

## 3. Linear equations

- linear equations
- example: polynomial interpolation
- applications
- geometrical interpretation

# Linear equations

$m$  equations in  $n$  variables  $x_1, x_2, \dots, x_n$ :

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

in matrix form:  $Ax = b$ , where

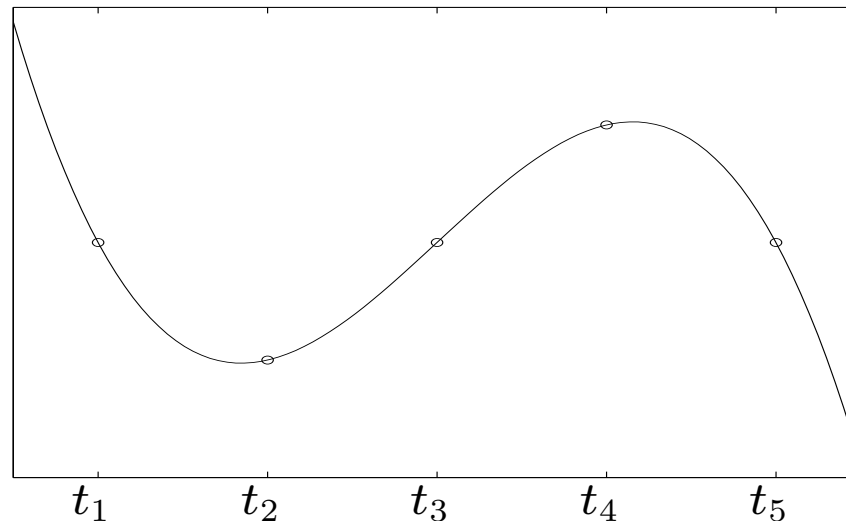
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

## Example: polynomial interpolation

fit a polynomial

$$p(t) = x_1 + x_2t + x_3t^2 + \cdots + x_nt^{n-1}$$

through  $n$  points  $(t_1, y_1), \dots, (t_n, y_n)$



problem data (parameters):  $t_1, \dots, t_n, y_1, \dots, y_n$

problem variables:  $x_1, \dots, x_n$

write out the conditions on  $x$ :

$$\begin{aligned} p(t_1) &= x_1 + x_2 t_1 + x_3 t_1^2 + \cdots + x_n t_1^{n-1} = y_1 \\ p(t_2) &= x_1 + x_2 t_2 + x_3 t_2^2 + \cdots + x_n t_2^{n-1} = y_2 \\ &\vdots \\ p(t_n) &= x_1 + x_2 t_n + x_3 t_n^2 + \cdots + x_n t_n^{n-1} = y_n \end{aligned}$$

in matrix form:  $Ax = b$  with

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$n$  linear equations in  $n$  variables

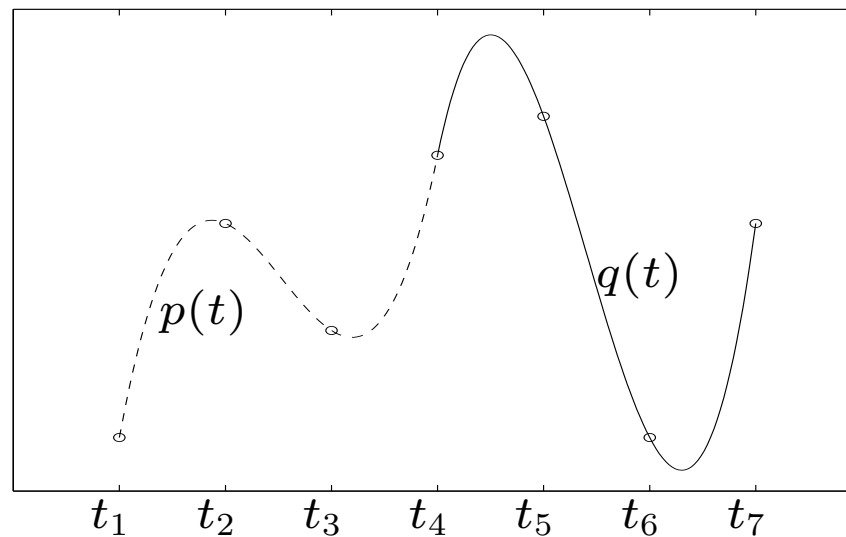
## Exercise

express as a set of linear equations: find two cubic polynomials

$$p(t) = c_0 + c_1t + c_2t^2 + c_3t^3, \quad q(t) = d_0 + d_1t + d_2t^2 + d_3t^3$$

that satisfy the following properties:

