# 18. Linear optimization

- linear program
- examples
- geometrical interpretation
- extreme points
- simplex method

# Linear program

$$
\begin{array}{ll}
\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
& \cdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m
\end{array}
$$

- $n$ optimization (decision) variables $x_1$, . . . , $x_n$
- $m$ linear inequality constraints

**matrix notation**

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b
\end{array}
$$

the inequality between vectors $Ax \leq b$ is interpreted elementwise

# Production planning

- a company makes three products, in quantities $x_1$, $x_2$, $x_3$ per month
- profit per unit is $1.0$ (for product 1), $1.4$ (product 2), $1.6$ (product 3)
- products use different amounts of resources (labor, material, . . . )

fraction of total available resource needed per unit of each product

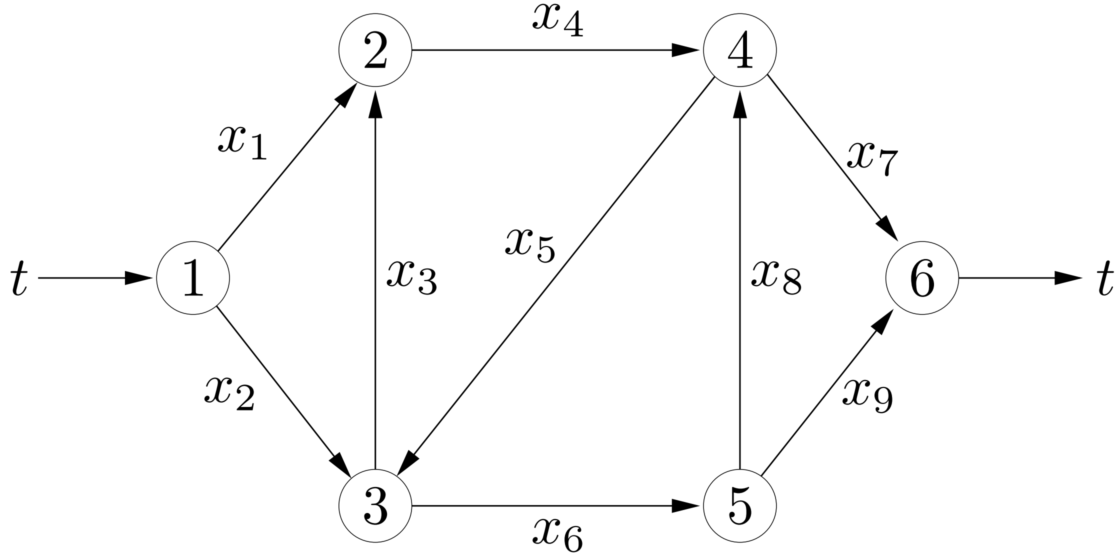
| | product 1 | product 2 | product 3 |
|---|---|---|---|
| resource 1 | 1/1000 | 1/800 | 1/500 |
| resource 2 | 1/1200 | 1/700 | 1/600 |

**optimal production plan**

$$\begin{array}{ll} \text{maximize} & x_1 + 1.4x_2 + 1.6x_3 \\ \text{subject to} & (1/1000)x_1 + (1/800)x_2 + (1/500)x_3 \leq 1 \\ & (1/1200)x_1 + (1/700)x_2 + (1/600)x_3 \leq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{array}$$

solution: $x_1 = 462$, $x_2 = 431$, $x_3 = 0$

# Network flow optimization




capacity constraints

$$
\begin{aligned}
0 \le x_1 \le 3 \\
0 \le x_2 \le 2 \\
0 \le x_3 \le 1 \\
0 \le x_4 \le 2 \\
0 \le x_5 \le 1 \\
0 \le x_6 \le 3 \\
0 \le x_7 \le 3 \\
0 \le x_8 \le 1 \\
0 \le x_9 \le 1
\end{aligned}
$$

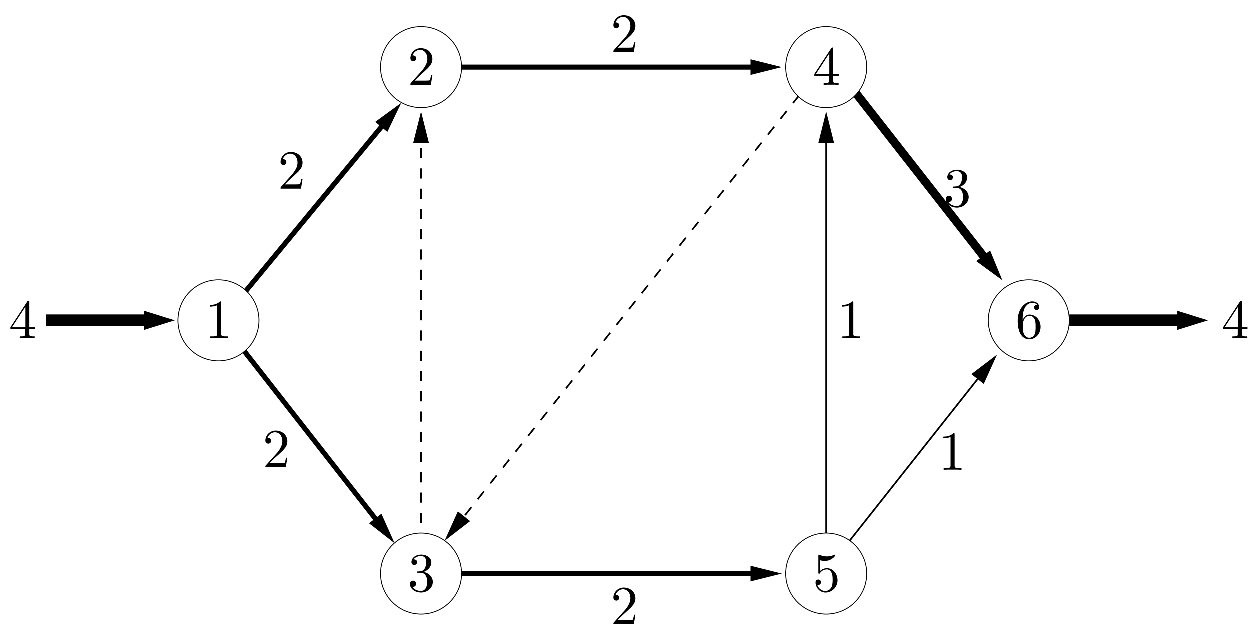
| | |
|---|---|
| maximize | $t$ |
| subject to | flow conservation at nodes |
| | capacity constraints on the arcs |

## linear programming formulation

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & t = x_1 + x_2, \quad x_1 + x_3 = x_4, \quad \text{et cetera} \\ & 0 \leq x_1 \leq 3, \quad 0 \leq x_2 \leq 2, \quad \text{et cetera} \end{array}$$

