On the Mixing Time of Geographical Threshold Graphs

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Abstract We study the mixing time of random graphs generated by the geographical threshold graph (GTG) model, a generalization of random geometric graphs (RGG). In a GTG, nodes are distributed in a Euclidean space, and edges are assigned according to a threshold function involving the distance between nodes as well as randomly chosen node weights. The motivation for analyzing this model is that many real networks (e.g., wireless networks, the Internet, etc.) need to be studied by using a "richer" stochastic model (which in this case includes both a distance between nodes and weights on the nodes). We specifically study the mixing times of random walks on 2-dimensional GTGs near the connectivity threshold. If the weight distribution function decays with $\Pr[W \geq x] = o(1/x^{2+\nu})$ for an arbitrarily small constant $\nu > 0$ then the mixing time is $O(n)$.

1 Introduction

In recent years, we have witnessed the development of numerous approaches to study the structure of large real-world technological and social networks, and to optimize processes on these networks. Large networks, such as the Internet, World Wide Web, phone call graphs, infections disease contacts and financial transactions, have provided new challenges for modeling and analysis [Bon05]. As an example, Web graphs may have billions of nodes and edges, which implies that processing and extracting information on these large sets of data, is 'hard' [APR02]. Extensive theoretical and experimental research has been done in web-graph modeling, attempting to capture both the structure and dynamics of the web graph [KRR$^+$00,BA99,ACL00,BRST01,CF01].

In general, a particularly fertile approach is to consider the network as an instance of an ensemble, arising from a suitable random generative model. Since the seminal papers on the evolution of uniform random graph model [ER59,ER60], many other models have been proposed to better capture the structure seen in real-world networks, which are systematically covered in [Dur06]. One straightforward example is the random geometric graph (RGG) model, where nodes are

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placed uniformly at random in a Euclidean space and edges are placed between any two nodes within a threshold distance. For further study of RGGs, see the monograph by Penrose [Pen03]. The RGGs have the advantage of describing many aspects of systems such as sensor networks, while avoiding unnecessary detail. However, they fail to capture heterogeneity in the network.

Geographical threshold graphs (GTGs) are a generalization of RGGs. Heterogeneity in the network is provided via a richer stochastic model that nevertheless preserves much of the simplicity of the RGG model. GTGs assign to nodes both a location and a weight. The weight may represent a quantity such as transmission power in a wireless network or influence in a social network. Edges are placed between two nodes if a symmetric function of their weights and the distance between them exceeds a certain threshold [BK07].

Structural properties of GTGs, such as connectivity, clustering coefficient, degree distribution, diameter, existence and absence of the giant component, chromatic number have been recently analyzed [BHP09,BHP07,BMP09]. These properties are not merely of theoretical importance, but also play an important role in applications. In communication networks, connectivity implies the ability to reach all parts of the network. In packet routing, diameter gives the minimal number of hops needed for transmission between two arbitrary nodes. In the case of epidemics, the existence or absence of the giant component controls whether the epidemic spreads or is contained. When treating the node colors as the different radio channels or frequencies, the chromatic number gives the minimal number of channels needed so that neighboring radios do not interfere with each other.

Herein, we consider random walks on GTGs near the connectivity threshold. Random walks (or more formally, Markov chains) on large networks have many applications. For example, random walks model the spread of disease or the dispersion of information [BGPS06]. The *mixing time* of a random walk is the expected number of random steps that are required to guarantee that the current distribution is close to the stationary distribution. Mixing times are an essential tool in both theory and practice: for example, see the recent survey of Diaconis [Dia09] on Markov chain Monte Carlo methods. In [BBS08], the authors derived the mixing time of exponential random graphs, a model extensively used in sociology, showing that the mixing time of the *Glauber dynamics* was $\Theta(n^2 \log n)$ (exponential) for the unimodal (multimodal) *Gibbs distribution*, respectively. For the definitions of the Glauber dynamics and Gibbs distribution see [BBS08].

The mixing time for RGG at the connectivity threshold has been determined. For the 2-dimensional RGG, Avin and Ercal [AE07] showed that the mixing time is $O(n)$. More recently, Cooper and Frieze [CF09] proved the mixing time is $\tilde{O}(n^{2/d})$ for $d \geq 3$ (in this notation, the logarithmic factors are surpressed). In this paper, we study the mixing times of random walks on 2-dimensional GTGs near the connectivity threshold. We provide a set of criteria on the distribution of node weights that guarantees that the mixing time is $O(n)$.

The GTG model is constructed from of a set of n nodes placed independently uniformly at random into the unit square in $\mathbf{R}^d$. A non-negative weight w_i, taken randomly and independently from a continuous probability distribution function $f(w) : \mathbf{R}_0^+ \to \mathbf{R}_0^+$, is assigned to each node v_i for $i \in \{1, 2, \ldots, n\}$. For nodes i, j at distance r_{ij}, the edge (i, j) exists if and only if the following connectivity relation is satisfied:

$$G(w_i, w_j)d(r_{ij}) \geq \theta_n \,. \tag{1}$$

Here $G(w_i, w_j)$ is the interaction strength between nodes, $d(r)$ is a decreasing function of r and θ_n is a given threshold parameter that depends on the size of the network. The interaction strength $G(w_i, w_j)$ is usually taken to be symmetric and either multiplicatively or additively separable, i.e., in the form of $G(w_i, w_j) = g(w_i)g(w_j)$ or $G(w_i, w_j) = g(w_i) + g(w_j)$. We use $d(r) = r^{-\nu}$ for some positive ν, which is typical for e.g., the path-loss model in wireless networks [BK07].

Some basic results have already been shown, e.g., the expected degree of a node with given weight w, for the case of uniformly distributed nodes over a unit space it has been shown [MMK05,BK07]. In both the multiplicative and additive case of $G(w, w')$, questions of diameter, connectivity, and topology control have been addressed [BK07].

Here we restrict ourselves to the case of $g(w) = w$, $\nu = 2$, and nodes distributed uniformly over a two-dimensional space. For analytical simplicity we take the space to be a unit torus. We concentrate on the analysis of the additive model, i.e., when the connectivity relation for nodes i, j is given by

$$\frac{w_i + w_j}{r_{ij}^2} \geq \theta_n. \tag{2}$$

The connectivity threshold for such graphs has been determined. Let

$$F(x) = \Pr[w \leq x] = \int_0^x f(w)dw$$

be the cumulative density function (cdf) for the weights.

