frac{\varepsilon}{\ln\ln n}\Big(\ln\ln\ln n + B - 1 - o(1)\Big)\Big\} \qquad (17) \\ &= \exp\Big\{\ln n\Big(-1 + \frac{B - s + o(1)}{\ln\ln n}\Big)\Big\}. \end{aligned}$$

Hence, by Eq. (4) and the union bound, and for an arbitrary positive constant δ by taking $s \geq B + \delta$ it follows that $\mathbf{Pr}(M < (1+\varepsilon)\ln n/\ln\ln n)$ with probability $\geq 1 - e^{-\frac{\ln n}{\ln\ln n}(\delta - o(1))}$. Thus, for any positive δ, with high probability, that is probability $\geq 1 - e^{-\frac{\ln n}{\ln\ln n}(\delta - o(1))}$, the chromatic number satisfies

$$\chi(G_n) \leq \frac{\ln n}{\ln\ln n}\Big(1 + \frac{\ln\ln\ln n + B + \delta + o(1)}{\ln\ln n}\Big).$$

4.3 Comparison of Bounds.

Let us now optimize the constants $\eta = 1 + \ln\lambda$ and $B = \ln(e\mu) = 1 + \ln\mu$ to minimize the gap between lower and upper bounds on $\chi(G_n)$. We define $s_1 = \sup_{\alpha\in[0,1]} \eta$ and $s_2 = \inf_{\alpha\in[0,1]} B$. By using the definition of $\alpha = \mathbf{Pr}(W \leq w_0)$, we obtain

$$\begin{aligned} s_1 &= 1 + \sup_{\alpha\in[0,1]} \ln\lambda \qquad (18) \\ &= 1 + \sup_{\alpha\in[0,1]} \ln((1-\alpha)nr_0^2\pi) \end{aligned}$$

$$
\begin{aligned}
&= 1 + \sup_{\alpha\in[0,1]} \ln\Big(\frac{\pi}{2}\frac{n}{\theta_n}(1-\alpha)w_0\Big) \\
&= 1 + \ln\frac{\pi}{2} + \sup_{w_0\geq 0} \ln((1-F(w_0))w_0) \\
&= 1 + \ln\frac{\pi}{2} + \ln\big(\sup_{w_0\geq 0} w_0(1-F(w_0))\big).
\end{aligned}
$$

For the other bound, $s_2 = 1 + \inf_{\alpha\in[0,1]} \ln\mu$, and μ is bounded by

$$
\begin{aligned}
\mu &\leq \sum_{j=-1}^{\infty} \mathbf{E}(Z_j) \\
&\leq \pi(12\sqrt{2})^2 \sum_{j=-1}^{\infty} 4^{j+1}(1-F(4^j)),
\end{aligned}
$$

so

$$
s_2 \leq 1 + \ln\Big(1152\pi \sum_{j=-1}^{\infty} 4^j(1-F(4^j))\Big). \tag{19}
$$

Note that the conditions imposed on the weight distribution in Lemma 3.3 are $\mathbf{Pr}(W_i > x) = O(x^{-\gamma})$ for some $\gamma > 1$. Then for $j \geq 0$ it may be helpful to write $1 - F(4^j) = O(4^{-\gamma j}) \leq D4^{-\gamma j}$, with the constant D given by $D = \max_j 4^{\gamma j}(1-F(4^j))$. In that case,

$$
\begin{aligned}
s_2 &\leq 1 + \ln\Big(1152\pi\Big(\frac{1}{4} + \sum_{j=0}^{\infty} 4^j D 4^{-\gamma j}\Big)\Big) \\
&= 1 + \ln\Big(1152\pi\Big(\frac{1}{4} + D\sum_{j=0}^{\infty} 4^{(1-\gamma)j}\Big)\Big) \\
&= 1 + \ln\Big(1152\pi\Big(\frac{1}{4} + \frac{D}{1-4^{1-\gamma}}\Big)\Big).
\end{aligned}
$$

Now the lower and upper bounds on $\chi(G_n)$, respectively,

$$
\frac{\ln n}{\ln\ln n}\Big(1 + \frac{\ln\ln\ln n + s_1}{\ln\ln n}\Big) \leq \chi(G_n)
$$

and

$$
\chi(G_n) \leq \frac{\ln n}{\ln\ln n}\Big(1 + \frac{\ln\ln\ln n + s_2}{\ln\ln n}\Big),
$$

give us the size of the gap

$$
C\ln n/(\ln\ln n)^2. \tag{20}
$$

Finally, the constant C, specified in the abstract, is

$$
C = s_2 - s_1,
$$

where s_1 and s_2 are as above.

