Infinite Precision Analysis of the Fast QR Decomposition RLS Algoritm

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***Abstract*— This work develops relations for the mean squared value of internal variables in the Fast QRD-RLS. The objective is to derive relations based on known characteristics of input signals that predict the behavior of the internal quantities of the algorithm. It is shown that the Fast and Conventional QRD-RLS algorithms have some variables in common, and thus previous results of the infinite precision analysis of the Conventional algorithm remain valid for the Fast version. Conditions for avoiding overflow in fixed-point implementations are presented. Simulation results are also shown.**

I. Introduction

In 1990, John Cioffi introduced the first QR Decomposition based Fast Recursive Least Squares (FQRD-RLS) Algorithm [1]. It has been claimed that this algorithm has very nice numerical properties [1], [2], but a formal proof for this result has not yet been developed. In fact, this algorithm is is expected to be stable, since it uses many of the steps found in the Conventional QRD-Algorithm. Recent studies have shown that the conventional QRD-Algorithm possesses good numerical properties [3].

However, for characterizing the mean squared values of deviations in the internal quantities of an algorithm caused by quantization effects, an infinite precision analysis is first required, providing relations for the mean squared values of the variables themselves [4]. Besides this, infinite precision analysis is required to determine the dynamic range of the internal variables that can be used to control overflow and to choose registers lengths in fixed-point implementations.

The notation used here is the same on [2]. On table I, the equations for calculating different variables of the FQRD-RLS algorithm are shown.

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II. Mean Squared Values of the Internal Variables in the FQRD-RLS Algorithm

A. Mean Squared Value of $x_{q,i}$

According to equation (50), if an order $N = 1$ is used for the algorithm, it is possible to write

$$\begin{aligned} a_1(k) &= x(k)\cos\theta_0(k) \\ &- \lambda^{1/2}x_{q,0}(k-1)\sin\theta_0(k) \quad (1) \\ x_{q,0}(k) &= x(k)\sin\theta_0(k) \\ &+ \lambda^{1/2}x_{q,0}(k-1)\cos\theta_0(k) \quad (2) \\ e_{aq}(k) &= a_1(k)\cos\theta_1(k) \\ &- \lambda^{1/2}x_{q,1}(k-1)\sin\theta_1(k) \quad (3) \\ x_{q,1}(k) &= a_1(k)\sin\theta_1(k) \\ &+ \lambda^{1/2}x_{q,1}(k-1)\cos\theta_1(k) \quad (4) \end{aligned}$$

Assuming that $\cos\theta_0(k)$, $x(k)$, $x_{q,0}(k-1)$ and $\sin\theta_0(k)$ are almost uncorrelated with each other, that $x(k)$ and $\sin\theta_0(k)$ are zero mean, and that all these variables are statistically stationary [3], it is possible to write from equations (1) and (2)

$$\begin{aligned} E\{a_1^2(k)\} &= E\{x^2(k)\}E\{\cos^2\theta_0(k)\} \\ &+ \lambda E\{x_{q,0}^2(k)\}E\{\sin^2\theta_0(k)\} \quad (5) \\ E\{x_{q,0}^2(k)\} &= E\{x^2(k)\}E\{\sin^2\theta_0(k)\} \\ &+ \lambda E\{x_{q,0}^2(k)\}E\{\cos^2\theta_0(k)\} \quad (6) \end{aligned}$$

