

The Structure of Geographical Threshold Graphs

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Abstract. We analyze the structure of random graphs generated by the geographical threshold model. The model is a generalization of random geometric graphs. Nodes are distributed in space, and edges are assigned according to a threshold function involving the distance between nodes as well as randomly chosen node weights. We show how the degree distribution, percolation and connectivity transitions, clustering coefficient and diameter relate to the threshold value and weight distribution. We give bounds on the threshold value guaranteeing the existence and absence of a giant component, connectivity and disconnectivity of the graph, and small diameter. Finally, we consider the clustering coefficient for nodes with degree l , finding that its scaling is very close to $1/l$ when the node weights are exponentially distributed.

Key words: random graph, geographical threshold graph, giant component, connectivity, clustering coefficient.

1 Introduction

Large networks such as the Internet, World Wide Web, phone call graphs, infections disease contacts, and financial transactions have provided new challenges for modeling and analysis [Bon05]. For example, Web graphs may have billions of nodes and edges, which implies that the analysis on these graphs, i.e., processing and extracting information on these large sets of data, is “hard” [APR02]. Extensive theoretical and experimental research has been done in web-graph modeling. Early measurements suggested that the Internet exhibits a power-law degree distribution [FFF99] and that the web graph also follows a power-law distribution in in- and out-degree of links [KKR⁺99]. Modeling approaches using random graphs have attempted to capture both the structure and dynamics of the web graph [KRR⁺00,BA99,ACL00,BRST01,CF01].

The study of random graphs began with the introduction of the uniform random graph model [ER59,ER60]. Since then many other models have been proposed to better capture the structure seen in real-world networks [Bol01,Dur06]. Some examples are random graph models with a given or expected degree sequence [MR95,CL06], threshold graphs [MP95,HSS06] with edges created according to a function of node

weights, or graphs with an underlying geometric structure, such as random geometric graphs [Pen03]. In this paper we study another recent addition to this collection of models: geographical threshold graphs (GTGs), a static model for networks that includes both geometric information and node weight information.

GTGs combine the geometric structure of random geometric graphs with node properties similar to threshold graphs. The properties of this graph ensemble have only recently begun to be studied [MMK05,BHP07,BK07]. One motivation for analyzing this model is that many real networks need to be studied by using a “richer” stochastic model than random geometric graphs. The GTG model has been applied, for instance, in the study of wireless ad hoc networks in systems where the wireless nodes have different communication ranges or battery power [BK07]. In that case, the weights represent available power or bandwidth of a wireless node. By varying the weights, properties such as the diameter or degree distribution can be tuned. Other possible applications of GTGs that are yet to be explored are epidemic modeling, where the weights could represent susceptibility or infectivity of an individual, or social networks where the weights might be related to affinity or attractiveness.

2 Geographical Threshold Graph Model

The GTG model is constructed from a set of n nodes placed independently and uniformly at random in a volume in \mathbf{R}^d . A non-negative weight w_i , taken randomly and independently from a probability distribution function $f(w) : \mathbf{R}_0^+ \rightarrow \mathbf{R}_0^+$, is assigned to each node v_i for $i \in [n]$. Let $F(x) = \int_0^x f(w)dw$ be the cumulative density function. For two nodes v_i and v_j at distance r , the edge (i, j) exists if and only if the following connectivity relation is satisfied:

$$G(w_i, w_j)h(r) \geq \theta_n, \quad (1)$$

where θ_n is a given threshold parameter that depends on the size of the network. The function $h(r)$ specifies the connection probability as a function of distance and is assumed to be decreasing in r . In the following we take $h(r) = r^{-\beta}$, for some positive β , which is typical for e.g., the path-loss model in wireless networks [BK07]. The interaction strength between nodes $G(w_i, w_j)$ is typically taken to be symmetric (to produce an undirected graph) and either multiplicatively or additively separable, i.e., in the form of $G(w_i, w_j) = g(w_i)g(w_j)$ or $G(w_i, w_j) = g(w_i) + g(w_j)$.

Some basic results have already been shown. For the case of uniformly distributed nodes over a unit space it has been shown [MMK05,BK07] that the expected degree of a node with weight w is

$$\mathbb{E}[k(w)] = \frac{n\pi^{d/2}}{\Gamma(d/2 + 1)} \int_w f(w') (h^{-1}(\theta_n/G(w, w')))^d dw', \quad (2)$$

where h^{-1} is the inverse of h . The degree distribution has been studied for specific weight distribution functions $f(w)$ [MMK05]. In both the multiplicative and additive case of $G(w, w')$, questions of diameter, connectivity, and topology control have been addressed [BK07].

Here we restrict ourselves to the case of $g(w) = w$, $\beta = 2$, and nodes distributed uniformly over a two-dimensional space. For analytical simplicity we take the space to be a unit torus, and use the additive model for the connectivity relation.

$$\frac{w_i + w_j}{r^2} \geq \theta_n. \quad (3)$$

Many of our techniques may be generalized to other cases in a straightforward manner. Finally, we impose the following relatively weak conditions on the weight distribution $f(w)$: (1) a finite mean $\mu = \mathbb{E}[w]$ and (2) a finite variance $\sigma^2 = \mathbb{E}[w^2] - \mathbb{E}[w]^2$. Some examples of GTG instances with exponential weight distribution $f(w) = e^{-w}$ are shown in Figure 1.

The paper is organized as follows. We first state a basic property concerning the degree distribution of GTGs. In Section 4, Theorems 1 and 2 provide bounds on θ_n for the absence and the existence of a giant component. Similarly, in Section 5, Theorems 4 and 3 provide bounds on θ_n for the graph being disconnected and connected. Section 6 gives upper bounds on the diameter, along with simulation results. Finally, in Section 7 we calculate the clustering coefficient, and discuss certain of its properties.

3 Degree Distribution

We start by stating the degree distribution in our GTG model. Let the position vector of the nodes be \mathbf{x} and the weight vector be \mathbf{w} . W.l.o.g. let us consider node v_1 . It is straightforward to show that the probability of v_1 having degree k , given weights \mathbf{w} , is

$$\Pr[d_1 = k | \mathbf{w}] = \binom{n-1}{k} \prod_{i=2}^{k+1} \text{Area}(B(x_i, r_{i1})) \prod_{j=k+2}^n (1 - \text{Area}(B(x_j, r_{j1}))), \quad (4)$$

where $\text{Area}(B(x_i, r_{i1}))$ is the area of the ball at center x_i with radius r_{i1} , and due to (3) the radii are given by

$$r_{i1} = \sqrt{\frac{w_1 + w_i}{\theta_n}} \quad (5)$$

for $i = 2, \dots, n$. After marginalization, it follows

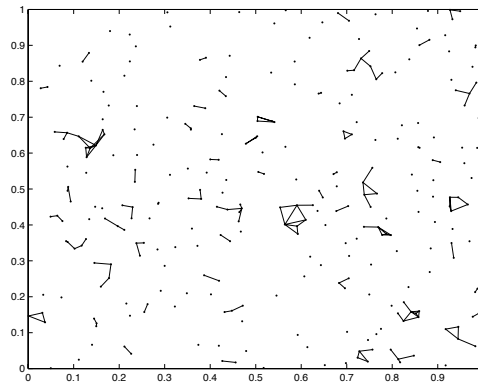
$$\begin{aligned} \Pr[d_1 = k | w_1] &= \left(\prod_{i=2}^n \int_{w_i} dw_i f(w_i) \right) \Pr[d_1 = k | \mathbf{w}] \\ &= \binom{n-1}{k} \left(\int_w dw f(w) \frac{\pi(w_1 + w)}{\theta_n} \right)^k \left(1 - \int_w dw f(w) \frac{\pi(w_1 + w)}{\theta_n} \right)^{n-1-k} \\ &= \binom{n-1}{k} \left(\frac{\pi(w_1 + \mu)}{\theta_n} \right)^k \left(1 - \frac{\pi(w_1 + \mu)}{\theta_n} \right)^{n-1-k}. \end{aligned} \quad (6)$$