Theorem 1 ([BHP07]). *Let G be a GTG on the unit torus with threshold function $\theta_n = cn/\log n$ where the constant $c < \sup_{\alpha \in (0,1)} \alpha F^{-1}(1-\alpha)/4$. Then G is connected whp*[1].

This paper characterizes the mixing time for a GTG at the connectivity threshold, provided that the weight distribution decays at an adequate rate.

Theorem 2. *Let G be a connected GTG with threshold function $\theta_n = cn/\log n$ on the unit torus. If the weight distribution $f(x)$ satisfies $Pr[W \geq x] = o(1/x^{2+\nu})$ for $\nu > 0$, and $c \leq \frac{1}{5}\min\left\{\sup_{\alpha\in(0,1/2]} \alpha F^{-1}(\alpha), \sup_{\alpha\in(1/2,1)} (1-\alpha)F^{-1}(\alpha)\right\}$, then the mixing time of G is $O(n)$.*

Note that $\Pr[W \geq x] = o(1/x^{2+\nu})$ also ensures that the expectation of the weights is finite. Our proof can be adapted for higher dimensions, giving a mixing bound of $\tilde{O}(n^{2/d})$, where this notation surpresses logarithmic factors.

[1] We will use the notation "with high probability" and denote it as *whp*, meaning with probability at least $1 - o(1/n)$.

The rest of the paper is organized as follows. In Section 2 we derive upper and lower bounds on the maximal weight. Then we find upper and lower bounds on the node degrees, and show that $E(G) = \Theta(n \log n)$. In Section 3 we construct a family of canonical paths for the GTG. In Section 4, we prove our main result that the mixing time for a GTG near the threshold for connectivity is $O(n)$. In Appendix A, we exemplify our results with two different weight functions. Our first example is the exponential weight distribution $f(w) = e^{-w}$ with cumulative density function $F(x) = 1 - e^{-x}$. Our second example is a power law cumulative density function $F(x) = 1 - x^{-\gamma}$ where $\gamma > 2$. In Appendix B, we sketch the adaptation of our proofs for the d-dimensional case, $d \geq 3$ when the weights are governed by the exponential distribution.

2 Node weights and node degrees in GTG

In this section, we determine the upper and lower bounds on the maximal weight $W_{\max}$ in GTG. Then we derive the upper and lower bounds on the degrees of the nodes in GTG near the connectivity threshold. Finally, we show that $|E(G)| = \Theta(n \log n)$ for these connected GTGs. We use $\omega(1)$ to denote an arbitrarily slowly increasing function of n.

The maximal weight $W_{\max}$ satisfies $\Pr[W_{\max} \leq x] = F(x)^n$, since the weights are independently distributed. Consider a continuous weight distribution $f(w)$ with cdf $F(x)$. Our goal is to find two thresholds w_1, w_2, such that $\Pr[W_{\max} \leq w_1] = o(1)$ and $\Pr[W_{\max} \geq w_2] = o(1)$. Define $\rho(x) := -\log(1 - F(x))$, or equivalently, let $F(x)$ satisfy $F(x) = 1 - e^{-\rho(x)}$. $F(x)$ is continuous and increasing, so $\rho(x)$ is continious and increasing as well with $\rho(0) = 0$ and $\rho(\infty) = \infty$. We have

$$\begin{aligned}\lim_{n\to+\infty} \Pr[W_{\max} \leq x] &= \lim_{n\to+\infty} (1 - e^{-\rho(x)})^n \;=\; \lim_{n\to+\infty} e^{-n/e^{\rho(x)}} \\ &= \begin{cases} 0, \text{ for } \rho(x) = \log n - \omega(1) \\ 1, \text{ for } \rho(x) = \log n + \omega(1). \end{cases}\end{aligned}$$

In order to obtain the thresholds w_1 and w_2, we 'invert' $\rho(x) = \log n \pm \omega(1)$. We have $\rho^{-1}(y) = F^{-1}(1 - e^{-y})$, and the thresholds now follow:

$$w_1 = F^{-1}(1 - e^{-(\log n - \omega(1))}) = F^{-1}\Big(1 - \frac{\omega(1)}{n}\Big), \tag{3}$$

$$w_2 = F^{-1}(1 - e^{-(\log n + \omega(1))}) = F^{-1}\Big(1 - \frac{1}{n\omega(1)}\Big). \tag{4}$$

Finally, the maximal weight $W_{\max}$ satisfies

$$\lim_{n\to+\infty} \Pr[W_{\max} \in (w_1, w_2)] = 1, \tag{5}$$

where w_1, w_2 are given by Eq (3), Eq (4), respectively.

Let us now determine the upper and lower bounds for the node degrees. We consider the GTG around the connectivity regime, as described in Theorem 1.

As shown in [BHP07], the degree d_i of a node v_i with weight w_i follows the binomial distribution

$$d_i(\cdot|w_i) \sim \mathrm{Bin}(n-1, p(w_i)) \text{ where } p(w_i) = \frac{\pi}{\theta_n}(w_i + \mu) \tag{6}$$

where $\mu = \mathrm{E}[W]$ is the expected node weight.

Lemma 1. *Let G be a connected GTG with threshold function $\theta_n = cn/\log n$. For any constant c_0 such that $c_0 > 2c/\Big(1 - \sqrt{\frac{c}{\mu\pi}}\Big)$, whp all nodes $v \in V(G)$ satisfy $d(v) \in I_{GTG}$, where*

$$I_{GTG} = \Big[\frac{2\pi\mu}{c_0}\log n, \frac{\pi}{c}F^{-1}\Big(1 - \frac{1}{n\omega(1)}\Big)(1+o(1))\log n\Big].$$

Proof. We apply the Chernoff bound on the degree of a node v with a given weight w:

$$\Pr\Big[d(v|w) \le \mathrm{E}[d(v|w)](1-\delta)\Big] \le \exp(-\mathrm{E}[d(v|w)]\delta^2/2).$$