Considering that $E\{\cos^2\theta_i(k)\} \approx \lambda$ and that $E\{\sin^2\theta_i(k)\} \approx 1-\lambda$ for $i = 0, \cdots, N$, where N is the order of the adaptive filter [3], and that $x(k)$ is zero-mean white gaussian noise with variance σ_x^2, the following values for the mean square values of $x_{q,0}(k)$ and $a_1(k)$ can be derived

$$E\{x_{q,0}^2(k)\} = \frac{\sigma_x^2}{1+\lambda} \quad (7)$$

$$E\{a_1^2(k)\} = \sigma_x^2\left[\frac{2\lambda}{1+\lambda}\right] \quad (8)$$

Using the same assumptions, it is possible to derive the following relations for the mean squared values of $e_{aq}(k)$ and $x_{q,1}(k)$

$$E\{e_{aq}^2(k)\} = E\{a_1^2(k)\}E\{\cos^2\theta_1(k)\} + \lambda E\{x_{q,1}^2(k)\}E\{\sin^2\theta_0(k)\} \quad (9)$$

$$E\{x_{q,1}^2(k)\} = E\{a_1^2(k)\}E\{\sin^2\theta_1(k)\} + \lambda E\{x_{q,1}^2(k)\}E\{\cos^2\theta_1(k)\} \quad (10)$$

Again, using the same relations for the mean squared values of cosines and sines, the following relations for $E\{e_{aq}^2(k)\}$ and $E\{x_{q,1}^2(k)\}$ are derived

$$E\{x_{q,1}^2(k)\} = \frac{\sigma_x^2}{1+\lambda}\left[\frac{2\lambda}{1+\lambda}\right] \quad (11)$$

$$E\{a_1^2(k)\} = \sigma_x^2\left[\frac{2\lambda}{1+\lambda}\right]^2 \quad (12)$$

In fact the above results can be generalized for an arbitrary order of the adaptive filter

$$E\{x_{q,i}^2(k)\} = \frac{\sigma_x^2}{1+\lambda}\left[\frac{2\lambda}{1+\lambda}\right]^i \quad (13)$$

$$E\{e_{aq}^2(k)\} = \sigma_x^2\left[\frac{2\lambda}{1+\lambda}\right]^{N+1} \quad (14)$$

B. Mean Squared Value of $e_a(k)$

According to (51), the mean of $e_a(k)$ can be calculated as follows

$$E\{e_a(k)\} = \lambda E\{e_a(k-1)\} + E\{e_{aq}^2(k)\} \quad (15)$$

Considering that $e_a(k)$ is stationary, it is possible to derive the following relation for the mean, using relation (14)

$$E\{e_a(k)\} = \frac{\sigma_x^2}{1-\lambda}\left[\frac{2\lambda}{1+\lambda}\right]^{N+1} \quad (16)$$

If it is considered that the variance of $e_a(k)$ can be neglected if compared to the mean, it is possible to find the following relation for $E\{e_a^2(k)\}$

$$E\{e_a^2(k)\} = \frac{\sigma_x^4}{(1-\lambda)^2}\left[\frac{2\lambda}{1+\lambda}\right]^{2(N+1)} \quad (17)$$

C. Mean Squared Value of $e_{b0}(k)$

According to equation (52), the following relation must be valid

$$E\{e_{b0}(k)\} = E\{\|\mathbf{x}_q(k)\|^2\} + E\{e_a(k)\} \quad (18)$$

since $\mathbf{Q}_\alpha(k)$ is an orthonormal transformation. The value of $E\{\|\mathbf{x}_q(k)\|^2\}$ can be calculated in many ways, and one is based on the summation of the mean squared values of the $\mathbf{x}_q(k)$ vector entries, as follows

$$E\{\|\mathbf{x}_q(k)\|^2\} = \frac{\sigma_x^2}{1+\lambda}\sum_{i=0}^{N}\left[\frac{2\lambda}{1+\lambda}\right]^i = \frac{\sigma_x^2}{1-\lambda}\left[1-\left[\frac{2\lambda}{1+\lambda}\right]^{N+1}\right] \quad (19)$$