That is, the degree distribution d_i , of a node v_i with weight w_i , follows the Binomial distribution

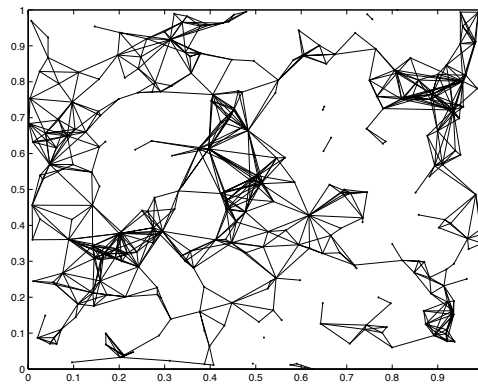
$$d_i(\cdot | w_i) \sim \text{Bin}(n-1, p_i) \quad (7)$$

where

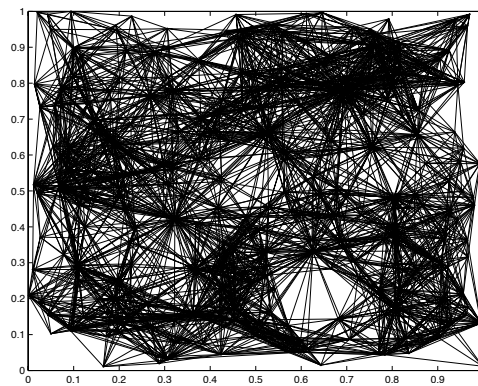
$$p_i = \frac{\pi}{\theta_n}(w_i + \mu). \quad (8)$$



(a)



(b)



(c)

Fig. 1. Instances of GTG with exponential weight distribution, for $n = 300$ at decreasing threshold parameter values (increasing mean degree): (a) $\theta_n/n = 2\pi$, well below the percolation transition; (b) $\theta_n/n = 1$, above the percolation but below the connectivity transition; (c) $\theta_n/n = 1/2e$, well above connectivity.

4 Giant Component

Definition 1 (Giant Component). *The giant component is a connected component with size $\Theta(n)$.*

In this section we analyze the conditions for the existence of the giant component, giving bounds on the threshold parameter value θ_n where it first appears. For $\theta_n = cn$, we specify positive constants $c' > c''$ and prove that whp, if $c > c'$ the giant component does not exist whereas if $c < c''$ the giant component exists.

We do not prove that there is a zero-one law governing the emergence of the giant component. However, given that the the probability of a giant component is 0 above c' and 1 below c'' , it would be rather surprising if the transition were not a sharp one.

4.1 Absence of Giant Component

Theorem 1. *Let $\theta_n = cn$ for $c > c'$, where $c' = 2\pi\mu$. Then whp there is no giant component in GTG.*

Proof. We start by introducing a slightly different GTG model from our usual one. Consider the space of possible node positions and weights, $S = \{(x, y, w) : x, y \in [0, 1], w \geq 0\}$. In the model defined above, we place n nodes in S , leading to a binomial degree distribution: call this the *Binomial GTG*. Let us now instead place nodes in S according to a spatial Poisson process with rate $nf(w)$, so that the expected number of nodes is n : call this the *Poisson GTG*. We will prove that the Poisson GTG does *not* have a giant component. It is straightforward to see that if the Binomial GTG had a giant component with nonvanishing probability, the Poisson GTG would as well. Thus, the Binomial GTG cannot have a giant component either.

The approach is similar to one given in [AS00]. Divide the nodes into three classes: alive, dead and neutral. Denote the number of alive nodes as Y_t . The algorithm works as follows. At time $t = 0$, designate one node (picked u.a.r.) as being alive and all others as neutral. Now, at each subsequent time step t , pick a node v_t u.a.r. from among those that are alive, and then consider all neutral nodes connected to v_t . Denote the number of these nodes as Z_t . Change these nodes from neutral to alive, and change v_t itself from alive to dead. The random variables Y_t, Z_t satisfy the following recursion relation: $Y_0 = 1$ and $Y_t = Y_{t-1} + Z_t - 1$, for $t \geq 1$. The number of alive nodes satisfies

$$Y_t - 1 = \sum_{i=1}^t Z_i - t. \tag{9}$$

The Z_i are independent random variables, since no prior information is involved in the designation of neutral nodes. We formalize this argument as follows. For a node $v_i = (x_i, y_i, w_i)$, define $S_i \subseteq S$ as the subspace of all positions and weights of nodes that can be connected to v_i , namely $S_i = \{(x, y, w) : x, y \in [0, 1], w \geq 0, (x - x_i)^2 + (y - y_i)^2 \leq (w + w_i)/\theta_n\}$. At time $t = 0$, any node within S_0 is a neutral node connected to v_0 . But at a subsequent time step t , nodes within any S_i for $i < t$ have already been designated alive, so only nodes in

$$B_t = S_t \setminus \bigcup_{i=0}^{t-1} S_i$$

can be neutral. Thus, at any two different time steps t and t' , neutral nodes are drawn from *disjoint* subspaces B_t and $B_{t'}$. Using this partitioning of S , we may now imagine an alternative procedure for generating an instance of a Poisson GTG. Rather than populating the entire space S according to a Poisson process at the outset, wait until time step t before populating subspace B_t . By the Principle of Deferred Decisions [Knu97], the two random processes are identical: given the weights and positions of nodes in B_0, \dots, B_{t-1} , no information can be inferred regarding the nodes in B_t . The Z_i are thus independent Poisson random variables whose means are the expected numbers of nodes in B_i . Given that $B_i \subseteq S_i$ and the expected population of S_i is np_i with p_i as defined in Eq. (8), Z_i satisfies the stochastic bound

$$\Pr[Z_i \geq k] \leq \Pr[\text{Po}(np_i) \geq k]. \quad (10)$$

Now consider the number of alive nodes, and let T be the largest t such that $Y_t > 0$. Then T is the size of the component containing v_0 , and the giant component exists if and only if $T = \Theta(n)$ with some nonvanishing probability. The variable T satisfies

$$\Pr[T \geq t] = \Pr[Y_t > 0] = \Pr[Y_t \geq 1] = \Pr\left[\sum_{i=1}^t Z_i \geq t\right] \leq \Pr\left[\sum_{i=1}^t \text{Po}(np_i) \geq t\right]. \quad (11)$$

Consider the threshold $\theta_n = cn$. Since the sum of independent Poisson random variables is itself Poisson distributed, we need to prove that $\Pr[\text{Po}(n \sum_{i=1}^t p_i) \geq t] \rightarrow 0$ for $t = \Theta(n)$, for some $c > 0$. For any positive constant $\varepsilon \in (0, 1)$, we use the following inequality

$$\begin{aligned} \Pr\left[\text{Po}\left(n \sum_{i=1}^t p_i\right) \geq t\right] &\leq \Pr\left[\text{Po}\left(n \sum_{i=1}^t p_i\right) \geq t \mid \sum_{i=1}^t w_i \in (1 \pm \varepsilon)t\mu\right] \\ &\quad + \Pr\left[\sum_{i=1}^t w_i \notin (1 \pm \varepsilon)t\mu\right] \end{aligned} \quad (12)$$

We will now bound the conditional probability using the concentration on the Poisson random variable [Pen03]. Define $\lambda = \mathbb{E}[n \sum_{i=1}^t p_i] = 2at\mu$, where $a = n\pi/\theta_n$. Given any $\varepsilon_0 \in (0, 1)$, for $t \rightarrow \infty$, i.e., $\lambda \rightarrow +\infty$, it follows that

$$\Pr[\text{Po}(\lambda) \notin (1 \pm \varepsilon_0)\lambda] \leq e^{-\lambda H(1-\varepsilon_0)} + e^{-\lambda H(1+\varepsilon_0)} \rightarrow 0, \quad (13)$$

where the function $H(x) = 1 - x + x \log x$, for $x > 0$. It is now sufficient to choose a small enough that $t > 2at\mu(1 + \varepsilon_0)$ for some positive constant ε_0 . This is equivalent to $1 > 2a\mu$, i.e., $c > 2\pi\mu$. It follows that $\Pr[\text{Po}(a \sum_{i=1}^t (w_i + \mu)) \geq t] = o(1)$ for $t = \theta_n$. Thus, the first right-hand term in Eq. (12) goes to zero.