- $p(t_1) = y_1, p(t_2) = y_2, p(t_3) = y_3$  ( $t_i, y_i$  given for  $i = 1, 2, 3$ )
- $q(t_5) = y_5, q(t_6) = y_6, q(t_7) = y_7$  ( $t_i, y_i$  given for  $i = 5, 6, 7$ )
- $p(t_4) = q(t_4), p'(t_4) = q'(t_4)$  ( $t_4$  is given)



# Applications

a set of linear equations  $Ax = b$  (with  $A$   $m \times n$ ) is

- *square* if  $m = n$
- *underdetermined* if  $m < n$
- *overdetermined* if  $m > n$

all three types arise in practice

broad categories of applications:

- analysis or simulation
- control or design
- estimation or inversion

# Analysis or simulation

- $y = Ax$  is a linear (or linearized) model of a physical system
- $x$  and  $y$  are quantities that describe the operation of the system (*e.g.*, currents & voltages, forces & displacements, . . . )

we know  $y$ ; we are interested in determining  $x$

## examples

- structure (*e.g.*, building, bridge, . . . ):  $y_i$ 's are forces or loads,  $x_i$ 's are the resulting displacements at specified points
- electrical circuit:  $y_i$ 's are values of voltage and current sources,  $x_i$ 's are the resulting voltages and currents in the circuit

# Control or design

- $x$  is vector of design parameters or inputs (which we can choose)
- $y$  is vector of results, or outcomes
- $y = Ax$  describes how input choices affect results

## problems

- find  $x$  so that  $y = y_{\text{des}}$
- find all  $x$ 's that result in  $y = y_{\text{des}}$  (*i.e.*, all designs that meet the specs)
- among  $x$ 's that satisfy  $y = y_{\text{des}}$ , find a small one (*i.e.*, find a small or efficient  $x$  that meets specifications)

**example: final position/velocity of mass from forces** (see p. 1-10)

relation between forces and final velocity/position:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 19/2 & 17/2 & \cdots & 1/2 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{10} \end{bmatrix}$$

- $x$ : sequence of forces
- $y_1$ : final position;  $y_2$ : final velocity

**problems:** given desired final position/velocity  $y_{\text{des}}$

- find force sequence  $x$  so that  $y = y_{\text{des}}$  (2 linear equations in 10 vars.)
- find all  $x$ 's that result in  $y = y_{\text{des}}$
- among  $x$ 's that satisfy  $y = y_{\text{des}}$ , find one with small  $\|x\|$

# Estimation or inversion

- $y_i$  is a measurement or sensor reading (which we know)
- $x_j$  is parameter to be estimated or determined
- $y = Ax$  describes how parameters affect measurements  
( $a_{ij}$  is the sensitivity of the  $i$ th sensor to the  $j$ th parameter)

## problems

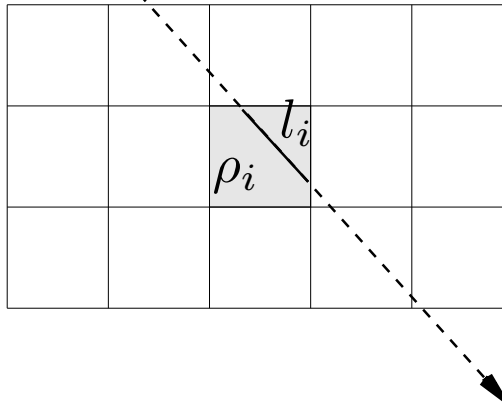
- find  $x$ , given  $y$
- find all  $x$ 's that result in  $y$  (*i.e.*,  $x$ 's consistent with measurements)
- among all  $x$ 's that result in  $y$ , find  $x$  closest to some prior guess  $x_{\text{prior}}$
- if there is no  $x$  such that  $y = Ax$ , find  $x$  such that  $y \approx Ax$  (if the sensor readings are inconsistent, find  $x$  which is almost consistent)

# Image reconstruction from projections

**problem:** estimate density  $\rho(x, y, z)$  of a 3-dimensional object

partition object in  $N$  small boxes ('pixels' or 'voxels') and assume density is constant on each pixel

X-ray beam with intensity  $I_0$



measured intensity  $I = I_0 e^{-\sum_{i=1}^N l_i \rho_i}$

$\rho_i$ : (unknown) density in pixel  $i$ ;