Substituting equations (19) and (16) in (18), it follows that

$$E\{e_{b0}(k)\} = \frac{\sigma_x^2}{1-\lambda} \quad (20)$$

If the variance of $e_{b0}(k)$ is much smaller than its mean, an approximation for its mean squared value is

$$E\{e_{b0}^2(k)\} = \frac{\sigma_x^4}{(1-\lambda)^2} \quad (21)$$

D. The mean squared value of $\gamma_N(k)$

According to [3], $E\{\gamma_N^2(k)\}$ is given by

$$E\{\gamma_N^2(k)\} = \lambda^{N+1} \quad (22)$$

E. The mean squared values of $\alpha_1(k)$ and $\alpha_2(k)$

Using (53), the averaging principle [5], and supposing that $\gamma_N(k)$, $e_{aq}^2(k)$ and $e_a(k)$ are uncorrelated and statistically stationary, the following relation can be derived

$$E\{\alpha_2^2(k)\} = \frac{E\{\gamma_N^2(k)\}E\{e_{aq}^2(k)\}}{E\{e_a(k)\}} \quad (23)$$

Substituting relations (22), (14) and (16) into the above equation, the following result is obtained

$$E\{\alpha_2^2(k)\} = \lambda^{N+1}(1-\lambda) \quad (24)$$

Based on equation (54), we can conclude that $\alpha_1(k)$ and $\alpha_2(k)$ have the same mean squared values, since $\mathbf{Q}_\alpha(k)$ is an orthonormal transformation and the elements of $\mathbf{g}_N(k)$ are assumed to be statistically stationary.

$$E\{\alpha_1^2(k)\} = \lambda^{N+1}(1-\lambda) \quad (25)$$

F. The mean squared values of $\cos^2\alpha_i(k)$ *and* $\sin^2\alpha_i(k)$

The calculation of the sines and cosines of $\alpha_i(k)$ uses an internal variable, $e_{b,j}(k)$, $j = 0, \cdots, N$ in the loop. The values of this variable are calculated according to the following difference equation

$$e_{b,N-i}(k) = x_{q,N-i}^2(k) + e_{b,N-i+1}(k) \tag{26}$$

The expected value of the solution for this difference equation is given as follows

$$E\{e_{b,N-i}(k)\} = \sum_{j=0}^{i} E\{x_{q,N-j}^2(k)\} + E\{e_a(k)\} \tag{27}$$

since $e_{b,N+1}(k) = e_a(k)$. Substituting relations (13) and (17) in the above equation, the following relation is obtained

$$E\{e_{b,N-i}(k)\} = \frac{\sigma_x^2}{1-\lambda}\left[\frac{2\lambda}{1+\lambda}\right]^{N-i} \tag{28}$$

It can be easily verified that the following relation is used for computing $\sin\alpha_i(k)$

$$\sin\alpha_i(k) = \frac{x_{q,N-i}(k)}{e_{b,N-i}^{1/2}(k)} \tag{29}$$

Using the averaging principle [5], and assuming that the numerator and denominator in the above relation can be considered almost uncorrelated for approximation purposes, it is possible to obtain

$$E\{\sin^2\alpha_i(k)\} = \frac{E\{x_{q,N-i}^2(k)\}}{E\{e_{b,N-i}(k)\}} \tag{30}$$

Substituting equations (13) and (28) in the above relation it follows after simple algebraic calculations that

$$E\{\sin^2\alpha_i(k)\} = \frac{1-\lambda}{1+\lambda} \tag{31}$$

Using the fundamental trigonometric relation $\sin^2\alpha_i(k) + \cos^2\alpha_i(k) = 1$, the mean squared value for the cosines follows

$$E\{\cos^2\alpha_i(k)\} = \frac{2\lambda}{1+\lambda} \tag{32}$$

G. Mean Squared Values of $g_{N,i}(k)$

The entries in $\mathbf{g}_N(k)$ are calculated recursively in time according to equation (54). It can be verified that the last entry in the left-side vector of this equation has its value changed after successive rotations. These values will be named $e_{n,i}(k)$, $i = 0, \cdots, N$.