Now consider the second right-hand term in Eq. (12). By the central limit theorem, for $t \rightarrow \infty$, $(\sum w_i - t\mu)/(\sqrt{t}\sigma)$ tends to the normal distribution $N(0, 1)$. Thus,

$$\Pr\left[\sum w_i \notin (1 \pm \varepsilon)t\mu\right] = \Pr\left[\frac{\sum w_i - t\mu}{\sqrt{t}\sigma} \notin (-\varepsilon, \varepsilon)\sqrt{t}\frac{\mu}{\sigma}\right] \rightarrow 0. \quad (14)$$

It follows that for $c > 2\pi\mu$, the probability of having a giant component goes to zero, which completes the proof of the theorem.

4.2 Existence of Giant Component

Theorem 2. *Let $\theta_n = cn$ for $c < c'' = \sup_{\alpha \in (0,1)} \alpha F^{-1}(1 - \alpha)/\lambda_c$, where $\pi\lambda_c \approx 4.52$ is the mean degree at which the giant component first appears in Random Geometric Graphs (RGG) [Pen03]. Then whp the giant component exists in GTG.*

Proof. For any constant $\alpha \in (0, 1)$, we prove that whp there are αn “high-weighted” nodes, all with weights greater than or equal to some s_n ; we state s_n later. Let X_i be the indicator of the event $W_i \geq s_n$. Then $\Pr[X_i = 1] = 1 - F(s_n) =: q$. Let $X = \sum_{i=1}^n X_i$ be the number of high-weighted nodes. Using the Chernoff bound $\Pr[X \leq (1 - \delta)E[X]] \leq \exp(-E[X]\delta^2/2)$, with $\delta = 1 - \alpha/q$,

$$\Pr[X \leq \alpha n] = \Pr[X \leq (1 - \delta)E[X]] \leq \exp(-n(q - \alpha)^2/(2q)) = n^{-\gamma} \quad (15)$$

for some constant $\gamma > 1$ satisfying $(q - \alpha)^2 = 2q\gamma \log n/n$. Solving that quadratic equation in q gives $q = \alpha + \Theta(\log n/n)$, so $F(s_n) = 1 - q = 1 - \alpha - \Theta(\log n/n)$. For any $\varepsilon > 0$ and n sufficiently large the following is satisfied

$$F^{-1}(1 - \alpha) \geq s_n \geq F^{-1}(1 - \alpha - \varepsilon). \quad (16)$$

Thus, let us define the sequence s_n by its limit

$$s_n \rightarrow F^{-1}(1 - \alpha) = \Theta(1). \quad (17)$$

Now we consider the set of αn high-weighted nodes. For each such node v_i with weight w_i , define its characteristic radius to be

$$r_i^2(w_i) = w_i/\theta_n. \quad (18)$$

Then it follows that any other high-weighted node v_j within this radius is connected to v_i , since the connectivity relation is satisfied:

$$(w_i + w_j)/r^2 \geq w_i/r_i^2 = \theta_n. \quad (19)$$

Let $\theta_n = cn$, where $c < \alpha F^{-1}(1 - \alpha)/\lambda_c$. For the radius r_i , whp it follows

$$r_i^2(w_i) = \frac{w_i}{\theta_n} \geq \frac{s_n}{\theta_n} > \frac{\lambda_c}{\alpha n}. \quad (20)$$

Let us therefore consider small circles, with a fixed radius r_0 s.t. $\sqrt{s_n/\theta_n} > r_0 > \sqrt{\lambda_c/(\alpha n)}$, around each of these αn nodes. A subgraph of this must be a RGG with mean degree $> \pi\lambda_c$, which whp contains a giant component. Since its size is $\Theta(\alpha n) = \theta_n$, it is a giant component of the GTG too. We may optimize the bound by taking the supremum of c over $\alpha \in (0, 1)$, and the theorem follows.

Corollary 1. *Given the recent bound [KY06] $\lambda_c \geq 4/3\sqrt{3}$ for RGG, $c'' \leq \sup_{\alpha \in (0,1)} \alpha F^{-1}(1 - \alpha)3\sqrt{3}/4$.*

4.3 Comparison of Upper and Lower Bounds

We again stress that we have not proven a zero-one law for the emergence of the giant component. However, if a sharp transition does indeed exist, c' and c'' provide bounds on its location.

Remark 1. If $\beta F^{-1}(1 - \beta)$ has an extremum for $\beta \in (0, 1)$, this occurs at

$$\beta = F^{-1}(1 - \beta) f(F^{-1}(1 - \beta)). \quad (21)$$

For example, for the exponential distribution $f(w) = \gamma \exp(-\gamma w)$, the maximum is at $\beta = 1/e$ giving a bound of $c'' = 1/(e\gamma\lambda_c)$.

Let us compare the constants for the absence and existence of giant component $c' = 2\pi\mu$ and $c'' = \sup_{\alpha \in (0,1)} \alpha F^{-1}(1 - \alpha)/\lambda_c$, respectively.

Claim. The following is satisfied $2\pi\mu \geq \sup_{\alpha \in (0,1)} \alpha F^{-1}(1 - \alpha)/\lambda_c$, that is, $c' \geq c''$.

Proof. We have $F : [0, +\infty) \rightarrow [0, 1)$. Since F is a bijection, the inverse exists $F^{-1} : [0, 1] \rightarrow [0, +\infty)$. Let the new variable x be $x = F^{-1}(1 - \alpha)$, and consequently $\alpha = 1 - F(x)$. The following is satisfied

$$\sup_{\alpha \in (0,1)} \alpha F^{-1}(1 - \alpha) = \sup_{x \in (0,\infty)} x(1 - F(x)) = \sup_{x \in (0,\infty)} x e^{-\rho(x)}. \quad (22)$$

Using the telescope formula, the expected value μ satisfies

$$\mu = \int_0^\infty (1 - F(y)) dy = \int_0^\infty e^{-\rho(y)} dy. \quad (23)$$

Let us consider two functions $f_1(x) = \int_0^x e^{-\rho(y)} dy$ and $f_2(x) = x e^{-\rho(x)}$. The first derivatives are $f_1(x)' = e^{-\rho(x)}$ and $f_2(x)' = e^{-\rho(x)}(1 - x\rho(x)')$. Then it follows $(f_1 - f_2)(x)' = e^{-\rho(x)} x \rho(x)' \geq 0$, and $(f_1 - f_2)(0) = 0$. For every $x \geq 0$ we have $f_1(x) \geq f_2(x)$. Taking the supremum over the last inequality and considering the fact that $2\pi\lambda_c > 1$, the claim follows.

5 Connectivity

Definition 2 (Connectivity). *The graph on n vertices is connected if the largest component has size n .*

In this section we analyze conditions for connectivity, giving bounds on the threshold parameter θ_n at where the entire graph first becomes connected. Similarly to our approach in the case of the giant component, for $\theta_n = cn/\log n$, we specify positive constants $c' > c''$ and prove that whp, if $c > c'$ the graph is disconnected whereas if $c < c''$ the graph is connected.

As in the case of the emergence of the giant component, it seems likely but has not been proven that there is a sharp phase transition at which GTGs become connected.

5.1 Connected Graph

Theorem 3. *Let $\theta_n = cn/\log n$ for $c < \sup_{\alpha \in (0,1)} \alpha F^{-1}(1-\alpha)/4$. Then the GTG is connected whp.*

Proof. The proof is divided into two parts. In the first part, we prove that a constant fraction of nodes αn are connected. In the second part we prove that the rest of the $(1-\alpha)n$ nodes are connected to the first set of αn nodes.