($t = x_1 + x_2$ is equivalent to inequalities $t \leq x_1 + x_2$, $t \geq x_1 + x_2$, . . . )

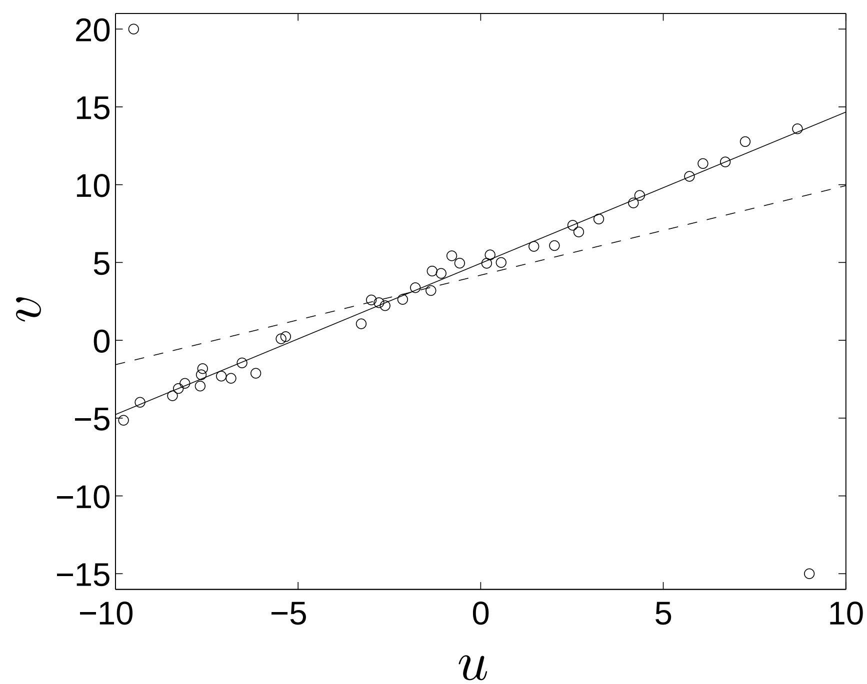
## solution



# Data fitting

fit a straight line to $m$ points $(u_i, v_i)$ by minimizing sum of absolute errors

$$\text{minimize} \quad \sum_{i=1}^{m} |\alpha + \beta u_i - v_i|$$




dashed: least-squares solution; solid: minimizes sum of absolute errors

**linear programming formulation**

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} y_i \\ \text{subjec to} & -y_i \leq \alpha + \beta u_i - v_i \leq y_i, \quad i = 1, \ldots, m \end{array}$$

- variables $\alpha$, $\beta$, $y_1$, . . . , $y_m$
- inequalities are equivalent to

$$y_i \geq |\alpha + \beta u_i - v_i|$$

- the optimal $y_i$ satisfies

$$y_i = |\alpha + \beta u_i - y_i|$$

## Terminology

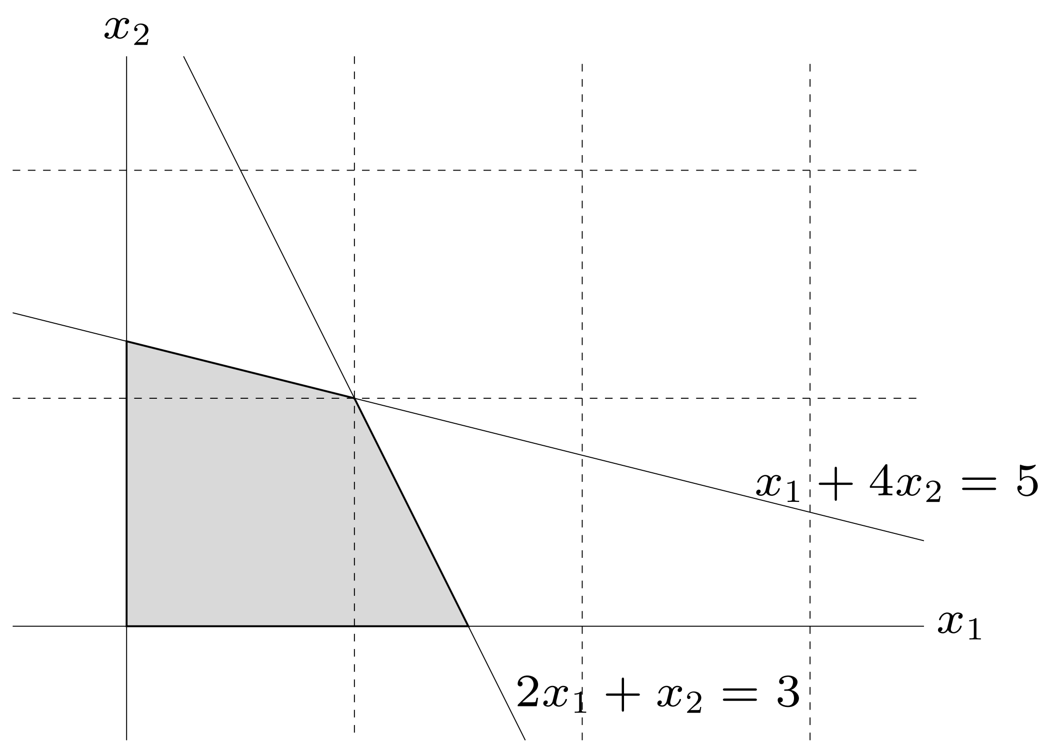
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

problem data: $n$-vector $c$, $m \times n$-matrix $A$, $m$-vector $b$

- $x$ is **feasible** if $Ax \leq b$
- **feasible set** is set of all feasible points
- $x^\star$ is **optimal** if it is feasible and $c^T x^\star \leq c^T x$ for all feasible $x$
- **optimal value**: $p^\star = c^T x^\star$
- **unbounded problem**: $c^T x$ is unbounded below on feasible set
- **infeasible probem**: feasible set is empty