By having $\mathrm{E}[d(v|w)] = (n-1)p(w)$ and choosing $\delta = 1 - \frac{2c/c_0}{1+\frac{w}{\mu}} > 0$, it follows

$$\begin{aligned}
\Pr\Big[d(v|w) \le \frac{2\pi\mu}{c_0}\log n\Big] &\le \exp(-\mathrm{E}[d(v|w)]\delta^2/2) \\
&\le \exp\Big(-\frac{\mu\pi}{c}\frac{\log n}{n}(n-1)\Big(1+\frac{w}{\mu}\Big)\Big(1 - \frac{2c/c_0}{1+\frac{w}{\mu}}\Big)^2\Big) \\
&= n^{-\frac{\mu\pi}{c}(1-\frac{1}{n})(1+\frac{w}{\mu})\Big(1-\frac{2c/c_0}{1+\frac{w}{\mu}}\Big)^2}.
\end{aligned} \tag{7}$$

Next, we will find conditions such that Eq. (7) is $o(1/n)$ for all $w \ge 0$ and sufficiently large n. For the sake of simplicity, let us denote $x = 1 + w/\mu \ge 1$, and consider the function $\phi(x) = \frac{\mu\pi}{c}x(1 - \frac{2c}{c_0 x})^2$. It follows that the minimum of $\phi(x)$ is attained at $x = 2c/c_0$. Moreover, $\phi(x)$ is strictly decreasing on $(0, 2c/c_0)$ and strictly increasing on $(2c/c_0, +\infty)$. By taking $2c/c_0 \le 1$ and $\phi(1) = \frac{\pi\mu}{c}(1 - 2c/c_0)^2 > 1$, or equivalently $c_0 > 2c/\Big(1 - \sqrt{c/\mu\pi}\Big)$, it follows that $\phi(x) > 1$ for $x \ge 1$. That is, Eq. (7) is $o(1/n)$, for sufficiently large n. Thus, the degree distribution satisfies

$$\begin{aligned}
\Pr\Big[d(v) \le \frac{2\pi\mu}{c_0}\log n\Big] &= \int dw f(w)\Pr\Big[d(v|w) \le \frac{2\pi\mu}{c_0}\log n\Big] \\
\le \int dw f(w) n^{-\frac{\mu\pi}{c}(1-\frac{1}{n})(1+\frac{w}{\mu})\Big(1-\frac{2c/c_0}{1+\frac{w}{\mu}}\Big)^2} &= o\Big(\int dw f(w) n^{-1}\Big) = o(1/n).
\end{aligned}$$

The union bound gives the lower bound on degree of the nodes in the graph.

We now obtain the upper bound. By Eq. (3) and Eq. (4), $F^{-1}(1 - \omega(1)/n) \le W_{\max} \le F^{-1}(1 - 1/n\omega(1))$. Moreover, the continuity of $F^{-1}(x)$, ensures that for any $\epsilon > 0$ and any function $\omega(1)$, there is sufficiently large $n = n(\epsilon)$, such that the upper and lower bounds on $W_{\max}$ are arbitrarily close $(w_2^t - w_1^t) \leq \epsilon$.

The degree of the node with maximal weight satisfies the binomial distribution $\mathrm{Bin}(n-1, (\pi/\theta_n)(W_{\max} + \mu))$, which is concentrated around its mean $(\pi/c)(1-1/n)(W_{\max}+\mu)\log n$. Finally, the union bound gives the upper bound on the degrees. □

We now partition the interval I_{GTG}. This partition is used here to calculate $|E(G)|$ and again in section 4 to bound the mixing time. Writing $F(x) = 1 - 1/h(x)$, we assume that $h(x) \gg x^{2+\nu}$ for some small constant $\nu > 0$. Our first interval contains the low weight nodes:

$$B = \{v \in V(G) \mid w_v \leq F^{-1}(1-\alpha)\} = \{v \in V(G) \mid F(w_v) \leq 1-\alpha\}. \tag{8}$$

By Eq. (3), $W_{\max} = F^{-1}(1 - \frac{1}{n\omega(1)}) = h^{-1}(n\omega(1))$.

We now partition the αn nodes with weights in $[F^{-1}(1-\alpha), W_{\max}]$. Let

$$\begin{aligned} a_0 &= W_{\max} \\ a_k &= \max\left\{F^{-1}\left(1 - a_{k-1}^{-(1+\epsilon)}\right), F^{-1}(1-\alpha)\right\}, \text{ for } k \geq 1. \end{aligned}$$

The a_k are only defined until we reach $F^{-1}(1-\alpha)$. Call this final index M. Our partition consists of the subintervals of the form $(a_k, a_{k-1}]$ for $1 \leq k \leq M$. Note that the indexing of our endpoints is the reverse of the standard convention.

For $1 \leq k \leq M$, let

$$A_k = \{v \in V(G) \mid w_v \in (a_k, a_{k-1}]\}. \tag{9}$$

The degree of $v \in A_k$ is $O((1+w_v)\log n) = O(a_{k-1}\log n)$. Furthermore, we have $\Pr[W > a_k] = 1/a_{k-1}^{1+\epsilon}$ and therefore $\mathrm{E}[|A_k|] = \Theta\left(n/a_{k-1}^{1+\epsilon}\right)$. By the Chebychev inequality, the actual value is tightly concentrated around its mean because $a_{k-1}^{1+\epsilon} \leq a_0^{1+\epsilon} = W_{\max}^{1+\epsilon} = (h^{-1}(n\omega(1))^{1+\epsilon} = o(n)$.

Lemma 2. $\sum_{k=0}^{M} a_k^{-\epsilon} = \Theta(1)$.

Proof. We use the general d'Alembert's convergence criterion: the sum of positive terms $\sum_{k=0}^{\infty} c_k$ is convergent if there exist a positive integer N and $\eta > 0$ such that $k > N$ guarantees $c_k/c_{k-1} < 1 - \eta$.