First part: Invoking the proof of the appearance of the giant component, there are αn nodes all with weights $\geq s_n \rightarrow F^{-1}(1-\alpha) = \Theta(1)$.

Let $\theta_n = cn/\log n$, where $c < \alpha F^{-1}(1-\alpha)\pi$. Analogously to r_t , define the connectivity radius r_c

$$r_c^2(w_i) = \frac{w_i}{\theta_n} \geq \frac{s_n}{\theta_n} > \frac{\log n}{\alpha\pi n}. \quad (24)$$

Similarly to Theorem 2 let us consider small circles around each of these αn nodes, and consider these nodes as a RGG. It is known that $r_n = \sqrt{\log n/(\pi n)}$ is the connectivity threshold in RGG [GK98]. The connectivity of RGG implies the connectivity of these αn nodes in our GTG.

Second part: Color the αn high-weighted nodes blue, and the remaining $(1-\alpha)n$ nodes red. Now let us tile our space into $n/(c_0 \log n)$ squares of size $c_0 \log n/n$. We state c_0 later. Consider any square S_i , and let B_i be the number of blue nodes in S_i . In expectation there are $E[B_i] = \alpha c_0 \log n$ blue nodes in each square. From the Chernoff bound it follows

$$\Pr[B_i \geq (1-\delta)\alpha c_0 \log n] \geq 1 - n^{-\alpha c_0 \delta^2/2}. \quad (25)$$

Let us consider one red node r . The node r belongs to some square S_i . Let M_r be the event that the red node r is connected to some blue node $b \in S_i$. Let the weights of r, b be w_r, w_b , respectively. The probability of the complement of M_r , conditioned on there being at least one blue node in S_i , is given by

$$\begin{aligned} \Pr[M_r^c | B_i \geq 1] &= \Pr[w_r + w_b \leq r^2 \theta_n] \leq \Pr[w_r + w_b \leq 2c_0 \frac{\log n}{n} c \frac{n}{\log n}] \\ &= \Pr[w_r + w_b \leq 2c_0 c]. \end{aligned} \quad (26)$$

As long as $F^{-1}(1-\alpha) > 2c_0 c$, $w_b > 2c_0 c$ and hence $\Pr[M_r^c | B_i \geq 1] = 0$. For large enough n it must hold that $(1-\delta)\alpha c_0 \log n > 1$, and so from Eq. (25),

$$\begin{aligned} \Pr[M_r^c] &\leq \Pr[M_r^c | B_i \geq (1-\delta)\alpha c_0 \log n] + \Pr[B_i < (1-\delta)\alpha c_0 \log n] \\ &\leq 0 + n^{-\alpha c_0 \delta^2/2}. \end{aligned} \quad (27)$$

If $\alpha c_0 \delta^2/2 \geq 1 + \varepsilon$ for some $\varepsilon > 0$, then by the union bound,

$$\Pr[\bigcup_r M_r^c] \leq \sum_r \Pr[M_r^c] \leq (1-\alpha)nn^{-(1+\varepsilon)} = (1-\alpha)n^{-\varepsilon}. \quad (28)$$

Finally, the probability that all red nodes are connected to the set of blue nodes is given by the following relation

$$\Pr[\bigcap_r M_r] = 1 - \Pr[\bigcup_r M_r^c] \geq 1 - (1-\alpha)n^{-\varepsilon} \rightarrow 1. \quad (29)$$

The requirements we have imposed on constants so far are: $c < \alpha F^{-1}(1 - \alpha)\pi$, $c < F^{-1}(1 - \alpha)/(2c_0)$ and $\alpha c_0 \geq 2(1 + \varepsilon)/\delta^2$. These conditions combine to give

$$c < \alpha F^{-1}(1 - \alpha) \min\left(\pi, \frac{\delta^2}{4(1 + \varepsilon)}\right). \quad (30)$$

Since $\alpha \in (0, 1)$, $\delta \in (0, 1)$ and $\varepsilon > 0$ are arbitrary, we obtain

$$c < \sup_{\alpha \in (0,1)} \alpha F^{-1}(1 - \alpha)/4. \quad (31)$$

5.2 Disconnected Graph

Theorem 4. *Let $\theta_n = cn/\log n$, where $c > \pi\mu$. Then the GTG is disconnected whp.*

Proof. For a node v_i , let Y_i be the indicator of the event that v_i is isolated. We will consider the sum

$$Y = \sum_{i=1}^n Y_i \quad (32)$$

and show that $\Pr[Y = 0] \rightarrow 0$. It will then follow that whp there is at least one isolated node and so the graph is disconnected.

From the binomial degree distribution in Eq. (6), the probability that v_i is isolated, conditional on w_i , is

$$\Pr[Y_i = 1 | w_i] = \left(1 - \frac{w_i + \mu}{\theta_n} \pi\right)^{n-1}. \quad (33)$$

Now define

$$\begin{aligned} p &\equiv \mathbb{E}[Y_i] = \Pr[Y_i = 1] \\ &= \int dw_i f(w_i) \left(1 - \frac{w_i + \mu}{\theta_n} \pi\right)^{n-1}. \end{aligned} \quad (34)$$

Applying the second moment method,

$$\begin{aligned} \Pr[Y = 0] &\leq \frac{\text{Var}[Y]}{\mathbb{E}[Y]^2} \\ &= \frac{\sum_i \text{Var}[Y_i] + \sum_{i \neq j} \text{Cov}[Y_i, Y_j]}{(np)^2}. \end{aligned} \quad (35)$$

The variance and covariance are given by

$$\text{Var}[Y_i] = \mathbb{E}[Y_i^2] - \mathbb{E}[Y_i]^2 = p - p^2 \quad (36)$$

$$\text{Cov}[Y_i, Y_j] = \mathbb{E}[Y_i, Y_j] - \mathbb{E}[Y_i]\mathbb{E}[Y_j] = \Pr[Y_i = 1, Y_j = 1] - p^2, \quad (37)$$

so

$$\begin{aligned} \Pr[Y = 0] &\leq \frac{n(p - p^2) + n(n-1)(\Pr[Y_i = 1, Y_j = 1] - p^2)}{(np)^2} \\ &< \frac{1}{np} + \frac{\Pr[Y_i = 1, Y_j = 1]}{p^2} - 1. \end{aligned} \quad (38)$$

Let us first consider the $1/(np)$ term.

Claim. Let $\theta_n = c \frac{n}{\log n}$, where c is a constant. If $c > \pi\mu$, then $1/(np) \rightarrow 0$.

Proof. Let ζ be any positive constant that satisfies $c > \pi(\mu + \zeta)$. Let $\theta_n = c \frac{n}{\log n}$. We have

$$\begin{aligned}
 p &= \int dwf(w) \left(1 - \frac{w + \mu}{\theta_n} \pi\right)^{n-1} \\
 &\geq \int_0^\zeta dwf(w) \left(1 - \frac{w + \mu}{\theta_n} \pi\right)^{n-1} \\
 &\geq F(\zeta) \left(1 - \frac{\mu + \zeta}{\theta_n} \pi\right)^{n-1} \\
 &= F(\zeta) \left(1 - \frac{\mu + \zeta}{cn} \pi \log n\right)^{n-1} \\
 &= F(\zeta) n^{-(\mu + \zeta)\pi/c} (1 + o(1)).
 \end{aligned}$$

Therefore, if $c > \pi(\mu + \zeta)$, $pn \geq F(\zeta)\omega(n)$ and so $1/(np) \rightarrow 0$ for $n \rightarrow \infty$.