# Example

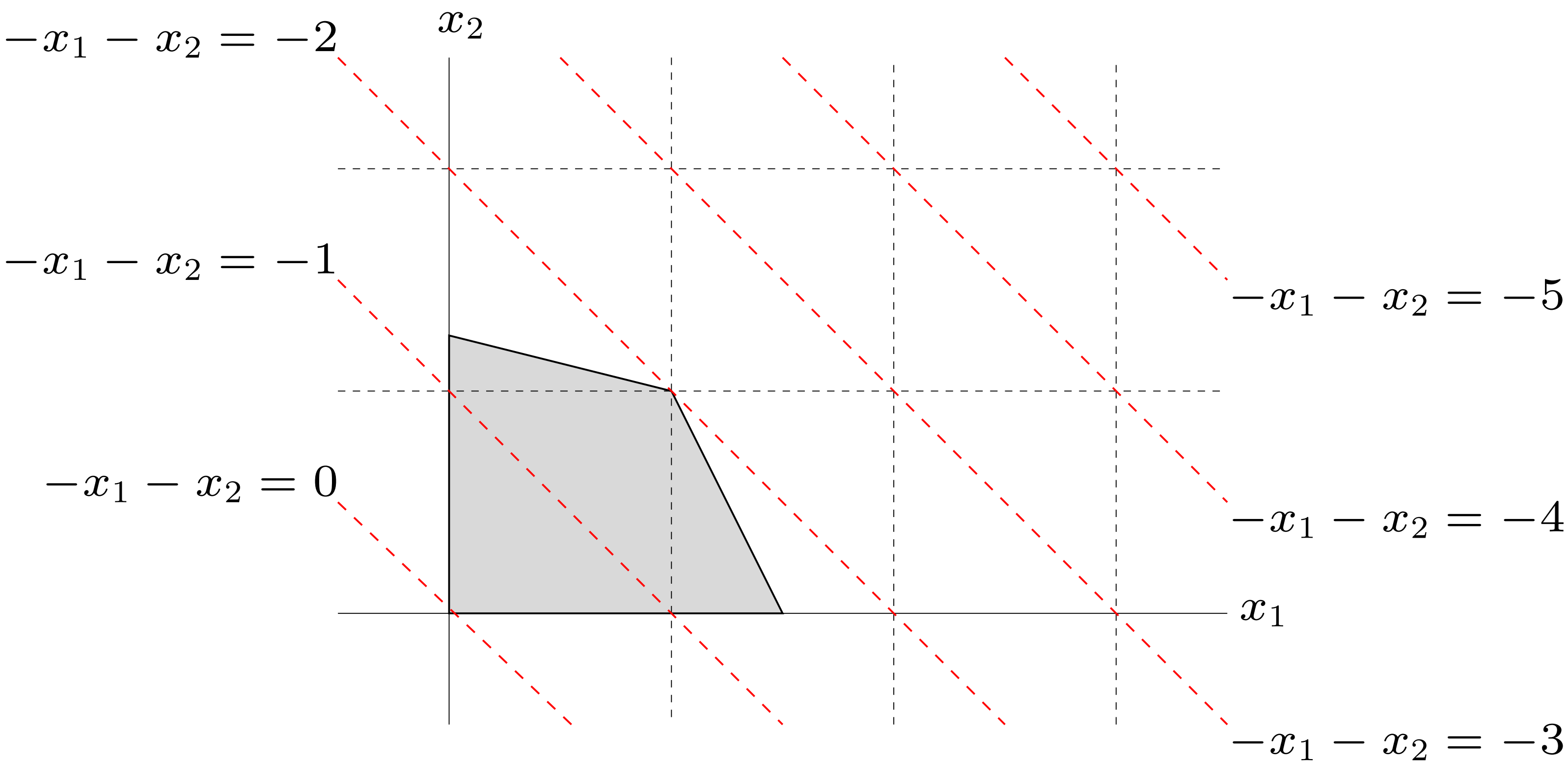
$$\begin{array}{ll} \text{minimize} & -x_1 - x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 3 \\ & x_1 + 4x_2 \leq 5 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{array}$$




feasible set is shaded

## solution

$$\begin{array}{ll} \text{minimize} & -x_1 - x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 3 \\ & x_1 + 4x_2 \leq 5 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{array}$$




