4. Theory of linear equations

- definitions: range and nullspace, left and right inverse
- nonsingular matrices
- positive definite matrices
- left- and right-invertible matrices
- summary

Range of a matrix

the **range** of an $m \times n$-matrix A is defined as

$$\mathbf{range}(A) = \{y \in \mathbf{R}^m \mid y = Ax \text{ for some } x \in \mathbf{R}^n\}$$

- the set of all m-vectors that can be expressed as Ax
- the set of all linear combinations of the columns of A
- the set of all vectors b for which $Ax = b$ is solvable

full range matrix

- A has a full range if $\mathbf{range}(A) = \mathbf{R}^m$
- if A has a full range then $Ax = b$ is solvable for every right-hand side b

Nullspace of a matrix

the **nullspace** of an $m \times n$-matrix A is defined as

$$\mathbf{nullspace}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\}$$

- the set of all vectors that are mapped to zero by $f(x) = Ax$
- the set of all vectors that are orthogonal to the rows of A
- if $Ax = b$ then $A(x + y) = b$ for all $y \in \mathbf{nullspace}(A)$

zero nullspace matrix

- A has a zero nullspace if $\mathbf{nullspace}(A) = \{0\}$
- if A has a zero nullspace and $Ax = b$ is solvable, the solution is unique

Example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

- the range of A is

$$\mathbf{range}(A) = \left\{ \begin{bmatrix} u \\ u+v \\ -v \end{bmatrix} \middle| u, v \in \mathbf{R} \right\}$$

- the nullspace of A is

$$\mathbf{nullspace}(A) = \left\{ \begin{bmatrix} u \\ u \\ -u \end{bmatrix} \middle| u \in \mathbf{R} \right\}$$

Left inverse

definitions

- C is a left inverse of A if $CA = I$
- a left-invertible matrix is a matrix with at least one left inverse

example

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

property: A is left-invertible $\Longleftrightarrow$ A has a zero nullspace

- '$\Rightarrow$': if $CA = I$ then $Ax = 0$ implies $x = CAx = 0$
- '$\Leftarrow$': see later (p.4-25)

Right inverse

definitions

- B is a right inverse of A if $AB = I$
- a right-invertible matrix is a matrix with at least one right inverse

example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

property: A is right-invertible $\Longleftrightarrow$ A has a full range

- '$\Rightarrow$': if $AB = I$ then $y = Ax$ has a solution $x = By$ for every y
- '$\Leftarrow$': see later (p.4-27)

Matrix inverse

if A has a left **and** a right inverse, then they are equal:

$$AB = I, \quad CA = I \qquad \Longrightarrow \qquad C = C(AB) = (CA)B = B$$

we call $C = B$ the **inverse** of A (notation: A^{-1})

example

$$A = \begin{bmatrix} -1 & 1 & -3 \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \qquad A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 4 & 1 \\ 0 & -2 & 1 \\ -2 & -2 & 0 \end{bmatrix}$$

Outline

- definitions: range and nullspace, left and right inverse
- **nonsingular matrices**
- positive definite matrices
- left- and right-invertible matrices
- summary

Nonsingular matrices

for a **square** matrix A the following properties are equivalent

1. A has a zero nullspace
2. A has a full range
3. A is left-invertible
4. A is right-invertible
5. $Ax = b$ has exactly one solution for every value of b

a square matrix that satisfies these properties is called **nonsingular**

- proof of 1 $\Leftrightarrow$ 2 follows on page 4-15
- we have already mentioned that 1 $\Leftrightarrow$ 3 (page 4-5), 2 $\Leftrightarrow$ 4 (page 4-6)
- $1, 2 \Leftrightarrow 5$ is by definition of nullspace and range
- 3, 4 imply that A is invertible, with unique inverse A^{-1} (see page 4-7)
- if A is nonsingular, the unique solution of $Ax = b$ is $x = A^{-1}b$