$l_i$ : (known) length of path through pixel  $i$  (up to a physical constant)

after taking logs: one linear equation in the variables  $\rho_i$ :

$$\sum_{i=1}^N l_i \rho_i = \log(I_0/I)$$

repeat measurement with different source locations and beam directions:  
 $M$  measurements yield  $M$  linear equations

$$\sum_{i=1}^N l_{ki} \rho_i = \log(I_0/I_k), \quad k = 1, \dots, M$$

- usually a huge number of variables (millions)
- underdetermined set of linear equations ( $M \ll N$ ): many solutions  $\rho_i$
- a popular method: pick a solution  $\rho$ , consistent with the measurements, and close to a prior guess

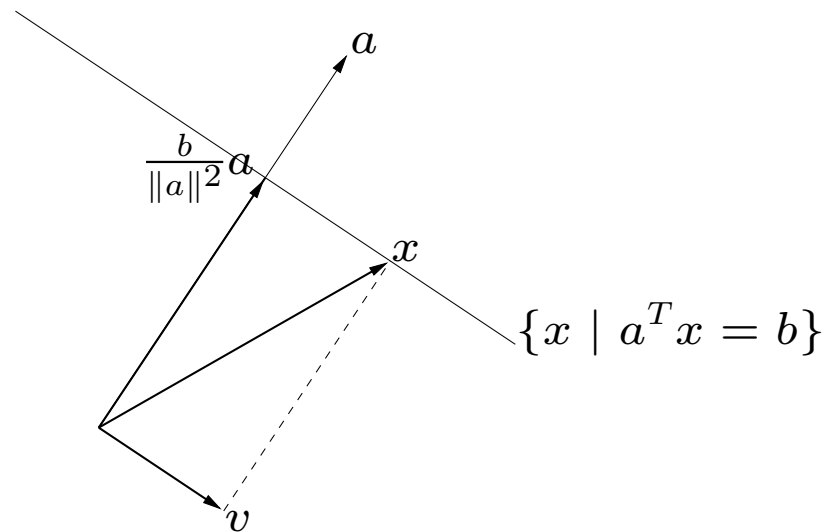
for example, choose  $\rho_i$  that minimizes  $\sum_{i=1}^N (\rho_i - \bar{\rho}_i)^2$  where  $\bar{\rho}_i$  is a prior guess of the density (*e.g.*, uniform density)

# Geometrical interpretation of $Ax = b$

**special case:** one equation in  $n$  variables ( $m = 1$ )

$$a^T x = b \quad (a \neq 0)$$

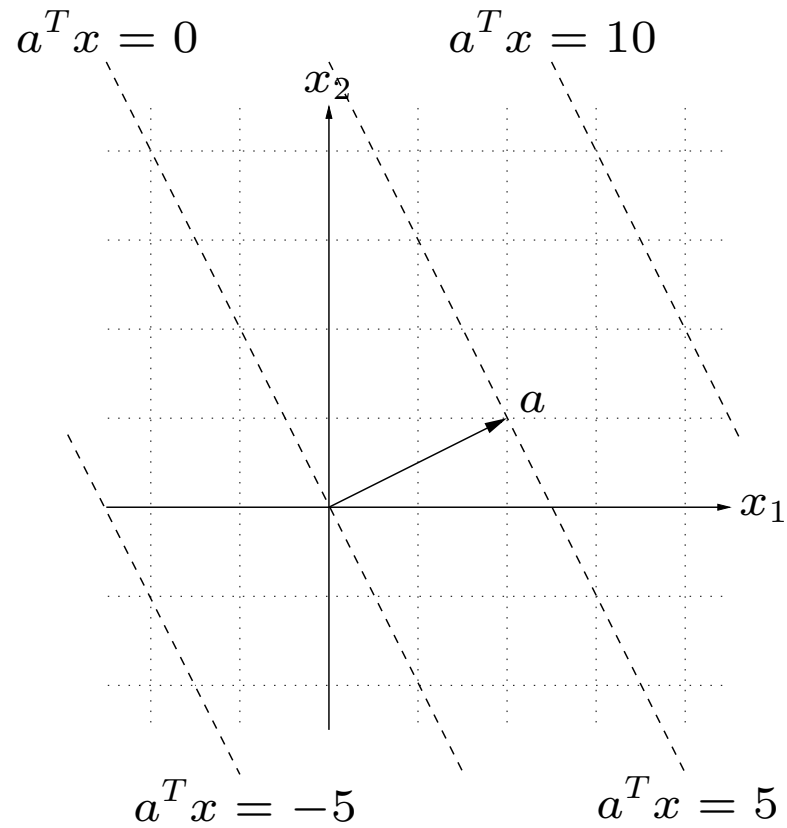
general solution:  $x = \frac{b}{\|a\|^2}a + v$  where  $v$  is an arbitrary vector with  $a^T v = 0$



the solution set is called a *(hyper-)plane*,  $a$  is the *normal vector*

for  $n = 2$ : a line perpendicular to  $a$ ;  $n = 3$ : a plane perpendicular to  $a$ .