Assuming an order $N = 2$ is used, the equations for updating $g_{N,i}(k)$ are as follows

$$\begin{aligned}
e_{n,0}(k) &= g_{N,0}(k-1)\sin\alpha_0(k) \\
&+ \alpha_2(k)\cos\alpha_0(k) && (33)\\
g_{N,0}(k) &= g_{N,1}(k-1)\cos\alpha_1(k) \\
&- e_{n,0}(k)\sin\alpha_1(k) && (34)\\
e_{n,1}(k) &= g_{N,1}(k-1)\sin\alpha_1(k) \\
&+ e_{n,0}(k)\cos\alpha_1(k) && (35)\\
g_{N,1}(k) &= g_{N,2}(k-1)\cos\alpha_2(k) \\
&- e_{n,1}(k)\sin\alpha_2(k) && (36)\\
e_{n,2}(k) &= g_{N,2}(k-1)\sin\alpha_2(k) \\
&+ e_{n,1}(k)\cos\alpha_2(k) && (37)\\
g_{N,2}(k) &= e_{n,2}(k) && (38)
\end{aligned}$$

Supposing that $g_i(k)$, $e_{n,i}(k)$, $i = 0, \cdots, N$ are zero mean and uncorrelated with each other, and substituting the relations (31) and (32) into the above equations, it is easy to show that

$$\begin{aligned}
E\{e_{n,0}^2(k)\} &= E\{g_{N,0}^2(k)\}\frac{1-\lambda}{1+\lambda} \\
&+ \lambda^{N+1}(1-\lambda)\frac{2\lambda}{1+\lambda} && (39)\\
E\{g_{N,0}^2(k)\} &= E\{g_{N,1}^2(k)\}\frac{2\lambda}{1+\lambda} \\
&+ E\{e_{n,0}^2(k)\}\frac{1-\lambda}{1+\lambda} && (40)\\
E\{e_{n,1}^2(k)\} &= E\{g_{N,1}^2(k)\}\frac{1-\lambda}{1+\lambda} \\
&+ E\{e_{n,0}^2(k)\}\frac{2\lambda}{1+\lambda} && (41)\\
E\{g_{N,1}^2(k)\} &= E\{g_{N,2}^2(k)\}\frac{2\lambda}{1+\lambda} \\
&+ E\{e_{n,1}^2(k)\}\frac{1-\lambda}{1+\lambda} && (42)\\
E\{e_{n,2}^2(k)\} &= E\{g_{N,2}^2(k)\}\frac{1-\lambda}{1+\lambda} \\
&+ E\{e_{n,1}^2(k)\}\frac{2\lambda}{1+\lambda} && (43)\\
E\{g_{N,2}^2(k)\} &= E\{e_{n,2}^2(k)\} && (44)
\end{aligned}$$

The above relations imply

$$E\{e_{n,i}^2(k)\} = E\{g_{N,i}^2(k)\} = \lambda^{N+1}(1-\lambda) \tag{45}$$

for $i = 0, \cdots, N$, when $N = 2$. In fact this can be easily generalized for an arbitrary order.

Using equation (55), it is possible to obtain

$$E\{\|\mathbf{g}_N(k)\|^2\} = 1 - E\{\gamma_N^2(k)\} = 1 - \lambda^{N+1} \qquad (46)$$

But, if equation (45) is used, another relation for $E\{\|\mathbf{g}_N(k)\|^2\}$ may be obtained

$$E\{\|\mathbf{g}_N(k)\|^2\} = (N+1)(1-\lambda)\lambda^{N+1} \qquad (47)$$

In fact, for λ close to 1 and large values of order N, equation (47) is a very good approximation for (46).

III. Conditions for Avoiding Overflow

By comparing the derived values for mean squared values of the internal variables in the FQRD-RLS algorithm, it is easy to verify that $E\{e_{b,i}(k)\}$, $i = 0, \cdots, N$ have the largest values if fixed-point arithmetic with fractional representation is used. Using (28), it is easy to notice that the critical case happens for $i = 0$, since $\frac{2\lambda}{1+\lambda} \leq 1$. Assuming that the internal variables must range between -1 and $+1$, we want that

$$E\{e_{b,0}(k)\} < 1 \qquad (48)$$

which implies, by using (28)

$$\sigma_x^2 < 1 - \lambda \qquad (49)$$

which is the same relation for avoiding overflow in the Conventional QRD-RLS algorithm [4].