optimal solution is $x^\star = (1, 1)$, optimal value is $p^\star = -2$

# Hyperplanes and halfspaces

**hyperplane**

solution set of one linear equation with nonzero coefficient vector $a$

$$a^T x = b$$

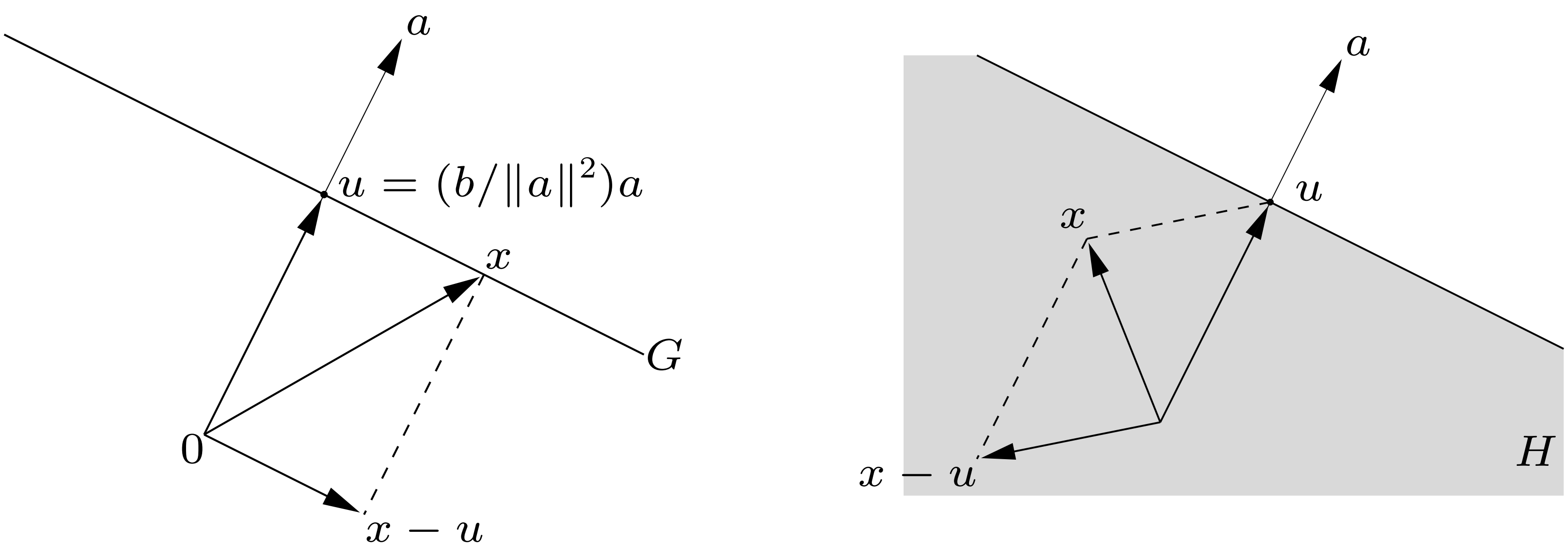
**halfspace**

solution set of one linear inequality with nonzero coefficient vector $a$

$$a^T x \leq b$$

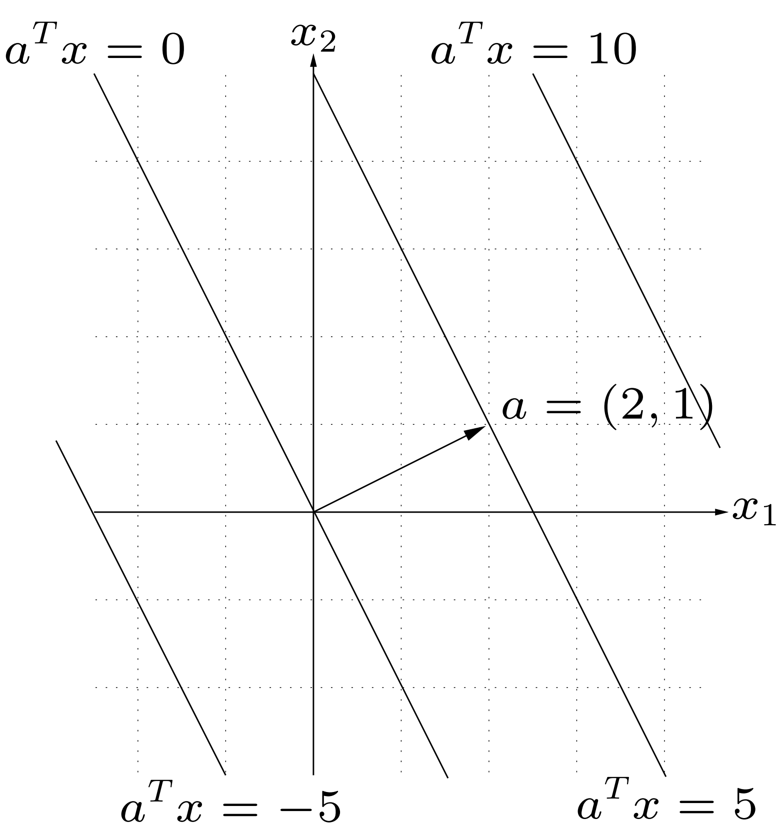
# Geometrical interpretation

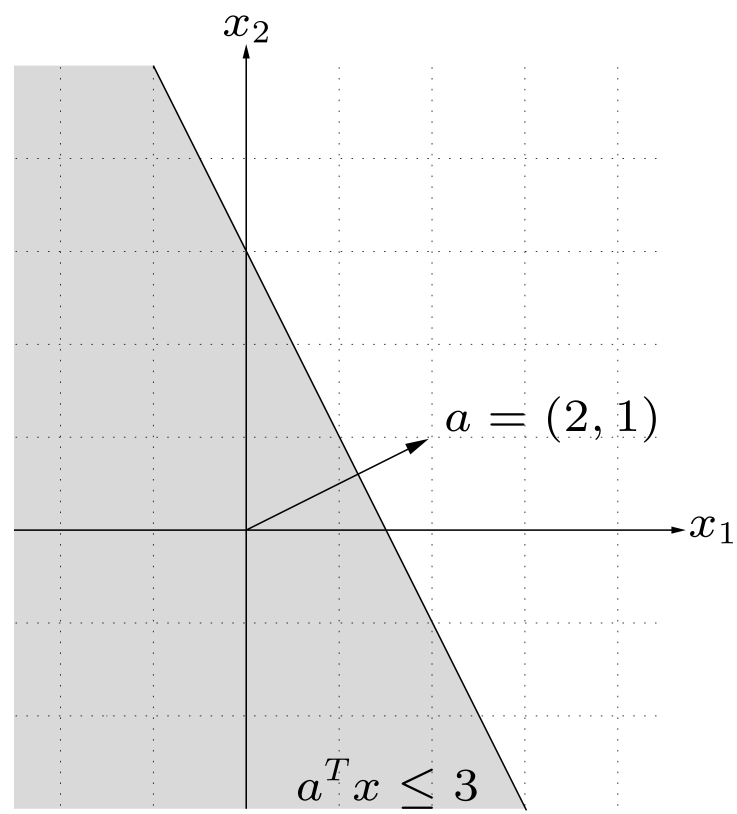
$G = \{x \mid a^T x = b\}$ $\qquad$ $H = \{x \mid a^T x \leq b\}$




- $u = (b/\|a\|^2)a$ satisfies $a^T u = b$
- $x$ is in $G$ if $a^T(x-u) = 0$, *i.e.*, $x - u$ is orthogonal to $a$
- $x$ is in $H$ if $a^T(x-u) \leq 0$, *i.e.*, angle $\angle(x-u, a) \geq \pi/2$

# Example



# Polyhedron

**definition:** the solution set of a finite number of linear inequalities

$$a_1^T x \leq b_1, \qquad a_2^T x \leq b_2, \qquad \ldots, \qquad a_m^T x \leq b_m$$