Examples

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

- A is nonsingular because it has a nonzero nullspace: $Ax = 0$ means

$$x_1 - x_2 + x_3 = 0, \qquad -x_1 + x_2 + x_3 = 0, \qquad x_1 + x_2 - x_3 = 0$$

  this is only possible if $x_1 = x_2 = x_3 = 0$

- B is singular because its nullspace is not zero:

$$Bx = 0 \quad \text{for } x = (1, 1, 1, 1)$$

Example: Vandermonde matrix

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \qquad \text{with } t_i \neq t_j \text{ for } i \neq j$$

we show that A is nonsingular by showing it has a zero nullspace

- $Ax = 0$ means $p(t_1) = p(t_2) = \cdots = p(t_n) = 0$ where

$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}$$

  $p(t)$ is a polynomial of degree $n-1$ or less

- if $x \neq 0$, then $p(t)$ can not have more than $n-1$ distinct real roots
- therefore $p(t_1) = \cdots = p(t_n) = 0$ is only possible if $x = 0$

Inverse of transpose and product

transpose

if A is nonsingular, then A^T is nonsingular and

$$(A^T)^{-1} = (A^{-1})^T$$

we write this is A^{-T}

product

if A and B are nonsingular and of the same dimension, then AB is nonsingular with inverse

$$(AB)^{-1} = B^{-1}A^{-1}$$

Schur complement

suppose A is $(k+1) \times (k+1)$ and partitioned as

$$A = \left[\begin{array}{cc} a_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]$$

(A_{22} has size $k \times k$, A_{12} has size $1 \times k$, A_{21} has size $k \times 1$)

definition: if $a_{11} \neq 0$ the Schur complement of a_{11} is the matrix

$$S = A_{22} - \frac{1}{a_{11}} A_{21} A_{12}$$

S has dimension $k \times k$

Schur complement and variable elimination

partitioned set of linear equations $Ax = b$

$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ B_2 \end{bmatrix} \tag{1}$$

- if $a_{11} \neq 0$, eliminating x_1 from the first equation gives

$$x_1 = \frac{b_1 - A_{12} X_2}{a_{11}} \tag{2}$$

- substituting x_1 in the other equations gives

$$S X_2 = B_2 - \frac{b_1}{a_{11}} A_{21} \tag{3}$$

hence, if $a_{11} \neq 0$, can solve (1) by solving (3) and substituting X_2 in (2)

consequences (for A with $a_{11} \neq 0$)

- (1) is solvable for any right-hand side iff (3) is solvable for any r.h.s.

$$A \text{ has a full range} \quad \Longleftrightarrow \quad S \text{ has a full range}$$

- with $b = 0$, only solution of (1) is $x = 0$ iff only solution of (3) is $X_2 = 0$

$$A \text{ has a zero nullspace} \quad \Longleftrightarrow \quad S \text{ has a zero nullspace}$$

Equivalence between ZN and FR conditions

a square matrix has a full range if and only if it has a zero nullspace

outline of proof by induction (see course reader for details)

- if A is 1×1, both properties are equivalent to $A \neq 0$
- assume result holds for matrices of order k; consider A of order $k+1$

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{with } A_{22} \text{ of size } k \times k$$

case 1. $a_{11} \neq 0$

$$\begin{aligned} A \text{ has a full range} \quad &\Longleftrightarrow \quad S \text{ has a full range} \\ &\Longleftrightarrow \quad S \text{ has a zero nullspace} \\ &\Longleftrightarrow \quad A \text{ has a zero nullspace} \end{aligned}$$

steps 1 and 3 follow from previous page; step 2 because S is $k \times k$

case 2. $a_{11} = 0$, $A_{21} \neq 0$

- reorder rows of A to obtain nonzero $1,1$ element
- reordering rows does not change zero nullspace or full range property
- result for case 1 implies A has full range iff it has zero nullspace

case 3. $a_{11} = 0$, $A_{21} = 0$, $A_{12} \neq 0$

- reorder columns of A to obtain nonzero $1,1$ element
- reordering does not change zero nullspace or full range property
- result for case 1 implies A has full range iff it has zero nullspace

case 4. $a_{11} = 0$, $A_{21} = 0$, $A_{12} = 0$

matrix does not have full range and does not have a zero nullspace

$$e_1 \notin \mathbf{range}(A), \qquad e_1 \in \mathbf{nullspace}(A)$$

Inverse of a nonsingular matrix

assume A has a full range and a nonzero nullspace

- A is right-invertible: partition X and I in $AX = I$ by columns

$$A \left[\begin{array}{cccc} X_1 & X_2 & \cdots & X_n \end{array} \right] = \left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \right]$$

  each equation $AX_i = e_i$ has a solution because A has a full range

- X is also a left-inverse: $AX = I$ implies $AXA = A$; therefore

$$A(XA - I) = 0$$

  if A has a zero nullspace, this implies $XA = I$

- from page 4-7, if X is a left and a right inverse, it is unique ($X = A^{-1}$)