IV. Simulation Results and Conclusions

Simulations were performed to show the accuracy of the obtained relations. The reference signal $d(k)$ is a 4-th order MA process generated by using the same inputs of the adaptive filter $x(k)$ added to a measurement noise modeled as gaussian white noise with zero mean and variance equal to -70 dB. The input signal $x(k)$ was modeled as a zero mean white gaussian noise of variance equal to -30 dB. The forgetting factor λ was chosen equal to 0.95 in order to prevent overflow. A total of 10 experiments with 10000 points each were run. The last 9900 samples were averaged to obtain the shown results.

Some of the results are shown on Table II. It is possible to verify that the experimental results are very close to the theoretical values obtained through the relations derived on this work. Analysing these results, we can conclude that the shown equations are sufficiently accurate for previewing the mean squared values of the internal variables in the FQRD-RLS algorithm.

The results shown on this work are important for characterizing the FQRD-RLS algorithm, providing relations for avoiding overflow and equations for the mean squared values of the internal variables in the algorithm.

TABLE I
FAST QRD-RLS ALGORITHM IN INFINITE PRECISION

$$\begin{bmatrix} e_{aq}(k) \\ \mathbf{x}_q(k) \end{bmatrix} = \mathbf{Q}_a(k-1) \begin{bmatrix} x(k) \\ \lambda^{1/2}\mathbf{x}_q(k-1) \end{bmatrix} \qquad (50)$$

$$e_a(k) = \lambda e_a(k-1) + e_{aq}^2(k) \qquad (51)$$

$$\begin{bmatrix} \mathbf{0} \\ \sqrt{e_{b0}(k)} \end{bmatrix} = \mathbf{Q}_\alpha(k) \begin{bmatrix} \mathbf{x}_q(k) \\ \sqrt{e_a(k)} \end{bmatrix} \qquad (52)$$

$$\alpha_2(k) = \gamma_N(k-1)\frac{e_{aq}(k)}{\sqrt{e_a(k)}} \qquad (53)$$

$$\begin{bmatrix} \alpha_1(k) \\ \mathbf{g}_N(k) \end{bmatrix} = \mathbf{Q}_\alpha(k) \begin{bmatrix} \mathbf{g}_N(k-1) \\ \alpha_2(k) \end{bmatrix} \qquad (54)$$

$$\begin{bmatrix} \gamma_N(k) \\ \mathbf{g}_N(k) \end{bmatrix} = \mathbf{Q}_a(k) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (55)$$

$$\begin{bmatrix} e_q(k) \\ \mathbf{y}_q(k) \end{bmatrix} = \mathbf{Q}_a(k) \begin{bmatrix} d(k) \\ \lambda^{1/2}\mathbf{y}_q(k-1) \end{bmatrix} \qquad (56)$$

$$e(k) = \gamma_N(k)e_q(k) \qquad (57)$$

TABLE II
RESULTS OF SIMULATIONS.

Value	Simulated (dB)	Calculated (dB)
$E\{x_{q,0}^2(k)\}$	-33.4	-32.9
$E\{x_{q,1}^2(k)\}$	-33.5	-33.0
$E\{x_{q,2}^2(k)\}$	-33.6	-33.1
$E\{x_{q,3}^2(k)\}$	-33.6	-33.2
$E\{x_{q,4}^2(k)\}$	-33.7	-33.4
$E\{e_{aq}^2(k)\}$	-30.5	-30.6
$E\{e_q^2(k)\}$	-34.9	-35.1
$E\{\alpha_2^2(k)\}$	-14.2	-14.1
$E\{\cos^2\alpha_i(k)\}$	-0.11	-0.10
$E\{\sin^2\alpha_i(k)\}$	-15.9	-16.4

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