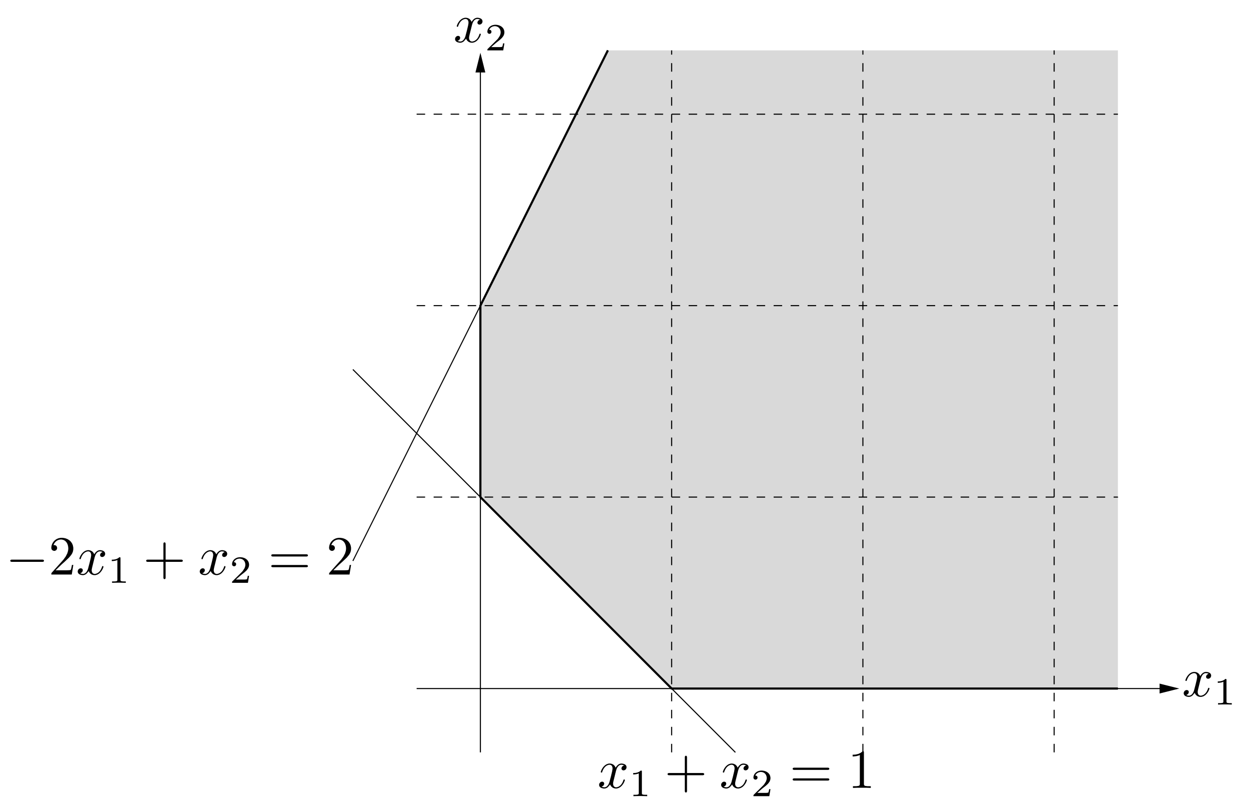
**matrix notation**: $Ax \leq b$ where

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

**geometrical interpretation**: intersection of $m$ halfspaces

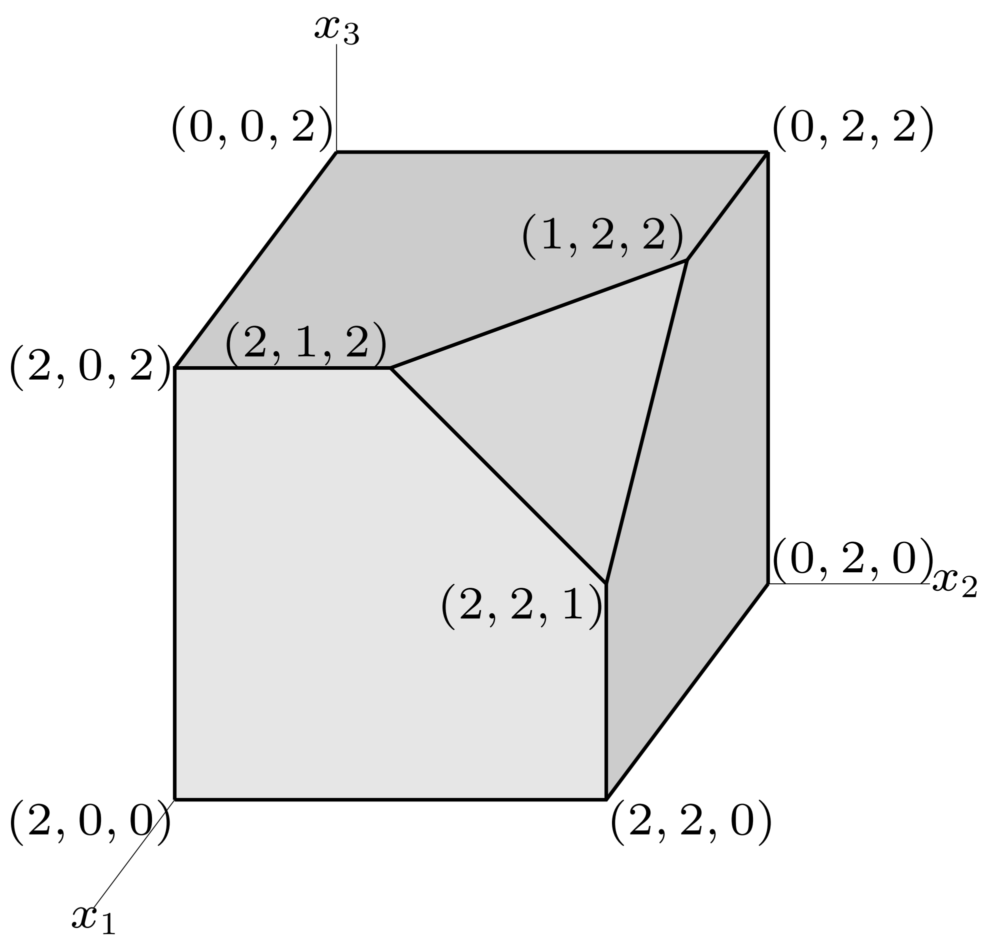
**example** $(n = 2)$

$$x_1 + x_2 \geq 1, \qquad -2x_1 + x_2 \leq 2, \qquad x_1 \geq 0, \qquad x_2 \geq 0$$



**example** $(n = 3)$

$$0 \le x_1 \le 2, \qquad 0 \le x_2 \le 2, \qquad 0 \le x_3 \le 2, \qquad x_1 + x_2 + x_3 \le 5$$



# Extreme points

let $\hat{x} \in \mathcal{P}$, where $\mathcal{P}$ is a polyhedron defined by inequalities $a_k^T x \leq b_k$

- if $a_k^T \hat{x} = b_k$, we say the $k$th inequality is **active** at $\hat{x}$
- if $a_k^T \hat{x} < b_k$, we say the $k$th inequality is **inactive** at $\hat{x}$

$\hat{x}$ is called an **extreme point** of $\mathcal{P}$ if

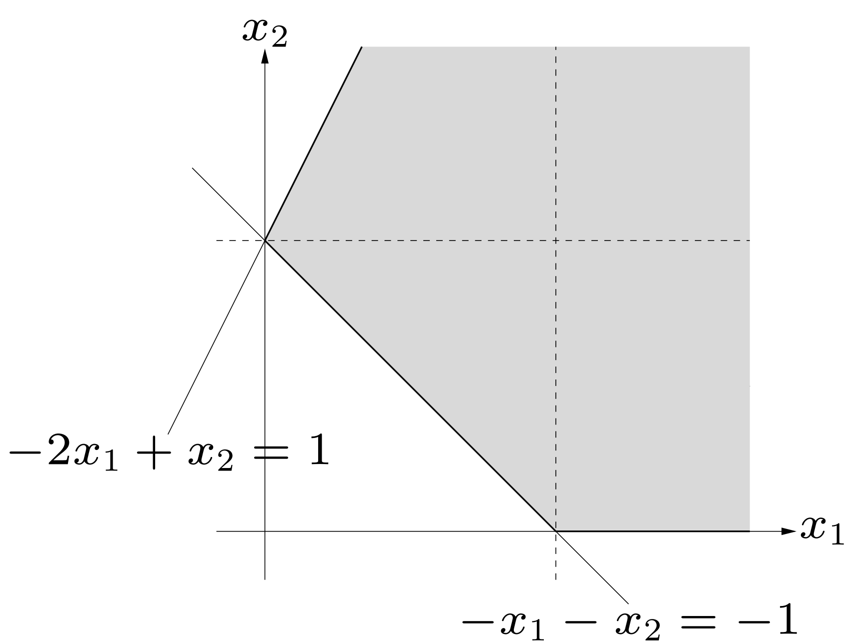
$$A_{I(\hat{x})} = \begin{bmatrix} a_{k_1}^T \\ a_{k_2}^T \\ \vdots \\ a_{k_p}^T \end{bmatrix} \text{ has a zero nullspace}$$

where $I(\hat{x}) = \{k_1, \ldots, k_p\}$ are the indices of the active constraints at $\hat{x}$

an extreme point $\hat{x}$ is **nondegenerate** if $p = n$, **degenerate** if $p > n$