Let $b_0 = F^{-1}(1-\alpha)$ and $b_k = h(b_{k-1})^{1/(1+\epsilon)}$ for $1 \leq k \leq M$. We have $b_k \leq a_{M-k-1}$ for $0 \leq k \leq M$. It follows that

$$\sum_{k=0}^{M} \frac{1}{a_k^{\epsilon}} \leq \frac{1}{a_M^{\epsilon}} + \sum_{k=0}^{\infty} \frac{1}{b_k^{\epsilon}} \leq 1 + \sum_{k=0}^{\infty} \frac{1}{b_k^{\epsilon}}.$$

and

$$\frac{1/b_k^{\epsilon}}{1/b_{k-1}^{\epsilon}} = \left(\frac{b_{k-1}}{b_k}\right)^{\epsilon} = \left(\frac{b_{k-1}}{h(b_{k-1})^{1/(1+\epsilon)}}\right)^{\epsilon} < 1 - \eta$$

for large k since $h(x) \gg x^{2+\nu} \gg x^{1+\epsilon}$. Therefore this sum converges to a constant (independent of n). □

Lemma 3. *If G is a GTG at the connectivity threshold then* $|E(G)| = \Theta(n \log n)$ *whp.*

Proof. Lemma 1 guarantees that all nodes have degree $\Omega(\log n)$ whp, so $|E(G)| = \Omega(n \log n)$. As for the upper bound, whp

$$\begin{aligned} 2|E(G)| &= \sum_{v \in B} \deg(v) + \sum_{k=1}^{M} \sum_{v \in A_k} \deg(v) \;=\; O\Big(\Big(|B| + \sum_{k=1}^{M} |A_k| \cdot a_{k-1}\Big) \log n\Big) \\ &= O\Big(\Big(\alpha n + \sum_{k=1}^{M} \frac{n}{a_{k-1}^{1+\epsilon}} \cdot a_{k-1}\Big) \log n\Big) \;=\; O(n \log n) \end{aligned}$$

by Lemma 2. □

3 Canonical paths for GTG

We employ canonical paths (as introduced in [JS96]) to calculate our bound on the mixing time. Throughout this section, we let G be a connected GTG with threshold function $\theta_n = cn/\log n$. For every ordered pair of nodes $u, v \in V(G)$ we choose a canonical path γ_{uv} between them. We define

$$\begin{aligned} \rho &= \max_{e=\{x,y\} \in E(G)} \frac{1}{\pi(x) P(x,y)} \sum_{\gamma_{uv} \ni e} \pi(u)\pi(v)|\gamma_{uv}| \\ &= \max_{e=\{x,y\} \in E(G)} \frac{1}{2|E(G)|} \sum_{\gamma_{uv} \ni e} \deg(u)\deg(v)|\gamma_{uv}| \end{aligned} \tag{10}$$

where $|\gamma_{uv}|$ is the length of the canonical path from u to v. As per [JS96] Proposition 12.1, the mixing time from node x satisfies

$$\tau_x(\delta) \le \rho\left(\log \pi(x)^{-1} + \log \delta^{-1}\right) \tag{11}$$

where $\delta > 0$. Our strategy for constructing the canonical paths is similar to the one used in [CF09] to bound the mixing time of random geometric graphs. We partition the unit square into a toric grid of $k \times k$ small squares, where k is specified below. A set of canonical paths for the grid will act as the framework for our canonical paths for the GTG. The canonical path from (a, b) to (c, d) is

$$(a, b), (a + 1 \bmod k, b), \ldots, (c, b), (c, b + 1 \bmod k), \ldots, (c, d). \tag{12}$$

Clearly the maximum length of a path is $2k$. Note that we always increment the index by $+1$ (even if there is a shorter path). While there are k^4 canonical paths, each edge appears in at most k^3 of them. Indeed if the canonical path from (a, b) to (c, d) traverses the edge $((i, j), (i+1, j)$ then $b = j$, leaving at most k^3 choices for a, c, d. Similarly, if we traverse the edge $((i, j), (i, j + 1)$ then $i = c$, leaving k^3 choices for a, b, d.

We begin to construct the canonical paths for our GTG. Let G be a GTG at the connectivity threshold $\theta_n = cn/\log n$ for $c < \sup_{\alpha\in(0,1)} \alpha F^{-1}(1-\alpha)/4$, as per Theorem 1. The proof of this result in [BHP09] establishes the following two facts. First, a constant fraction of the nodes αn have weights greater than $F^{-1}(1-\alpha)$. We let $H(G) = \{i \in V(G) \mid w_i > F^{-1}(1-\alpha)\}$ denote this set of *high weight nodes* and let $L(G) = \{i \in V(G) \mid w_i \geq F^{-1}(1-\alpha)\}$ denote the complementary set of *low weight nodes*. Second, every high weight node is connected to every node within the *critical radius*

$$r_c = \sqrt{\frac{\log(\alpha n)}{\pi \alpha n}}. \tag{13}$$

Tile $[0,1]^2$ into squares S_i with side length $k = \sqrt{c_0 \log n/n} = \Theta(1/r_c)$, where we state the conditions on constant c_0 later. We have a $k \times k = \Theta(r_c^{-1}) \times \Theta(r_c^{-1})$ grid, whose squares each have area $c_0 \log n/n$. Let $H(S_i)$ and $L(S_i)$ denote the high weight and low weight nodes in S_i, respectively.

Lemma 4. *There exist constants $\beta_0, \beta_1 > 0$ such that whp every square S satisfies $\beta_0 \log n \leq |L(S)| \leq \beta_1 \log n$ and $\beta_0 \log n \leq |H(S)| \leq \beta_1 \log n$.*

Proof. This proof is similar to the proof of Theorem 3 in [BHP09], and will use the same notation. Let $B_i = |H(S_i)|$ and $R_i = |L(S_i)|$. In expectation, there are $\mathrm{E}[B_i] = \alpha c_0 \log n$ high weight nodes within S_i. Using the lower and upper tail Chernoff bounds [Che81], it follows

$$\Pr[B_i \leq (1-\delta_1)\mathrm{E}[B_i]] \leq \exp\left(-\frac{\delta_1^2}{2}\mathrm{E}[B_i]\right)$$
$$\Pr[B_i \geq (1+\delta_2)\mathrm{E}[B_i]] \leq \exp\left(-\frac{\delta_2^2}{2(1+\delta_2)}\mathrm{E}[B_i]\right).$$

For any $\delta_2 \in [0,1]$, let $\delta_1 = \delta_2/\sqrt{1+\delta_2}$, and thus $\delta_1 \in [0,1]$. The number of high weighted nodes B_i within the square S_i satisfies

$$\Pr\left[B_i \in \left(1 - \frac{\delta_2}{\sqrt{1+\delta_2}}, 1+\delta_2\right)\mathrm{E}[B_i]\right] \geq 1 - 2\exp\left(-\frac{\delta_2^2}{2(1+\delta_2)}\mathrm{E}[B_i]\right)$$
$$= 1 - 2n^{-\alpha c_0 \delta_2^2/(1+\delta_2)}.$$