Outline

- definitions: range and nullspace, left and right inverse
- nonsingular matrices
- **positive definite matrices**
- left- and right-invertible matrices
- summary

Positive definite matrices

definitions

- A is **positive definite** if A is symmetric and

$$x^T Ax > 0 \text{ for all } x \neq 0$$

- A is **positive semidefinite** if A is symmetric and

$$x^T Ax \geq 0 \text{ for all } x$$

note: if A is symmetric of order n, then

$$x^T Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i>j} a_{ij} x_i x_j$$

Examples

$$A = \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 9 & 6 \\ 6 & 3 \end{bmatrix}$$

- A is positive definite:

$$x^T A x = 9x_1^2 + 12x_1x_2 + 5x_2^2 = (3x_1 + 2x_2)^2 + x_2^2$$

- B is positive semidefinite but not positive definite:

$$x^T B x = 9x_1^2 + 12x_1x_2 + 4x_2^2 = (3x_1 + 2x_2)^2$$

- C is not positive semidefinite:

$$x^T C x = 9x_1^2 + 12x_1x_2 + 3x_2^2 = (3x_1 + 2x_2)^2 - x_2^2$$

Example

$$A = \begin{bmatrix} 1 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

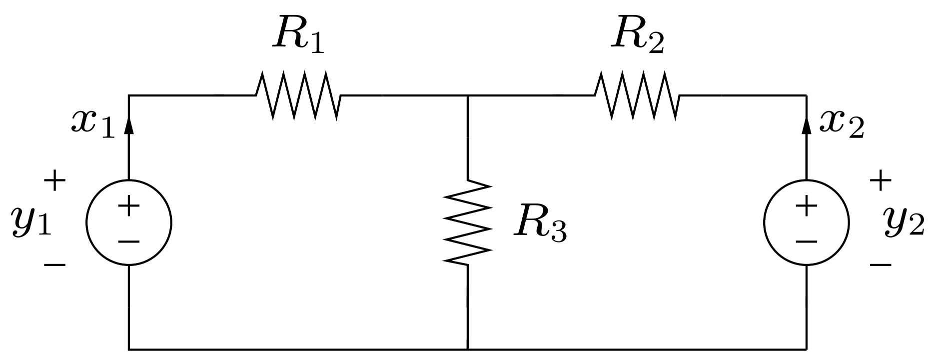
A is positive semidefinite:

$$x^T A x = (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2 \geq 0$$

A is not positive definite:

$$x^T A x = 0 \text{ for } x = (1, 1, \ldots, 1)$$

Resistor circuit



circuit model: $y = Ax$ with

$$A = \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix} \qquad (R_1, R_2, R_3 > 0)$$

interpretation of $x^T A x = y^T x$

$x^T A x$ is the power delivered by the sources, dissipated by the resistors

A is positive definite, $i.e.$, $x^T Ax > 0$ for all nonzero x

- proof from physics:

  power dissipated by the resistors is positive unless both currents are zero

- algebraic proof:

$$\begin{aligned} x^T Ax &= (R_1 + R_3)x_1^2 + 2R_3x_1x_2 + (R_2 + R_3)x_2^2 \\ &= R_1x_1^2 + R_2x_2^2 + R_3(x_1 + x_2)^2 \\ &\geq 0 \end{aligned}$$

  and $x^T Ax = 0$ only if $x_1 = x_2 = 0$

Properties of positive definite matrices

- A is nonsingular

  proof: $x^TAx > 0$ for all nonzero x, hence $Ax \neq 0$ if $x \neq 0$

- the diagonal elements of A are positive

  proof: $a_{ii} = e_i^TAe_i > 0$ (e_i is the ith unit vector)