## example

$$-x_1 - x_2 \le -1, \qquad -2x_1 + x_2 \le 1, \qquad -x_1 \le 0, \qquad -x_2 \le 0$$




- $\hat{x} = (1, 0)$ is an extreme point

$$A_{I(\hat{x})} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

- $\hat{x} = (0, 1)$ is an extreme point

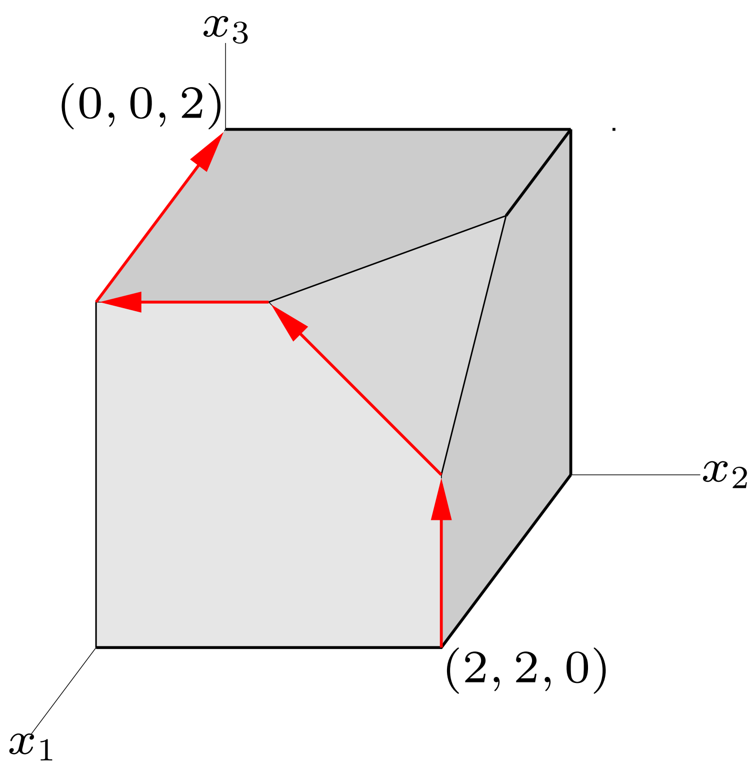
$$A_{I(\hat{x})} = \begin{bmatrix} -1 & -1 \\ -2 & 1 \\ -1 & 0 \end{bmatrix}$$

- $\hat{x} = (1/2, 1/2)$ is not an extreme point

$$A_{I(\hat{x})} = \begin{bmatrix} -1 & -1 \end{bmatrix}$$

# Simplex algorithm

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 - x_3 \\ \text{subject to} & 0 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq 2, \quad 0 \leq x_3 \leq 2 \\ & x_1 + x_2 + x_3 \leq 5 \end{array}$$




move from one extreme point to another extreme point with lower cost

# One iteration of the simplex method

suppose $\hat{x}$ is a nondegenerate extreme point

renumber constraints so that active constraints are $1, \ldots, n$

**active constraints** at $\hat{x}$:

$$a_1^T \hat{x} = b_1, \qquad \ldots, \qquad a_n^T \hat{x} = b_n$$

**inactive constraints** at $\hat{x}$:

$$a_{n+1}^T \hat{x} < b_{n+1}, \qquad \ldots, \qquad a_m^T \hat{x} < b_m$$

**matrix of active constraints:** define $I = I(\hat{x}) = \{1, \ldots, n\}$

$$A_I = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \qquad \text{(an } n \times n \text{ nonsingular matrix)}$$

**step 1 (test for optimality)** solve

$$A_I^T z = c$$

*i.e.*, find $z$ that satisfies

$$\sum_{k=1}^{n} z_k a_k = c$$

if $z_k \leq 0$ for all $k$, then $\hat{x}$ is optimal and we return $\hat{x}$

proof. consider any feasible $x$:

$$c^T x = \sum_{k=1}^{n} z_k a_k^T x \geq \sum_{k=1}^{n} z_k b_k = \sum_{k=1}^{n} z_k a_k^T \hat{x} = c^T \hat{x}$$

(the inequality follows from $z_k \leq 0$, $a_k^T x \leq b_k$)

**step 2 (compute step)** choose a $j$ with $z_j > 0$ and solve

$$A_I v = -e_j$$

$i.e.$, find $v$ that satisfies

$$a_j^T v = -1, \qquad a_k^T v = 0 \quad \text{for } k = 1, \ldots, n \text{ and } k \neq j$$

for small $t > 0$, $\hat{x} + tv$ is feasible and has a smaller cost than $\hat{x}$

proof:

$$\begin{aligned} a_j^T(\hat{x} + tv) &= b_j - t \\ a_k^T(\hat{x} + tv) &= b_k \quad \text{for } k = 1, \ldots, n \text{ and } k \neq j \end{aligned}$$

and for sufficiently small positive $t$

$$a_k^T(\hat{x} + tv) \leq b_k \quad \text{for } k = n+1, \ldots, m$$

finally, $c^T(\hat{x} + tv) = c^T\hat{x} + tz_j < c^T\hat{x}$

**step 3 (compute step size)** maximum $t$ such that $\hat{x} + tv$ is feasible, $i.e.$,

$$a_k^T\hat{x} + ta_k^Tv \leq b_k, \quad k = 1, \dots, m$$

the maximum step size is

$$t_{\max} = \min_{k: a_k^Tv > 0} \frac{b_k - a_k^T\hat{x}}{a_k^Tv}$$

(if $a_k^Tv \leq 0$ for all $k$, then the problem is unbounded below)