By using the spatial independence among the squares S_i, that is, the fact that nodes are tossed over the squares independently, it follows

$$\Pr\left[\bigcap_i \left\{B_i \in \left(1 - \frac{\delta_2}{\sqrt{1+\delta_2}}, 1+\delta_2\right)\mathrm{E}[B_i]\right\}\right] \geq \left(1 - 2n^{-\alpha c_0 \delta_2^2/(1+\delta_2)}\right)^{n/(c_0 \log n)}.$$

Taking the limit of the last expression as $n \to +\infty$, for $c_0 \geq (1+\delta_2)(\alpha\delta_2^2)$, we obtain the concentration on B_i, for each square S_i:

$$\lim_{n\to\infty} \Pr\left[\bigcap_i \left\{B_i \in \left(1 - \frac{\delta_2}{\sqrt{1+\delta_2}}, 1+\delta_2\right)\alpha c_0 \log n\right\}\right] = 1.$$

We now prove the concentration on the number of low weight nodes R_i, within each square S_i. In expectation, there are $\mathrm{E}[R_i] = (1-\alpha)c_0 \log n$ low weight nodes, within S_i. Thus we can apply the Chernoff tail bounds on R_i, and analogously to the previous analysis we obtain

$$\lim_{n\to\infty} \Pr\left[\bigcap_i \left\{ R_i \in \left(1 - \frac{\delta_2}{\sqrt{1+\delta_2}}, 1+\delta_2\right)(1-\alpha)c_0 \log n \right\}\right] = 1.$$

Finally, we can guarantee the concentration of both the high and low weight nodes by taking $c_0 \geq \max\left\{\frac{1+\delta_2}{\alpha\delta_2^2}, \frac{1+\delta_2}{(1-\alpha)\delta_2^2}\right\}$, that is,

$$c_0 \geq \begin{cases} \frac{1+\delta_2}{\alpha\delta_2^2} \text{ if } \alpha \in (0, 1/2) \\ \frac{1+\delta_2}{(1-\alpha)\delta_2^2} \text{ if } \alpha \in [1/2, 1). \end{cases} \tag{14}$$

So the lemma holds with $\beta_0 = \left(1 - (\delta_2)(\sqrt{1+\delta_2})\right)\min\{\alpha, 1-\alpha\}$ and $\beta_1 = (1+\delta_2)\max\{\alpha, 1-\alpha\}$. □

Two squares are *adjacent* if they share either a horizontal or vertical side.

Corollary 1. *Let S_i, S_j be adjacent squares. The number of edges between $H(S_i)$ and $H(S_j)$ is $\Omega(\log^2 n)$ provided that $c \leq \frac{1}{5}\min\{\sup_{x\in(1/2,1)}(1-x)F^{-1}(x),$ $\sup_{x\in(0,1/2]} xF^{-1}(x)\}$.*

Proof. Consider any high weight node $b_i \in H(S_i)$ and any high weight node $b_j \in H(S_j)$, with the weights w_{b_i} and w_{b_j}, respectively. The distance $r_{b_ib_j}$ between b_i and b_j is at most $\sqrt{5c_0 \log n/n}$. Consider the connectivity relation

$$\frac{w_{b_i} + w_{b_j}}{r_{b_ib_j}^2} \geq \frac{F^{-1}(1-\alpha) + F^{-1}(1-\alpha)}{(\sqrt{5c_0 \log n/n})^2} = \frac{2F^{-1}(1-\alpha)}{5c_0}\frac{n}{\log n}.$$

The high weight nodes b_i, b_j are connected with probability one if $(w_{b_i}+w_{b_j})/r_{b_ib_j}^2 \geq \theta_n = cn/\log n$, which is guaranteed if $2F^{-1}(1-\alpha)/(5c_0) \geq c$. Using Eq. (14), we require

$$c \leq \frac{2}{5}\frac{\delta_2^2}{1+\delta_2}\begin{cases} \alpha F^{-1}(1-\alpha) \text{ if } \alpha \in (0, 1/2) \\ (1-\alpha)F^{-1}(1-\alpha) \text{ if } \alpha \in [1/2, 1). \end{cases}$$

Since $\delta_2 \in (0,1)$ is arbitrary, with $\sup_{\delta_2\in(0,1)} \delta_2^2/(1+\delta_2) = 1$, the conditions combine to give

$$\begin{aligned} c &\leq \frac{1}{5}\min\Big\{\sup_{\alpha\in(0,1/2)} \alpha F^{-1}(1-\alpha), \sup_{\alpha\in[1/2,1)} (1-\alpha)F^{-1}(1-\alpha)\Big\} \\ &= \frac{1}{5}\min\Big\{\sup_{x\in(1/2,1)} (1-x)F^{-1}(x), \sup_{x\in(0,1/2]} xF^{-1}(x)\Big\}. \end{aligned}$$

□

We employ a randomized procedure for picking canonical paths. This procedure will guarantee that no edge appears in more than $\Theta(k^3) = \Theta(r_c^{-3}) =$

$\Theta((n/\log n)^{3/2})$ canonical paths. We now denote squares as $S_{a,b}$ for $1 \leq a, b \leq k$, where (a,b) is the location in the $k \times k$ grid. By Lemma 4, we have $\beta_0 \log n \leq |L(S_{a,b})| \leq \beta_1 \log n$ and $\beta_0 \log n \leq |H(S_{a,b})| \leq \beta_1 \log n$ for constants $\beta_0, \beta_1 > 0$. For each square $S_{a,b}$, evenly partition $L(S_{a,b})$ into β_0 sets $\{L_i(S_{a,b})\}_{i=1}^{\beta_0}$. Each set in this partition will contain either $\lfloor \beta_1/\beta_0 \rfloor$ or $\lceil \beta_1/\beta_0 \rceil$ low weight nodes. Finally, choose a distinct high weight node $h_i \in H(S_{a,b})$ for $1 \leq i \leq k$. The high weight node h_i is the *high weight representative* for the nodes in $L_i(S_{a,b})$. Note that each h_i represents at most $\lceil \beta_1/\beta_0 \rceil = O(1)$ nodes.