Next, we will show that $\Pr[Y_i = 1, Y_j = 1]/p^2 = o(1)$. Consider the joint probability conditional on a set of weights \mathbf{w} . Denoting the neighborhood relation by $v_i \sim v_j$,

$$\begin{aligned}
 \Pr[Y_i = 1, Y_j = 1 | \mathbf{w}] &= \Pr[v_i \sim v_j, \bigcap_{k \neq i, j} v_i \sim v_k, v_j \sim v_k | \mathbf{w}] \\
 &= \Pr[v_i \sim v_j | w_i, w_j] \Pr[\bigcap_{k \neq i, j} v_i \sim v_k, v_j \sim v_k | v_i \sim v_j, \mathbf{w}] \\
 &= \Pr[v_i \sim v_j | w_i, w_j] \prod_{k \neq i, j} \Pr[v_i \sim v_k, v_j \sim v_k | v_i \sim v_j, \mathbf{w}]. \quad (39)
 \end{aligned}$$

We now use the easily verified property that given events A , B and C that depend on \mathbf{w} ,

$$\Pr[B^c, C^c | A^c, \mathbf{w}] = 1 - \Pr[B | \mathbf{w}] - \Pr[C | \mathbf{w}] + (1 - \Pr[A | B, C, \mathbf{w}]) \frac{\Pr[B, C | \mathbf{w}]}{\Pr[A^c | \mathbf{w}]}. \quad (40)$$

Let $a = \Pr[v_i \sim v_j | w_i, w_j]$, $b = \Pr[v_i \sim v_k | w_i, w_k]$, $c = \Pr[v_j \sim v_k | w_j, w_k]$ and define the *clustering coefficient*

$$K = \Pr[v_i \sim v_j | v_i \sim v_k, v_j \sim v_k, w_i, w_j, w_k]. \quad (41)$$

Then,

$$\Pr[Y_i = 1, Y_j = 1 | \mathbf{w}] = (1 - a) \prod_{k \neq i, j} \left[1 - b - c + (1 - K) \frac{bc}{1 - a}\right]. \quad (42)$$

Note that $a = (w_i + w_j)\pi/\theta_n$, and similarly for b and c .

In Section 7, we show (Lemma 2) that if $w_i, w_j, w_k \leq \hat{w} = (1 - 3\sqrt{3}/4\pi)\theta_n/2\pi$, then $K \geq a$. Thus, under those conditions,

$$\begin{aligned}
 1 - b - c + (1 - K) \frac{bc}{1 - a} &\leq 1 - b - c + bc \\
 &= (1 - b)(1 - c). \quad (43)
 \end{aligned}$$

Now average Eq. (42) over all weights. It follows from the finite variance of $f(w)$ that for any constant M , $F(M\theta_n) = 1 - o(1/n)$, and so

$$\begin{aligned}
\Pr[Y_i = 1, Y_j = 1] &= \int f(w_i)dw_i \int f(w_j)dw_j (1-a) \left(\int f(w_k)dw_k \left[1 - b - c + (1-K) \frac{bc}{1-a} \right] \right)^{n-2} \\
&= \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j (1-a) \left(\int_0^{\hat{w}} f(w_k)dw_k \left[1 - b - c + (1-K) \frac{bc}{1-a} \right] \right)^{n-2} (1+o(1)) \\
&\leq \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j (1-a) \left(\int_0^{\hat{w}} f(w_k)dw_k (1-b)(1-c) \right)^{n-2} (1+o(1)) \\
&= \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j \left\{ \left(1 - \frac{\pi}{\theta_n}(w_i + w_j) \right) \times \right. \\
&\quad \left. \left(\int_0^{\hat{w}} f(w_k)dw_k \left(1 - \frac{\pi}{\theta_n}(w_i + w_k) \right) \left(1 - \frac{\pi}{\theta_n}(w_j + w_k) \right) \right)^{n-2} \right\} \\
&= \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j \left\{ \left(1 - \frac{\pi}{\theta_n}(w_i + w_j) \right) \times \right. \\
&\quad \left. \left(1 - \frac{\pi}{\theta_n}(w_i + w_j + 2\mu) + \frac{\pi^2}{\theta_n^2}(w_i w_j + \mu(w_i + w_j) + \mu^2 + \sigma^2) \right)^{n-2} \right\}. \tag{44}
\end{aligned}$$

Now consider p^2 . Using the fact that μ , σ and $1/2 - (\hat{w} + \mu)\pi/\theta_n$ are all $\Theta(1)$,

$$\begin{aligned}
p^2 &= \int f(w_i)dw_i \int f(w_j)dw_j \left(1 - \frac{\pi}{\theta_n}(w_i + \mu) \right)^{n-1} \left(1 - \frac{\pi}{\theta_n}(w_j + \mu) \right)^{n-1} \\
&\geq \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j \left(1 - \frac{\pi}{\theta_n}(w_i + \mu) \right)^{n-1} \left(1 - \frac{\pi}{\theta_n}(w_j + \mu) \right)^{n-1} \tag{45} \\
&= \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j \left(1 - \frac{\pi}{\theta_n}(w_i + w_j + 2\mu) + \frac{\pi^2}{\theta_n^2}(w_i w_j + \mu(w_i + w_j) + \mu^2) \right)^{n-1} \\
&= \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j \left(1 - \frac{\pi}{\theta_n}(w_i + w_j + 2\mu) + \frac{\pi^2}{\theta_n^2}(w_i w_j + \mu(w_i + w_j) + \mu^2 + \sigma^2) \right)^{n-1} (1 - o(1)) \\
&> \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j \left(1 - \frac{\pi}{\theta_n}(w_i + w_j + 2\mu) \right) \times \\
&\quad \left(1 - \frac{\pi}{\theta_n}(w_i + w_j + 2\mu) + \frac{\pi^2}{\theta_n^2}(w_i w_j + \mu(w_i + w_j) + \mu^2 + \sigma^2) \right)^{n-2} (1 - o(1)) \\
&= \int_0^{\hat{w}} f(w_i)dw_i \int_0^{\hat{w}} f(w_j)dw_j \left(1 - \frac{\pi}{\theta_n}(w_i + w_j) \right) \times \\
&\quad \left(1 - \frac{\pi}{\theta_n}(w_i + w_j + 2\mu) + \frac{\pi^2}{\theta_n^2}(w_i w_j + \mu(w_i + w_j) + \mu^2 + \sigma^2) \right)^{n-2} (1 - o(1)). \tag{46}
\end{aligned}$$

Finally, this gives the desired ratio

$$\frac{\Pr[Y_i = 1, Y_j = 1]}{p^2} < 1 + o(1). \tag{47}$$

By the second moment method it then follows that $\Pr[Y = 0] < o(1)$.

6 Diameter

In this section we analyze the diameter of GTG, and provide an upper bound on it. In the design of large networks, such as the Internet, wireless networks, etc., it is desirable to achieve low latency in the graph (i.e., the hop-count between any pair of nodes in the network is small). In other words, a graph with a small diameter is desired. We give the conditions on the threshold function θ_n such that the graph has a desired diameter in general. Furthermore, we derive the conditions on θ_n , in terms of the cumulative distribution function on weights $F(w)$, such that $diam$ belongs to the classes $diam = O(1)$, $diam = O(\log^q n)$ and $diam = O(\sqrt{n/\log n})$, respectively. These classes correspond to an ultra-low, low and high latency network, respectively. Particularly, for these three classes, we give the exact expressions on θ_n in the case of the exponentially distributed weights.

Here, we give the upper bound on the diameter in the GTG model. Let u and v be two arbitrary nodes. Let us construct the sequence of adjacent squares $S_1, S_2, \dots, S_{O(1/x)}$, of the size $x \times x$, linking u and v , such that u and v are the centers of the first and last squares, respectively⁴ (see Fig. 2). The geometric distance between any two nodes is $r \leq \Theta(1)$. Thus, there are $O(1/x)$ squares on the straight path $u - v$ in total.

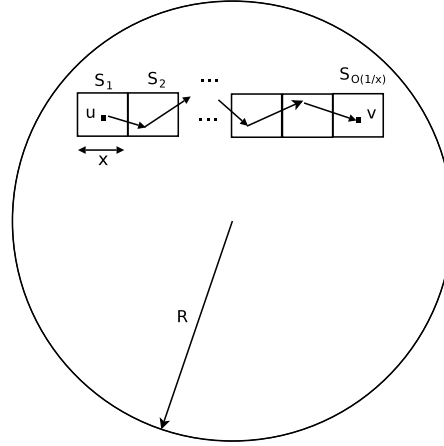


Fig. 2. Illustration of our diameter proof technique: a sequence of adjacent squares of size $x \times x$ link an arbitrary pair of nodes u and v in a unit-area disc.