**example** ( $n = 2$ ,  $a = (2, 1)$ )



**general case:**  $m$  equations in  $n$  variables

$$Ax = b$$

write  $A$  in terms of its rows:

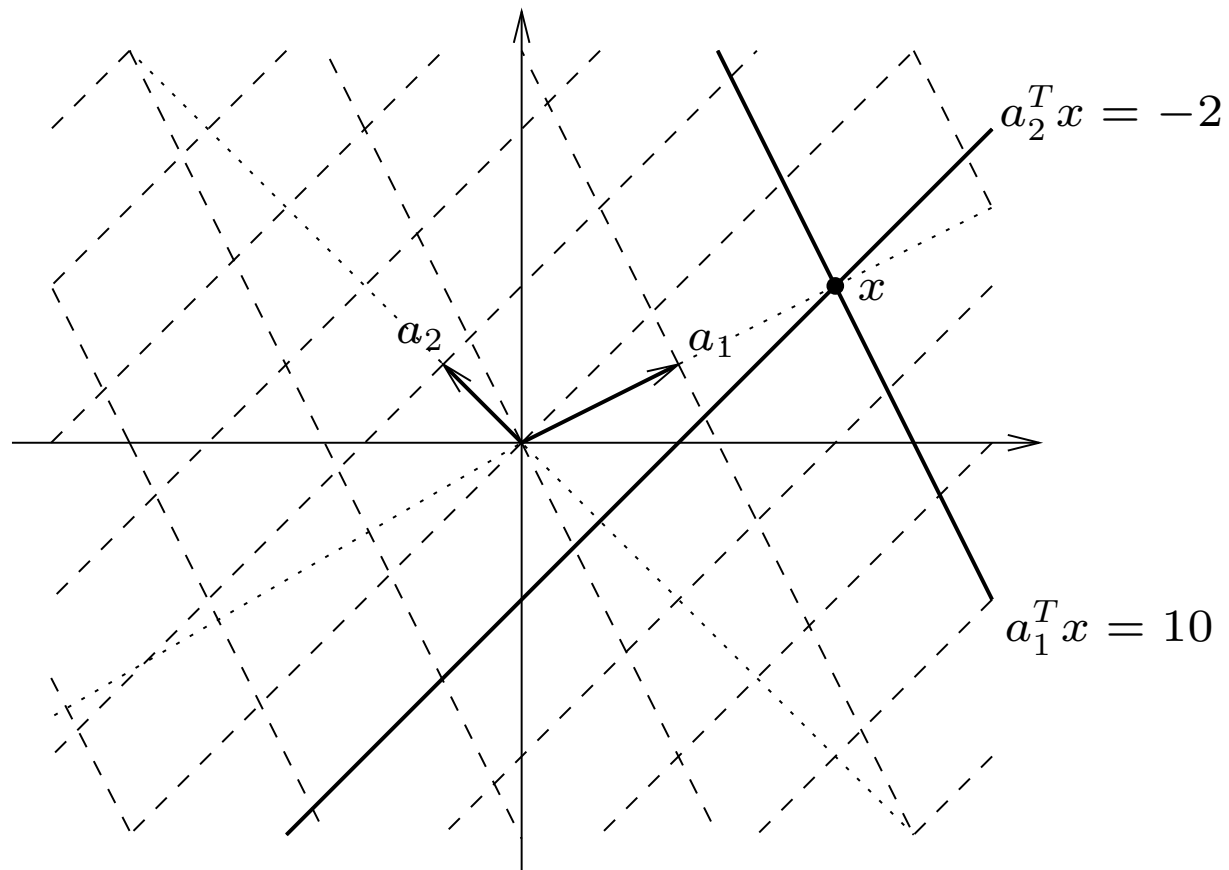
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \quad \text{where } a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$$

then  $Ax = b$  can be written as

$$a_1^T x = b_1, \quad a_2^T x = b_2, \quad \dots, \quad a_m^T x = b_m$$

the solution set is the *intersection* of  $m$  hyperplanes  $a_i^T x = b_i$   
(assuming the rows  $a_i$  are nonzero)

**example** ( $n = 2$ ):  $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ ,  $b = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$



solution is  $x = (4, 2)$