Consider any ordered pair of nodes (x,y). Let $x \in S_{a,b}$ and $y \in S_{c,d}$. We choose a canonical path from x to y as follows. Use the toric grid to identify the sequence of squares in the canonical path: $S_{a,b}$, $S_{a+1,b}$, …, $S_{c,b}$, $S_{c,b+1}$, …, $S_{c,d}$. For brevity, call these squares $S_0, S_1, \ldots, S_t$. If x is a low weight node, then set x_0 to be the high weight representative for x. If x is a high weight node, set $x_0 = x$. For $1 \leq i \leq t-1$, choose x_{i+1} to be a random high weight node in S_i. Set $x_t = y$ if y is a high weight node, otherwise using the high weight representative for y. Our canonical (x,y) path is $x, x_0, x_1, \cdots x_t, y$. If $x_0 = x$ or $x_t = y$, we remove the repeated node from the path.

Lemma 5. *Every edge in G appears in at most $O((n/\log n)^{3/2})$ canonical paths.*

Proof. Let Z_{xy} denote the number of times the edge xy is chosen. If both x and y are low weight nodes then $Z_{xy} = 0$ (we always move to a high weight node from a low weight node). When x is low weight and y is high weight, then the edge xy can only be used when x and y are in the same square. In this case, the edge xy can only be chosen when the canonical path has x as one of its endpoints. Therefore $Z_{xy} \leq 2(n-1)$.

Now suppose that both x and y are high weight nodes. The edge xy is only used if x, y are in the same square or in adjacent squares. First, we consider high weight x, y in the same square S. If $x = h_i$ and $y = h_j$, then xy will be used only by paths between $L_i(S) \cup \{x\}$ and $L_j(S) \cup \{y\}$. If one or both is not a high weight representative, then this edge will be used even fewer times. Therefore $Z_{xy} \leq (1 + \lceil \beta_1/\beta_0 \rceil)^2 = O(1)$ whp.

Consider $x \in S$ and $y \in T$ in adjacent squares S, T. Let $\mathcal{P}_{xy} = \{\gamma_{uv} \mid xy \in \gamma_{uv}\}$. We consider four cases, according to the locations of u and v. If $u \in S$ and $v \in T$ then just as in the previous case, whp there are at most $(1 + \lceil \beta_1/\beta_0 \rceil)^2 = O(1)$ such paths.

The total number of paths γ_{uv} with $u \in S$ and $v \notin T$ is $O(n)$. Indeed, x must be the high weight representative of u, which gives $1 + \lceil \beta_1/\beta_0 \rceil = O(1)$ choices for u, and there are $O(n)$ end nodes for paths that start in S_1. Similarly, the total number of paths with $u \notin S$ and $v \in T$ is $O(n)$.

The remaining case is when $u \notin S$ and $v \notin S$. This means that both x and y were chosen randomly to realize the toric canonical path. Let Z'_{xy} denote the number of times the edge xy is chosen as the random edge from S to T for some canonical path. Let $\mathcal{P}(S,T) = \{\gamma_{uv} \mid \gamma_{uv}$ traverses from S to T and $u \notin S$ and

$v \notin T\}$. Then whp

$$\mathrm{E}[Z'_{xy}] = \sum_{\gamma_{uv} \in \mathcal{P}(S,T)} \Pr(xy \in \gamma_{uv}) \le r_c^{-3} (\beta_1 \log n)^2 \cdot \frac{1}{(\beta_0 \log n)^2} = \left(\frac{\beta_1}{\beta_0}\right)^2 r_c^{-3}.$$

Indeed, the number of canonical paths for the toric $k \times k$ grid which pass from S to T is $O(r_c^{-3})$. Each toric path corresponds to at most $(\beta_1 \log n)^2$ paths in the GTG. Since both x and y are internal nodes of these paths, they were each chosen uniformly and independently with probability at most $1/(\beta_0 \log n)$.

Using Chernoff for this binomial distribution,

$$\begin{aligned}\Pr\left[|Z'_{xy} - \mathrm{E}[Z'_{xy}]| \ge \epsilon \mathrm{E}[Z'_{xy}]\right] &\le 2\exp\Big(-\frac{\epsilon^2}{3}\mathrm{E}[Z'_{xy}]\Big) \;\le\; 2\exp\Big(-\frac{\epsilon^2}{3}\big(\frac{\beta_1}{\beta_0}\big)^2 r_c^{-3}\Big) \\ &= 2\exp\Big(-\frac{\epsilon^2}{3}\big(\frac{\beta_1}{\beta_0}\big)^2\big(\frac{n}{\alpha\pi\log n}\big)^{3/2}\Big) \;=\; o(e^{-n})\end{aligned}$$

The union bound now gives

$$\begin{aligned}&\Pr\left[\wedge_{x,y}\left(|Z'_{xy} - \mathrm{E}[Z'_{xy}]| \ge \epsilon\mathrm{E}[Z'_{xy}]\right)\right] \\ &\le \sum_{x,y} \Pr\left[|Z'_{xy} - \mathrm{E}[Z'_{xy}]| \ge \epsilon\mathrm{E}[Z'_{xy}]\right] \;\le\; n^2 o(e^{-n}) \to 0.\end{aligned}$$

Therefore whp, every edge between high weight nodes in adjacent squares is used by $(1 \pm \epsilon) r_c^{-3} = \Theta((n/\log n)^{3/2})$ canonical paths. In this case, $\mathrm{E}[Z_{xy}] = \Theta((n/\log n)^{3/2})$. □

4 The mixing time for GTG

In this section, we prove that the mixing time for a GTG near the threshold for connectivity is $O(n)$.