- Schur complement $S = A_{22} - (1/a_{11})A_{21}A_{21}^T$ is positive definite, where

$$A = \begin{bmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix}$$

  proof: take any $v \neq 0$ and $w = -(1/a_{11})A_{21}^Tv$

$$v^TSv = \begin{bmatrix} w & v^T \end{bmatrix} \begin{bmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} > 0$$

Gram matrix

a **Gram matrix** is a matrix of the form

$$A = B^T B$$

properties

- A is positive semidefinite

$$x^T A x = x^T B^T B x = \|Bx\|^2 \geq 0 \quad \forall x$$

- A is positive definite if and only if B has a zero nullspace

- A is positive definite if and only if B^T has a full range (proof in reader)

Outline

- definitions: range and nullspace, left and right inverse
- nonsingular matrices
- positive definite matrices
- **left- and right-invertible matrices**
- summary

Left-invertible matrix

$$A \text{ is left-invertible} \quad \Longleftrightarrow \quad A \text{ has a zero nullspace}$$

proof

- '$\Rightarrow$' part: if $BA = I$ then $Ax = 0$ implies $x = BAx = 0$
- '$\Leftarrow$' part: if A has a zero nullspace then A^TA is positive definite and

$$B = (A^TA)^{-1}A^T$$

  is a left inverse of A

Dimensions of a left-invertible matrix

- if A is $m \times n$ and left-invertible then $m \geq n$
- in other words, a left-invertible matrix is square ($m = n$) or tall ($m > n$)

proof: assume $m < n$ and partition $XA = I$ as

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} X_1A_1 & X_1A_2 \\ X_2A_1 & X_2A_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

with X_1 and A_1 of size $m \times m$, X_2 $(n - m) \times m$, and A_2 $m \times (n - m)$

this is impossible:

- from 1,1-block: $X_1A_1 = I$ means $X_1 = A_1^{-1}$
- from 1,2-block $X_1A_2 = 0$: multiplying on the left with A_1 gives $A_2 = 0$
- from 2,2-block $X_2A_2 = I$: a contradiction with $A_2 = 0$

Right-invertible matrix

$$A \text{ is right-invertible} \quad \Longleftrightarrow \quad A \text{ has a full range}$$

proof

- '$\Rightarrow$' part: if $AC = I$ then $Ax = b$ has a solution

$$x = Cb$$

  for every value of b

- '$\Leftarrow$' part: if A has a full range then AA^T is positive definite and

$$C = A^T(AA^T)^{-1}$$

  is a right inverse of A

dimensions: right-invertible $m \times n$ matrix is square or wide ($m < n$)

Orthogonal matrices

definition (page 2-16) A is orthogonal if

$$A^T A = I$$

properties (with A orthogonal of size $m \times n$)

- A is left-invertible with left-inverse A^T
- A is tall ($m > n$) or square ($m = n$)
- if A is square then $A^{-1} = A^T$ and $AA^T = I$
- if A is tall, $AA^T \neq I$

Outline

- definitions: range and nullspace, left and right inverse
- nonsingular matrices
- positive definite matrices
- left- and right-invertible matrices
- **summary**

Summary: left-invertible matrix

the following properties are equivalent

1. A has a zero nullspace
2. A has a left inverse
3. $A^T A$ is positive definite
4. $Ax = b$ has at most one solution for every value of b

- we will refer to such a matrix as **left-invertible**
- a left-invertible matrix must be square or tall

Summary: right-invertible matrix

the following properties are equivalent

5. A has a full range

6. A is right-invertible

7. AA^T is positive definite

8. $Ax = b$ has at least one solution for every value of b

- we will refer to such a matrix as **right-invertible**

- a right-invertible matrix must be square or wide

Summary: nonsingular matrix

for square matrices, properties 1–8 are equivalent

- such a matrix is called **nonsingular** (or invertible)
- for nonsingular A, left and right inverses are equal and denoted A^{-1}
- if A is nonsingular then $Ax = b$ has a unique solution

$$x = A^{-1}b$$