**step 4 (update $\hat{x}$)**

$$\hat{x} := \hat{x} + t_{\max}v$$

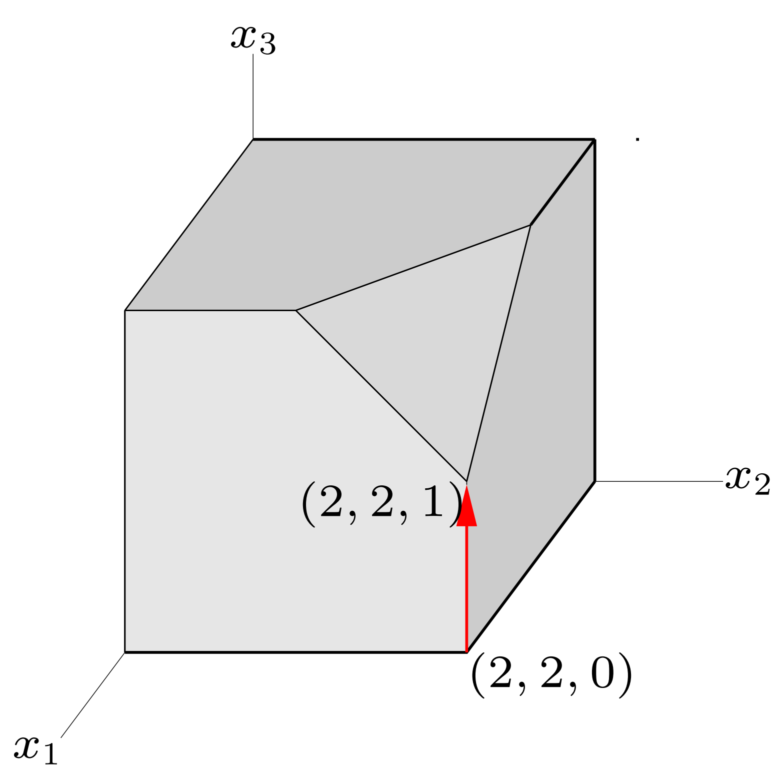
# Example

| | |
|---|---|
| minimize | $x_1 + x_2 - x_3$ |
| subject to | $0 \le x_1 \le 2, \quad 0 \le x_2 \le 2, \quad 0 \le x_3 \le 2$ |
| | $x_1 + x_2 + x_3 \le 5$ |

at $\hat{x} = (2, 2, 0)$

$$A_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

1. $z = A_I^{-T} c = (1, 1, 1)$
2. for $j = 3$, $v = A_I^{-1}(0, 0, -1) = (0, 0, 1)$
3. $\hat{x} + tv$ is feasible for $t \le 1$



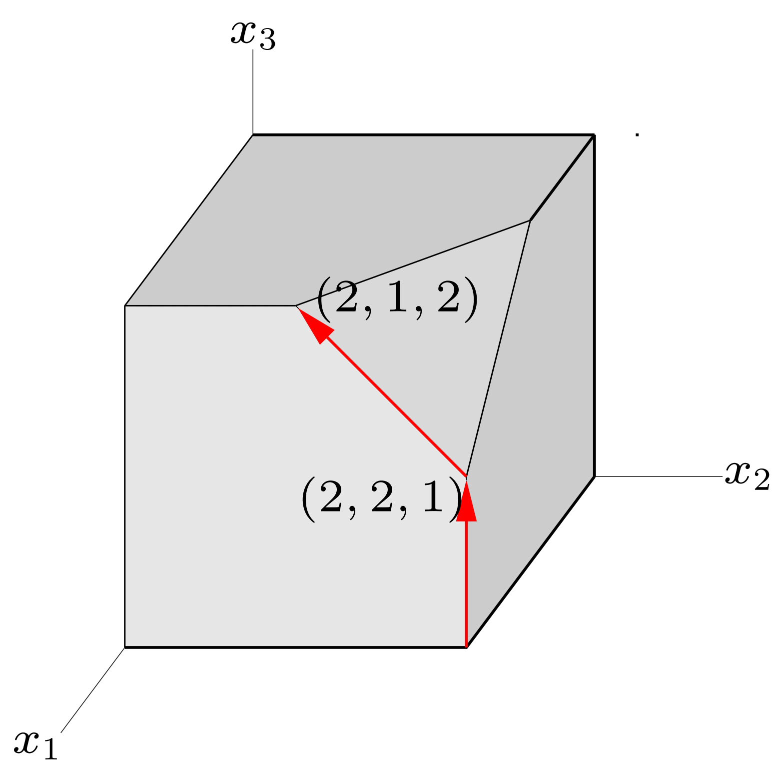
# Example

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 - x_3 \\ \text{subject to} & 0 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq 2, \quad 0 \leq x_3 \leq 2 \\ & x_1 + x_2 + x_3 \leq 5 \end{array}$$

at $\hat{x} = (2, 2, 1)$

$$A_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

1. $z = A_I^{-T} c = (2, 2, -1)$
2. for $j = 2$, $v = A_I^{-1}(0, -1, 0) = (0, -1, 1)$
3. $\hat{x} + tv$ is feasible for $t \leq 1$



# Example

$$
\begin{array}{ll}
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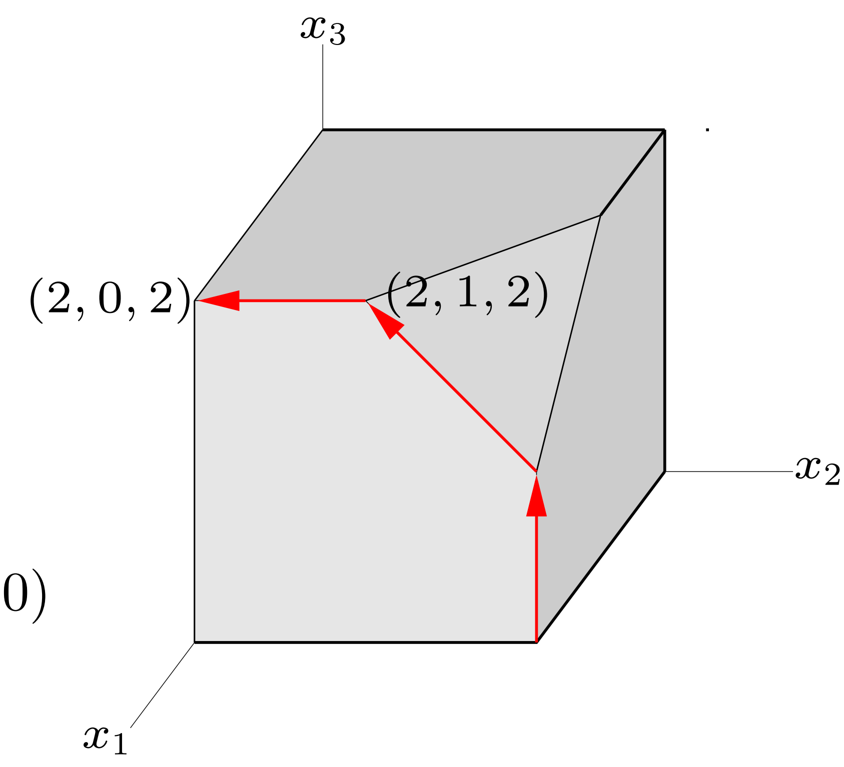
\text{subject to} & 0 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq 2, \quad 0 \leq x_3 \leq 2 \\
& x_1 + x_2 + x_3 \leq 5
\end{array}
$$

at $\hat{x} = (2, 1, 2)$

$$
A_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}
$$

1. $z = A_I^{-T} c = (0, -2, 1)$
2. for $j = 3$, $v = A_I^{-1}(-1, 0, 0) = (0, -1, 0)$
3. $\hat{x} + tv$ is feasible for $t \leq 1$