4.4 Examples of Bounds.

Here we compare lower and upper bounds for the following two weight distributions: (i) exponential and (ii) power-law.

(i) Exponential weight distribution: $f(w) = e^{-w}$, for $w \geq 0$, and thus $F(w) = 1 - e^{-w}$.

Since $\sup_{w_0 \geq 0} w_0(1 - F(w_0)) = 1/e$ is attained at $w_0 = 1$, then Eq. (18) yields

$$s_1 = \ln(\pi/2) \approx 0.4516. \tag{21}$$

On the other hand, Eq. (19) yields

$$s_2 \leq 1 + \ln\Big(1152\pi \sum_{j=-1}^{\infty} 4^j \exp(-4^j)\Big) \approx 8.7411, \tag{22}$$

giving $C < 8.29$.

(ii) Power-law weight distribution: $f(w) = w^{-\beta}$, for $w \geq 1$, and thus $F(w) = 1 - 1/w^{\beta-1}$.

Since $\sup_{w_0 \geq 1} w_0(1 - F(w_0)) = \sup_{w_0 \geq 1} 1/w_0^{\beta-2} = 1$, attained at $w_0 = 1$ and for $\beta \geq 2$, then Eq. (18) yields

$$s_1 = 1 + \ln(\pi/2) \approx 1.4516. \tag{23}$$

On the other hand, noting that $Z_{-1} = 0$ since no weights are less than 1 here, Eq. (19) becomes

$$\begin{aligned} s_2 &\leq 1 + \ln\Big(1152\pi \sum_{j=0}^{\infty} 4^{j(2-\beta)}\Big) \qquad (24) \\ &= 1 + \ln \frac{1152\pi}{1 - 4^{2-\beta}} \\ &\approx 9.1940 - \ln(1 - 4^{2-\beta}). \end{aligned}$$

If for instance $\beta = 3$, this last bound is ≈ 9.4817, giving $C < 8.04$.

5 Conclusion

In this work, we have derived the chromatic number and proposed a coloring algorithm on GTG, for the case of $\theta_n = \Theta(n)$, that is, when the mean degree is constant. It naturally arises, that we are interested into the values of the chromatic number for denser and sparser GTGs. A particularly interesting case would be to show χ around the connectivity regime. The connectivity threshold has been derived to be $\theta_n = \Theta(n/\ln n)$, [2]. However, the methods that we have used here rely heavily on techniques that work for random geometric graphs of equivalent degree. It is unclear whether those techniqes would apply near the connectivity threshold, because the limiting connectivity regime in RGG, when the typical vertex degree grows logarithmically, is of special interest and is already "hard" [10].

References

[1] C. Avin and G. Ercal, *On the cover time and mixing time of random geometric graphs*, Theor. Comput. Sci., vol. 380, issues 1–2, pp. 2–22, 2007.

[2] M. Bradonjić, A. A. Hagberg, and A. G. Percus, *Giant Component and Connectivity in Geographical Threshold Graphs*, Proceedings of the 5th Workshop on Algorithms and Models for the Web-Graph (WAW2007), pp. 209–216, 2007.

[3] M. Bradonjić, A. A. Hagberg, and A. G. Percus, *The Structure of Geographical Threshold Graphs*, Internet Mathematics, vol. 4, 2009. To appear.

[4] M. Bradonjić and Joseph Kong, *Wireless Ad Hoc Networks with Tunable Topology*, Proceedings of the 45th Annual Allerton Conference on Communication, Control and Computing, 2007.

[5] P. Gupta and P. R. Kumar, *Critical power for asymptotic connectivity*, Proceedings of the 37th IEEE Conference on Decision and Control, vol. 1, pp. 1106–1110, 1998.

[6] A. Goel, S. Rai, and B. Krishnamachari, *Sharp thresholds For monotone properties in random geometric graphs*, Proceedings of the thirty-sixth annual ACM Symposium on Theory of Computing (STOC 2004), pp. 580–586, 2004.

[7] C. McDiarmid, *Random channel assignment in the plane*, Random Struct. Algorithms, vol. 22, no. 2, pp. 187–212, 2003.

[8] C. McDiarmid and T. Müller, *On the chromatic number of random geometric graphs.* Submitted.

[9] T. Müller, *Twopoint concentration in random geometric graphs*, Combinatorica, Volume 28, Issue 5, Pages 529-545, 2009.

[10] M. D. Penrose, *Random Geometric Graphs*, Oxford University Press, 2003.