Lemma 6. *Our choice of canonical paths gives $\rho = O(n/\log n)$ when $Pr[W \ge x] = o(1/x^{2+\nu})$ for any constant $\nu > 0$.*

Proof. Recall that $|E(G)| = \Theta(n \log n)$ and $\max_{u,v \in V(G)} |\gamma_{uv}| = O((n/\log n)^{1/2})$. By equation (10),

$$\rho \le O\left(\frac{1}{n^{1/2}(\log n)^{3/2}}\right)\left(\max_{e=\{x,y\}\in E(G)} \sum_{\gamma_{uv} \ni e} \deg(u)\deg(v)\right).$$

Fix an edge $e = \{x, y\}$ between high weight nodes in adjacent squares. Without loss of generality, assume that these squares share a vertical boundary. Let $\sigma(B, B)$ denote the contribution of paths between low weight nodes to $\sum_{\gamma_{uv} \ni e} \deg(u)\deg(v)$. We have

$$\mathrm{E}[\sigma(B,B)] = O\Big(\big(\sqrt{\frac{\log n}{n}}n\log n\big)\big(n \log n\big)\,(1/\log n)^2\Big) = O\big(n^{3/2}(\log n)^{1/2}\big).$$

Indeed, if $e \in \gamma_{uv}$ then u must appear in the a horizontal strip of width $\sqrt{\log n/n}$ and length 1. This strip contains $\Theta(\sqrt{n \log n})$ low weight nodes whp. These nodes have degree $\Theta(\log n)$. The second term counts every low weight node as a potential end point, each one contributing $\Theta(\log n)$ for its degree. The $(\log n)^{-2}$ reflects the random choice of edge to traverse from one square to the next. We are guaranteed that $\sigma(B,B)$ is concentrated around its mean by Chebychev's inequality. So $\sigma(B,B) = O(n^{3/2}(\log n)^{1/2})$ whp.

We consider the contribution of paths between high weight nodes by using the partition $A_1, A_2, \ldots, A_M$ specified in equation (9). Arguing similarly to the calculation above, the contribution to ρ from paths between nodes in A_r and A_s where $0 \leq r, s \leq M$ is whp

$$\begin{aligned}\sigma(A_r, A_s) &= O\Big(\Big(\sqrt{\frac{\log n}{n}}\frac{n}{a_r^{1+\epsilon}}(a_r \log n)\Big)\Big(\frac{n}{a_s^{1+\epsilon}}(a_s \log n)\Big)\big(1/\log n\big)^2\Big) \\ &= O\Big(\frac{n^{3/2}(\log n)^{1/2}}{a_r^{\epsilon} a_s^{\epsilon}}\Big).\end{aligned}$$

Here the number of nodes in A_r that are in the horizontal strip of width $\sqrt{\log n/n}$ is concentrated around its expected value

$$\frac{\sqrt{n \log n}}{a_r^{1+\epsilon}} \geq \frac{\sqrt{n \log n}}{W_{\max}^{1+\epsilon}} = \frac{\sqrt{n \log n}}{(n\omega(1))^{(1+\epsilon)/(2+\nu)}} = \omega(\sqrt{\log n})$$

as far as $\epsilon < \nu/2$. Next we consider the paths between low weight and high weight nodes: whp

$$\sigma(B, A_s) = O\Big(\Big(\sqrt{\frac{\log n}{n}} n \log n\Big)\Big(\frac{n(a_s \log n)}{a_s^{1+\epsilon}}\Big)\big(1/\log n\big)^2\Big) = O\Big(\frac{n^{3/2}(\log n)^{1/2}}{a_s^{\epsilon}}\Big),$$

and similarly, $\sigma(A_r, B) = O(n^{3/2}(\log n)^{1/2}/a_r^{\epsilon})$. Finally, by Lemma 2, we have whp

$$\begin{aligned}\rho &= O\Big(\frac{n}{\log n}\Big(\sum_{i=1}^{M}\frac{1}{a_i^{\epsilon}} + \sum_{j=1}^{M}\sum_{k=1}^{M}\frac{1}{a_j^{\epsilon}a_k^{\epsilon}}\Big)\Big) \\ &= O\Big(\frac{n}{\log n}\Big(\sum_{i=1}^{M}\frac{1}{a_i^{\epsilon}} + \Big(\sum_{j=1}^{M}\frac{1}{a_j^{\epsilon}}\Big)^2\Big)\Big) \;=\; O\Big(\frac{n}{\log n}\Big).\end{aligned}$$

□

Proof of Theorem 2 Equation (11) gives $\tau_x(\delta) \leq \rho\left(\log \pi(x)^{-1} + \log \delta^{-1}\right)$. The previous lemma ensures that $\rho = O(n/\log n)$. Meanwhile, we have $\pi(x) = \deg(x)/2|E(G)| = \Omega(\log n/(n \log n)) = \Omega(n)$ by Lemma 1 and Lemma 3. In summary, $\tau_x(\delta) = O(n + n \log \delta^{-1}/\log n) = O(n)$ for $\delta = 1/n$. □

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A Characteristics of Geometric Threshold Graph for Example Weight Distributions

We describe the relevant characteristics of GTGs for two different weight distributions: one that decays exponentially and a second that decays polynomially.

A.1 Exponential Weight Distribution

Our first example is the exponential weight distribution $f(w) = e^{-w}$ with cumulative density function $F(x) = 1 - e^{-x}$. Inverting the cdf gives $F^{-1}(x) = -\log(1-x)$.

We first discuss the weights and degrees of the nodes in the GTG, as described in Section 2. As per equaion (5), the maximum weight satisfies

$$W_{\max} \in [\log n - \omega(1), \log n + \omega(1)]$$

whp. Lemma 1 guarantees that whp all the node degrees are in the interval

$$I_{GTG} = \left[\Theta(\log n), \Theta(\log^2 n)\right].$$

Next, we partition the interval I_{GTG}. By equation (8), the cutoff for low weight nodes is $F^{-1}(1-\alpha) = \log \alpha^{-1}$. We partition the high weight nodes into disjoint subsets as specified in equation (9). Note that $F^{-1}\left(1 - x^{-(1+\epsilon)}\right) = \log(x^{1+\epsilon}) = \Theta(\log x)$. Therefore, ignoring leading constants, the sequence of endpoints (in descenting order) is

$$\big(\log n, \log\log n, \log\log\log n, \ldots, \log^{(M)} n\big)$$

where $M \leq \log^* n$, also known as the iterative log function. This is the number of iterations of the log function required to obtain a result is less than 1.

Finally, we calculate upper bound on the constant c required by Corollary 1 in Section 3. For the exponential distribution, taking $c \leq (5e)^{-1}$ is sufficient.

A.2 Power Law Cumulative Density Function

We now give a parallel characterization of our second example: a power law cumulative density function $F(x) = 1 - x^{-\gamma}$ where $\gamma > 2$. Inverting this cdf gives $F^{-1}(x) = (1-x)^{-1/\gamma}$.