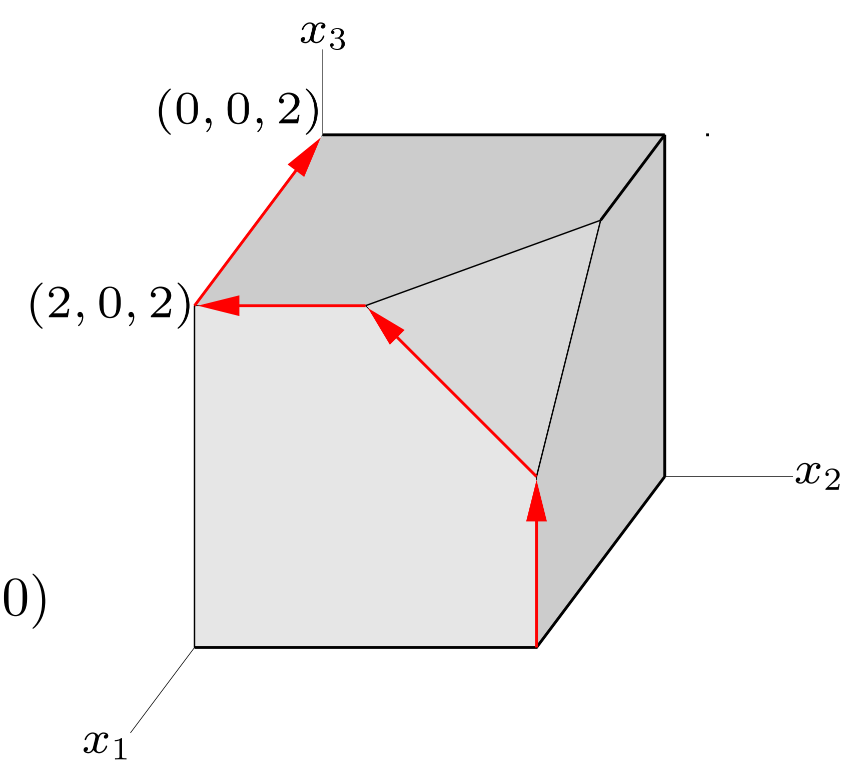
# Example

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 - x_3 \\ \text{subject to} & 0 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq 2, \quad 0 \leq x_3 \leq 2 \\ & x_1 + x_2 + x_3 \leq 5 \end{array}$$

at $\hat{x} = (2, 0, 2)$

$$A_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. $z = A_I^{-T} c = (1, -1, -1)$
2. for $j = 1$, $v = A_I^{-1}(-1, 0, 0) = (-1, 0, 0)$
3. $\hat{x} + tv$ is feasible for $t \leq 2$



# Example

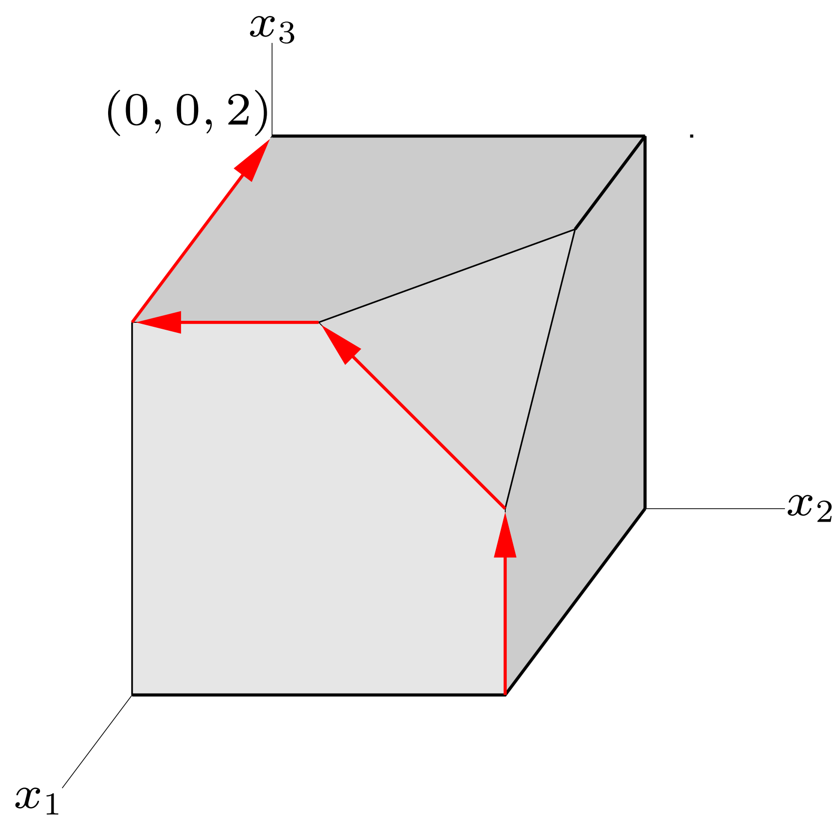
$$\begin{array}{ll} \text{minimize} & x_1 + x_2 - x_3 \\ \text{subject to} & 0 \le x_1 \le 2, \quad 0 \le x_2 \le 2, \quad 0 \le x_3 \le 2 \\ & x_1 + x_2 + x_3 \le 5 \end{array}$$

at $\hat{x} = (0, 0, 2)$

$$A_I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$z = A_I^{-T} c = (-1, -1, -1)$

therefore, $\hat{x}$ is optimal



# Practical aspects

- at each iteration, we solve two sets of linear equations

$$A_I^T z = c, \qquad A_I v = -e_j$$

  using an LU factorization or sparse LU factorization of $A_I$

- implementation requires a 'phase 1' to find first extreme point
- simple modifications handle problems with degenerate extreme points
- very large LPs (several 100,000 variables and constraints) are routinely solved in practice