Let V_i be the number of nodes that lie within the square S_i , for $i = 1, 2, \dots, O(1/x)$. We have $E[V_i] = nx^2$. Using the Chernoff bound, the following is satisfied

$$\Pr[V_i \leq (1 - \delta)E[V_i]] \leq e^{-E[V_i]\delta^2/2}. \quad (48)$$

Taking $\delta = 1/2$, we get $\Pr[V_i \leq nx^2/2] \leq e^{-nx^2/8}$, i.e., in each square S_i , there are at least $nx^2/2$ nodes whp.

⁴ The centers of the squares lie on the straight line $u - v$.

Let M_i be the event that in a square S_i , there is at least one node with weight $w \geq s_n$. We will specify s_n later. Now, we derive the lower bound on the probability $\Pr[M_i]$. This probability is greater than probability conditioned on the event that there are at least $nx^2/2$ nodes in S_i , i.e.

$$\begin{aligned} \Pr[M_i] &\geq \Pr[M_i | V_i \geq nx^2/2] \Pr[V_i \geq nx^2/2] \\ &\geq (1 - \Pr[W \leq s_n]^{nx^2/2}) (1 - e^{-nx^2/8}) \\ &= (1 - F(s_n)^{nx^2/2}) (1 - e^{-nx^2/8}). \end{aligned} \quad (49)$$

We now explain how we choose s_n such that any two neighboring squares S_j and S_{j+1} are connected by an edge (i.e., there are two connected nodes $a \in S_j$ and $b \in S_{j+1}$). Let weights of a and b be w and w' , respectively. We showed that in any square S_i there is at least one node with weight $\geq s_n$, whp. We want that the connectivity relation Eq.(1) for nodes a and b is satisfied. Maximal distance $\|a - b\|$ between a pair of nodes is $\|a - b\| \leq x\sqrt{5}$. Conditioned on the events that weights w, w' are greater than s_n we have the following relation for the connectivity of nodes a and b

$$\Pr[a \sim b | w, w' \geq s_n] \geq \Pr[2s_n/r^2 \geq \theta_n]. \quad (50)$$

Let us choose get $s_n = \Theta(x^2\theta_n)$. If an arbitrary pair of nodes (u, v) is connected by a path of nodes belonging to the squares $S_1, S_2, \dots, S_{O(1/x)}$, then the following relation on *diam* is satisfied

$$\begin{aligned} \Pr[\text{diam} = O(1/x)] &\geq \Pr[\bigcap_{i=1}^{O(1/x)} M_i] \\ &= \left((1 - e^{-nx^2/8}) (1 - F(s_n)^{nx^2/2}) \right)^{O(1/x)}, \end{aligned}$$

since the nodes, as well as weights, are distributed independently. Now, the lemma on the diameter follows.

Lemma 1. *Let the cumulative weight distribution function be $F(w)$ in GTG model. Let x and a sequence $s_n = \Theta(x^2\theta_n)$ be such that*

$$\lim_{n \rightarrow \infty} \left(1 - F(s_n)^{nx^2/2} \right)^{1/x} = 1. \quad (51)$$

Then, whp $\text{diam} = O(1/x)$.

Proof. Proof follows from the previous discussion.

6.1 Some Classes of Diameter

We now analyze and state conditions on θ_n such that $\text{diam} = O(1)$, $\text{diam} = O(\log^q n)$ and $\text{diam} = O(\sqrt{n}/\log n)$. We finally work out the case when the weight distribution is exponential, $f(w) = e^{-w}$, $w \geq 0$, (i.e., $F(w) = 1 - e^{-w}$, $w \geq 0$) and derive the upper bound on the threshold function θ_n in this particular case. For some other weight distribution, the analysis would be similar. Our results are given by:

1. **Ultra-low Latency:** $diam = O(1)$. Let $x < 1$ be a constant and $s_n = \theta_n$. If $F(\theta_n)^n \rightarrow 0$, then $diam = O(1)$ whp. For the exponential weight distribution it follows that $\theta_n = o(\log n)$.
2. **Low Latency:** $diam = O(\log^q n)$. Let $x = 1/\log^q n$ and $s_n = \theta_n/\log^{2q} n$. If $F(\theta_n/\log^{2q} n)^{\frac{n}{2\log^{2q} n}} \log^q n \rightarrow 0$, then $diam = O(\log^q n)$ whp. For the exponential weight distribution it follows that $\theta_n = o\left((\log n)^{2q(1-(\log^{2q} n)/n)}\right)$.
3. **High Latency:** $diam = O(\sqrt{n/\log n})$. Let $x = \sqrt{\log n/n}$ and $s_n = \theta_n \log n/n$. If $\sqrt{n/\log n} F(\theta_n \log n/n)^{\log n} \rightarrow 0$, then $diam = O(\sqrt{n/\log n})$ whp. For the exponential weight distribution it follows that $\theta_n = o\left((n/\log n)^{1-1/(2\log n)}\right)$.

Here we prove of the previous claims.

Ultra-low Latency: $diam = O(1)$. For the diameter to be a constant, let $x < 1$ be a constant. Invoking Lemma 1, it follows that $diam = O(1)$ whp if and only if $1 - F(s_n)^{nx^2/2} \rightarrow 1$, i.e., if and only if $F(s_n)^n \rightarrow 0$. The condition on the size of $diam$ is given by the following claim, and we can derive θ_n such that $diam = O(1)$ whp.

Claim. If $F(\theta_n)^n \rightarrow 0$, then $diam = O(1)$ whp.

For the exponential weight distribution it follows that $F(\theta_n)^n = (1 - e^{-\theta_n})^n \rightarrow e^{-n/e^{\theta_n}}$. The last equation tends to 0 if and only if $n/e^{\theta_n} \rightarrow \infty$. That is,

Claim. For the exponential weight distribution $f(w) = e^{-w}$, the diameter in GTG is $diam = O(1)$ if $\theta_n = o(\log n)$.

Low Latency: $diam = O(\log^q n)$. Let us choose $x = 1/\log^q n$. Invoking Lemma 1, we obtain:

$$(1 - F(s_n)^{nx^2/2})^{1/x} = \left(1 - F(s_n)^{\frac{n}{2\log^{2q} n}}\right)^{\log^q n} \quad (52)$$

For $s_n \rightarrow 0$, the last expression tends to 1, if and only if

$$F(s_n)^{\frac{n}{2\log^{2q} n}} \log^q n \rightarrow 0, \quad (53)$$

by using $\lim_{t \rightarrow +\infty} (1 - 1/t)^t = 1/e$. The condition on the size of $diam$ is given by the following claim.

Claim. if $F((\log^q n)^{\alpha} \theta_n)^{\frac{n}{2\log^{2q} n}} \log^q n \rightarrow 0$, then $diam = O(\log^q n)$ whp.

For the exponential weight distribution, the following is to be satisfied

$$F(s_n)^{\frac{n}{2\log^{2q} n}} \log^q n = \log^q n (1 - e^{-s_n})^{\frac{n}{2\log^{2q} n}} \rightarrow s_n^{n/(2\log^{2q} n)} \log^q n \rightarrow 0, \quad (54)$$

or equivalently

$$s_n = o\left((\log n)^{-\frac{2q}{n} \log^{2q} n}\right). \quad (55)$$

Claim. For the exponential weight distribution $f(w) = e^{-w}$, the diameter in GTG is $diam = O(\log^q n)$ if $\theta_n = o\left((\log n)^{2q(1-(\log^{2q} n)/n)}\right)$.