We consider our results concerning node weights and node degrees. We have

$$W_{\max} \in \left[\left(\frac{n}{\omega(1)} \right)^{1/\gamma}, (n\omega(1))^{1/\gamma} \right]$$

whp by equation (5). By Lemma 1, whp all the node degrees are in the interval

$$I_{GTG} = [\Theta(\log n), \Theta((n\,\omega(1))^{1/\gamma} \log n)].$$

We separate the low weight nodes from the high weight nodes using the weight cutoff $F^{-1}(1-\alpha) = \alpha^{-1/\gamma}$. Next, we partition the high weight nodes as per equation (9). Note that $F^{-1}\left(1 - x^{-(1+\epsilon)}\right) = x^{(1+\epsilon)/\gamma}$. It follows that our sequence of endpoints is

$$((n\,\omega(1))^{1/\gamma}, (n\,\omega(1))^{\beta/\gamma}, (n\,\omega(1))^{\beta^2/\gamma}, \ldots, (n\,\omega(1))^{\beta^M/\gamma})$$

where $\beta = (1+\epsilon)/\gamma < 1$. Here M is the smallest integer such that $(n\omega(1))^{\beta^M/\gamma} \leq F^{-1}(1-\alpha) = \alpha^{-1/\gamma}$. The latter requirement is equivalent to the condition $M \geq (\log\log(n\omega(1)) - \log\log(\alpha^{-1}))/\log(\beta^{-1})$.

Finally, we calculate an upper bound on the constant c required by Corollary 1 in Section 3. Since $F^{-1}(x) = (1-x)^{-1/\gamma}$, by Corollary 1 it follows $c \leq (5 \cdot 2^{(\gamma-1)/\gamma})^{-1}$.

B GTGs in Higher Dimensions with Exponential Weight Distribution

Our arguments for the 2-dimensional GTG can be adapted for higher dimensions. In this appendix, we sketch the key modifications to obtain a mixing time for a d-dimensional GTG at the threshold for connectivity. For technical reasons related to Lemma 6, we restrict ourselves to the exponential weight distribution. Our results give a mixing time of $\tilde{O}(n^{2/d})$ (ignoring logarithmic factors). This matches the mixing result of [CF09] for the d-dimension RGG. We conjecture that this same mixing bound holds for any weight distribution that decays with $\Pr[w > x] = o(1/x^{d+\nu})$ for an arbitrary constant ν.

For a d-dimensional GTG, the connectivity relation of equation (1) becomes

$$\frac{w_i + w_j}{r^d} \geq \theta_n.$$

In place of equation (6), we have

$$d_i(\cdot|w_i) \sim \text{Bin}(n-1, p(w_i)) \text{ where } p(w_i) = \frac{\Upsilon_d}{\theta_n}(w_i + \mu)$$

where Υ_d is the volume of the unit ball in d dimensions

$$\Upsilon_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)} = \begin{cases} \pi^k/k!, & d = 2k \text{ even}, \\ 2^d k! \pi^k/d!, & d = 2k+1 \text{ odd}. \end{cases}$$

Moreover, we must replace any instance of π with Υ_d. For example, the formulation of Lemma 1 for d dimension is that for $c_0 > 2c/\left(1 - \sqrt{c/\mu\Upsilon_d}\right)$, the degrees are contained in the interval

$$I_{GTG} = \left[\frac{2\Upsilon_d \mu}{c_0} \log n, \frac{\Upsilon_d}{c} F^{-1}\left(1 - \frac{1}{n\omega(1)}\right)(1+o(1)) \log n\right].$$

The creation of the canonical paths for a grid in the d-dimensional unit torus in analogous to the canonical path in equation (12). The canonical path from $(a_1, a_2, \ldots, a_d)$ to $(b_1, b_2, \ldots, b_d)$ is

$$\begin{gathered}(a_1, a_2, \ldots, a_d),\ (a_1+1, a_2, \ldots, a_d),\ \ldots,\ (b_1-1, a_2, \ldots, a_d),\ (b_1, a_2, \ldots, a_d), \\ (b_1, a_2+1, \ldots, a_d),\ \ldots,\ (b_1, b_2, \ldots, b_d - 1),\ (b_1, b_2, \ldots, b_d).\end{gathered}$$

Each path has length at most dk. Each edge is used in at most k^{d+1} paths. Indeed, any path than includes the edge from $(i_1, \ldots i_t, \ldots i_d)$ to $(i_1, \ldots i_t{+}1, \ldots i_d)$ must start at $(l_1, \ldots l_t, j_{t+1}, \ldots, j_d)$ and end at $(j_1, \ldots j_t{-}1, l'_t, \ldots, l'_d)$ for some $l_1, \ldots l_t$ and $l'_t, \ldots, l'_d$.

The d-dimensional equivalent of the critical radius in equation (13) is

$$r_c = \left(\frac{\log(\alpha n)}{\Upsilon_d \alpha n}\right)^{1/d}$$

We create a d-dimensional grid of cubes with side length $k = \Theta(1/r_c)$, where the leading constant is chosen to be small enough to guarantee the following two facts. First, each cube will contain $\Theta(\log n)$ low weight nodes and $\Theta(\log n)$ high weight nodes. Second, there are $\Omega(\log^2 n)$ edges between pairs of high weight nodes drawn from adjacent cubes, where we consider two cubes to be adjacent if they share a $(d-1)$-dimensional boundary.

As in Section 3, we employ a randomized procedure to realize our GTG canonical paths, using the grid of cubes as a framework. We can guarantee that every edge is used in at most $\Theta(r_c^{d+1})$ canonical paths. Using the fact that the weights are governed by the exponential distribution, we can adapt Lemma 6 to obtain

$$\rho = O\left(\frac{\log n}{n} r_c^{-d-2}\right) = O\left(\left(\frac{n}{\log n}\right)^{2/d}\right).$$

In particular, the exponential weight function guarantees that the number of nodes with weights in A_k (defined in equation (9)) is concentrated around its mean. This upper bound on ρ gives a mixing time of

$$\tau_x(\delta) = O\left(n^{2/d}(\log n)^{1-2/d} + \left(\frac{n}{\log n}\right)^{2/d} \log \delta^{-1}\right) = \tilde{O}\left(n^{2/d}\right)$$

for $\delta = 1/n$.