High Latency: $diam = O(\sqrt{n/\log n lemma})$. Let us choose $x = \sqrt{\log n/n}$. Invoking Lemma 1, we get

$$(1 - F(s_n)^{nx^2/2})^{1/x} = (1 - F(s_n)^{\log n})\sqrt{\frac{n}{\log n}} \quad (56)$$

It can be shown that the last expression tends to 1 if and only if $\sqrt{n/\log n}F(s_n)^{\log n} \rightarrow 0$, by using $\lim_{t \rightarrow +\infty} (1 - 1/t)^t = 1/e$. The condition on the size of $diam$ is given by the following claim.

Claim. If $\sqrt{n/\log n}F((\log n/n)^{\alpha/2}\theta_n)^{\log n} \rightarrow 0$, then $diam = O(\sqrt{n/\log n})$ whp.

For the exponential weight distribution, the following is to be satisfied

$$\sqrt{n/\log n}F(s_n)^{\log n} = \sqrt{n/\log n}(1 - e^{-s_n})^{\log n} \rightarrow \sqrt{n/\log n}s_n^{\log n} \rightarrow 0, \quad (57)$$

or equivalently

$$s_n = o\left((\log n/n)^{1/(2\log n)}\right). \quad (58)$$

Claim. For the exponential weight distribution $f(w) = e^{-w}$, the diameter in GTG is $diam = O(\sqrt{n/\log n})$ if $\theta_n = o\left((n/\log n)^{\alpha/2-1/(2\log n)}\right)$.

Simulations are done for the additive case of GTG with the path-loss exponent $\beta = 3$, (not $\beta = 2$) for the cases of $diam = O(\log^{1.5} n)$ and $diam = O(\sqrt{n/\log n})$. Exponentially distributed weights with mean 1 are used. The network sizes simulated are $n = \{100, 200, 500, 1000, 2000, 10000\}$. The threshold values θ_n for the two cases are obtained by invoking previous claims.



Fig. 3. (a) For the case of $diam = O(\log^q n)$, with $q = 1.5$, the analytical solid curve is the upper bound on $diam(n)$. Thus, our simulation results match with theoretical predictions, since the simulation points all lie below the analytical curve.

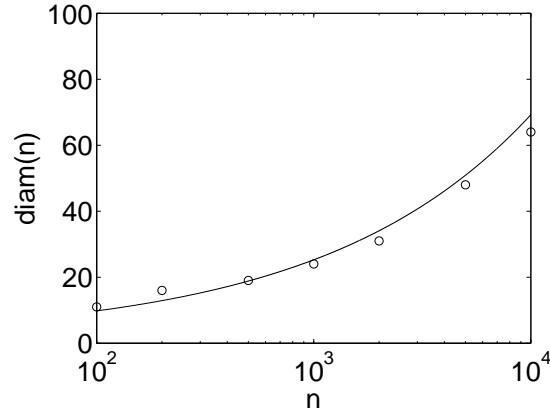


Fig. 4. (b) For the case of $diam = O(\sqrt{n/\log n})$, the solid curve plots the upper bound on $diam(n)$, and this bound exactly matches with the experimental values.

7 Clustering Coefficient

Let us consider in more detail the clustering coefficient defined in Section 5, namely the neighbor probability

$$K(w_i, w_j, w_k) = \Pr[v_i \sim v_j | v_i \sim v_k, v_j \sim v_k, w_i, w_j, w_k]. \quad (59)$$

Let $x = \sqrt{(w_i + w_j)/\theta_n}$, $y = \sqrt{(w_j + w_k)/\theta_n}$ and $z = \sqrt{(w_i + w_k)/\theta_n}$. Then, if d_{ij} represents the distance between points i and j ,

$$\begin{aligned} K &= \Pr[d_{ij} \leq x | d_{jk} \leq y, d_{ik} \leq z] \\ &= \frac{1}{\pi z^2} \int_0^z \Pr[d_{ij} \leq x | d_{jk} \leq y, d_{ik} = r] 2\pi r dr \\ &= \frac{1}{\pi^2 y^2 z^2} \int_0^z A(r) 2\pi r dr \end{aligned} \quad (60)$$

where $A(r)$ is the overlap area of a disc of radius x , centered at i , and a disc of radius y , centered at k . Now consider a triangle $\triangle ABC$, with sides $AB = r$, $AC = x$, $BC = y$, $\angle CAB = \alpha$, $\angle ABC = \beta$. Following arguments similar to those in [DC02], there are three possible cases for $A(r)$:

$$A(r) = \begin{cases} \pi[\min(x, y)]^2 & r \leq |x - y| \\ x^2(\alpha - \sin \alpha \cos \alpha) + y^2(\beta - \sin \beta \cos \beta) & |x - y| < r < x + y \\ 0 & r \geq x + y \end{cases} \quad (61)$$

where

$$\alpha = \cos^{-1}\left(\frac{r^2 + x^2 - y^2}{2xr}\right), \quad \beta = \cos^{-1}\left(\frac{r^2 - x^2 + y^2}{2yr}\right). \quad (62)$$

From the definitions of x , y and z , $|x - y| < z < x + y$. After some algebraic manipulation, one finds

$$\begin{aligned}
K &= \frac{x^2}{\pi} \left[\frac{1}{y^2} \cos^{-1} \left(\frac{z^2 + x^2 - y^2}{2xz} \right) + \frac{1}{x^2} \cos^{-1} \left(\frac{z^2 - x^2 + y^2}{2yz} \right) + \frac{1}{z^2} \cos^{-1} \left(\frac{x^2 + y^2 - z^2}{2xy} \right) \right. \\
&\quad \left. - \frac{x^2 + y^2 + z^2}{4x^2y^2z^2} \sqrt{2x^2y^2 + 2x^2z^2 + 2y^2z^2 - x^4 - y^4 - z^4} \right] \\
&= \frac{1}{\pi(w_i + w_k)(w_j + w_k)} \left\{ (w_i + w_j)(w_i + w_k) \cos^{-1} \left(\frac{w_i}{\sqrt{w_i + w_j}\sqrt{w_i + w_k}} \right) \right. \\
&\quad + (w_i + w_j)(w_j + w_k) \cos^{-1} \left(\frac{w_j}{\sqrt{w_i + w_j}\sqrt{w_j + w_k}} \right) \\
&\quad + (w_i + w_k)(w_j + w_k) \cos^{-1} \left(\frac{w_k}{\sqrt{w_i + w_k}\sqrt{w_j + w_k}} \right) \\
&\quad \left. - (w_i + w_j + w_k) \sqrt{w_i w_j + w_j w_k + w_k w_i} \right\}.
\end{aligned}$$

Note that while K is a function of the weights w_i , w_j and w_k , it is independent of θ_n . This reflects a similar property in random geometric graphs [DC02], where the clustering coefficient is independent of the graph's mean degree.

Written in terms of the connection probabilities a , b and c defined in Theorem 4, K is given by

$$\begin{aligned}
K\pi bc &= ab \cos^{-1} \frac{a+b-c}{2\sqrt{ab}} + bc \cos^{-1} \frac{-a+b+c}{2\sqrt{bc}} + ca \cos^{-1} \frac{a-b+c}{2\sqrt{ac}} \\
&\quad - \frac{a+b+c}{4} \sqrt{2ab + 2ac + 2ca - a^2 - b^2 - c^2}. \tag{63}
\end{aligned}$$

We now prove the bound on the clustering coefficient that we needed for Theorem 4.

Lemma 2. *If $w_i, w_j, w_k \leq \hat{w} = (1 - 3\sqrt{3}/4\pi)\theta_n/2\pi$ then $K \geq a$.*

Proof. Define

$$\begin{aligned}
S(a, b, c) &= \frac{K\pi}{a} \\
&= \frac{1}{a} \cos^{-1} \frac{-a+b+c}{2\sqrt{bc}} + \frac{1}{b} \cos^{-1} \frac{a-b+c}{2\sqrt{ac}} + \frac{1}{c} \cos^{-1} \frac{a+b-c}{2\sqrt{ab}} - \gamma \frac{a+b+c}{4abc},
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= \sqrt{2ab + 2ac + 2ca - a^2 - b^2 - c^2} \\
&= 2\sqrt{bc} \sqrt{1 - \left(\frac{-a+b+c}{2\sqrt{bc}} \right)^2} \tag{64}
\end{aligned}$$

It is easy to verify that $S = \pi$ when $w_i = w_j = w_k = \hat{w}$. We will now show that S is nonincreasing over the weights, and thus $S \geq \pi$ for all smaller values of w_i, w_j, w_k .

Consider the sign of the derivative

$$\begin{aligned}\frac{dS}{dw_i} &= \frac{\partial S}{\partial a} \frac{\partial a}{\partial w_i} + \frac{\partial S}{\partial b} \frac{\partial b}{\partial w_i} + \frac{\partial S}{\partial c} \frac{\partial c}{\partial w_i} \\ &= \frac{\pi}{\theta_n} \left(\frac{\partial S}{\partial a} + \frac{\partial S}{\partial b} \right).\end{aligned}\quad (65)$$

Since S is symmetric in a and b , it is sufficient to consider the sign of $\partial S/\partial a$:

$$\frac{\partial S}{\partial a} = \frac{\gamma}{4a^2bc}(-a+b+c) - \frac{1}{a^2} \cos^{-1} \frac{-a+b+c}{2\sqrt{bc}}.\quad (66)$$

Now let

$$t = \frac{-a+b+c}{2\sqrt{bc}} \in [0, 1].\quad (67)$$

Then,

$$\frac{\partial S}{\partial a} = \frac{1}{a^2} (t\sqrt{1-t^2} - \cos^{-1} t).\quad (68)$$

Given the function $\varphi(t) = t\sqrt{1-t^2} - \cos^{-1} t$ on $[0, 1]$, $\varphi'(t) = 2\sqrt{1-t^2} \geq 0$ and $\varphi(1) = 0$. It follows that $\partial S/\partial a \leq 0$, and so $dS/dw_i \leq 0$. Finally, S is symmetric in (w_i, w_j, w_k) , so it must be nonincreasing over each of the weights and bounded below by the value at $w_i = w_j = w_k = \hat{w}$. This completes the proof.

Finally, we mention the general clustering coefficient, K_l , defined by neighbor probability for a node with a given degree

$$\begin{aligned}K_l &= \Pr[v_i \sim v_j | d(v_k) = l, v_i \sim v_k, v_j \sim v_k] \\ &= \frac{\Pr[v_i \sim v_j, d(v_k) = l, v_i \sim v_k, v_j \sim v_k]}{\Pr[d(v_k) = l, v_i \sim v_k, v_j \sim v_k]} \\ &= \frac{\int d\mathbf{w} f(\mathbf{w}) \Pr[v_i \sim v_j, d(v_k) = l, v_i \sim v_k, v_j \sim v_k]}{\int d\mathbf{w} f(\mathbf{w}) \Pr[d(v_k) = l, v_i \sim v_k, v_j \sim v_k]}.\end{aligned}$$

Similarly to the analysis of the clustering coefficient K , it can be derived, that the general clustering coefficient is given by the ratio $K_l = I_n/I_d$, where

$$I_d = \int dw_n f(w_n) \left(\frac{\pi}{\theta_n} (\mu + w_n) \right)^l \left(1 - \frac{\pi}{\theta_n} (\mu + w_n) \right)^{n-l} \quad (69)$$

and

$$\begin{aligned}I_n &= \int \int \int dw_0 dw_1 dw_n f(w_0) f(w_1) f(w_n) (\pi y^2 \pi z^2 K(\mathbf{w})) \left(\frac{\pi}{\theta_n} (\mu + w_n) \right)^{l-2} \left(1 - \frac{\pi}{\theta_n} (\mu + w_n) \right)^{n-l} \\ &= \int \int \int dw_0 dw_1 dw_n f(w_0) f(w_1) f(w_n) \frac{\theta_n^2 y^2 z^2 K(\mathbf{w})}{(\mu + w_n)^2} \left(\frac{\pi}{\theta_n} (\mu + w_n) \right)^l \left(1 - \frac{\pi}{\theta_n} (\mu + w_n) \right)^{n-l} \\ &= \int dw_n \frac{f(w_n)}{(\mu + w_n)^2} \left(\frac{\pi}{\theta_n} (\mu + w_n) \right)^l \left(1 - \frac{\pi}{\theta_n} (\mu + w_n) \right)^{n-l} \\ &\quad \cdot \int \int dw_0 dw_1 f(w_0) f(w_1) (w_0 + w_n) (w_1 + w_n) K(\mathbf{w}).\end{aligned}\quad (70)$$

We evaluate the general clustering coefficient, $K_l = I_n/I_d$, by calculating the integrals numerically. The following figure shows the value of K_l , scaling almost perfectly as $\sim l^{-1}$, for the exponential weight distribution with mean 1 and parameters $n = 1000$ and $\theta_n = 1000$.

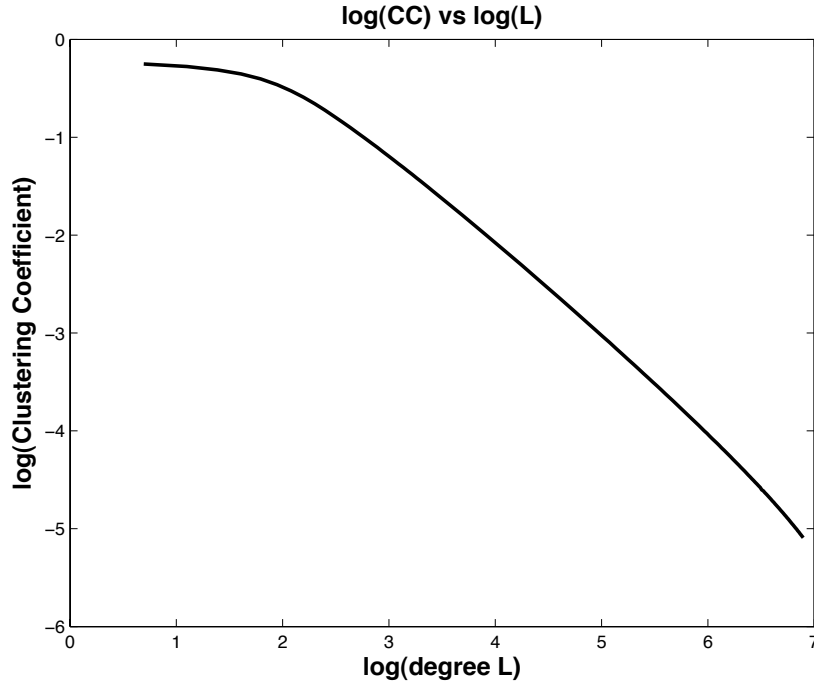


Fig. 5. Clustering Coefficient vs. Degree L

8 Summary

Geographical threshold graphs are a rich model with the possibility of controlling structural properties by choosing specific weight distributions and tuning threshold values. The model is a versatile one and can be used not only for the generation and analysis of web-graphs or large complex networks, but more generally for relation graphs in a large data set. If the data have a metric and can be mapped to nodes in Euclidean space, much of the foregoing analysis applies: one may hope to control structural properties of the data set by studying it as a GTG.

In this paper we have analyzed some of the structural properties. Given a node weight distribution $f(w)$ and threshold θ_n , the degree distribution can be easily calculated. We have shown the results on the absence and existence of the giant component by

giving a bound on the threshold θ_n . In Section 5 we presented upper and lower bounds for the constant c in the connectivity threshold $\theta_n = cn/\log n$. Finally (Section 7) we have derived a formula for the clustering coefficient in terms of the weight distribution and threshold, as well as discussed the general clustering coefficient, and evaluated it numerically. We also have provided an upper bound on the diameter of GTG. For our analysis we used the additive threshold function $(w_i + w_j)/r^2 \geq \theta_n$ for the connectivity relation, but our techniques may be generalized to other cases in a straightforward manner.

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