17. Ordinary differential equations

- initial value problem
- examples
- forward and backward Euler method

Initial value problem

first-order ordinary differential equation (ODE)

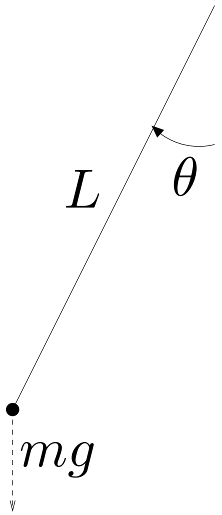
$$\frac{dx(t)}{dt} = f(x(t), t), \qquad x(0) = x_0$$

- t usually represents time
- f is a given function from $\mathbf{R}^n \times \mathbf{R}$ to $\mathbf{R}$
- x_0 is a given n-vector (the initial condition)
- $x(t) = (x_1(t), \ldots, x_n(t))$ is an unknown function from $\mathbf{R}$ to $\mathbf{R}^n$;

$$\frac{dx(t)}{dt} = \begin{bmatrix} dx_1(t)/dt \\ \vdots \\ dx_n(t)/dt \end{bmatrix}$$

other notation: $\dot{x}(t)$

Swinging pendulum



$$mL\ddot{\theta}(t) + cL\dot{\theta}(t) + mg\sin\theta(t) = 0$$

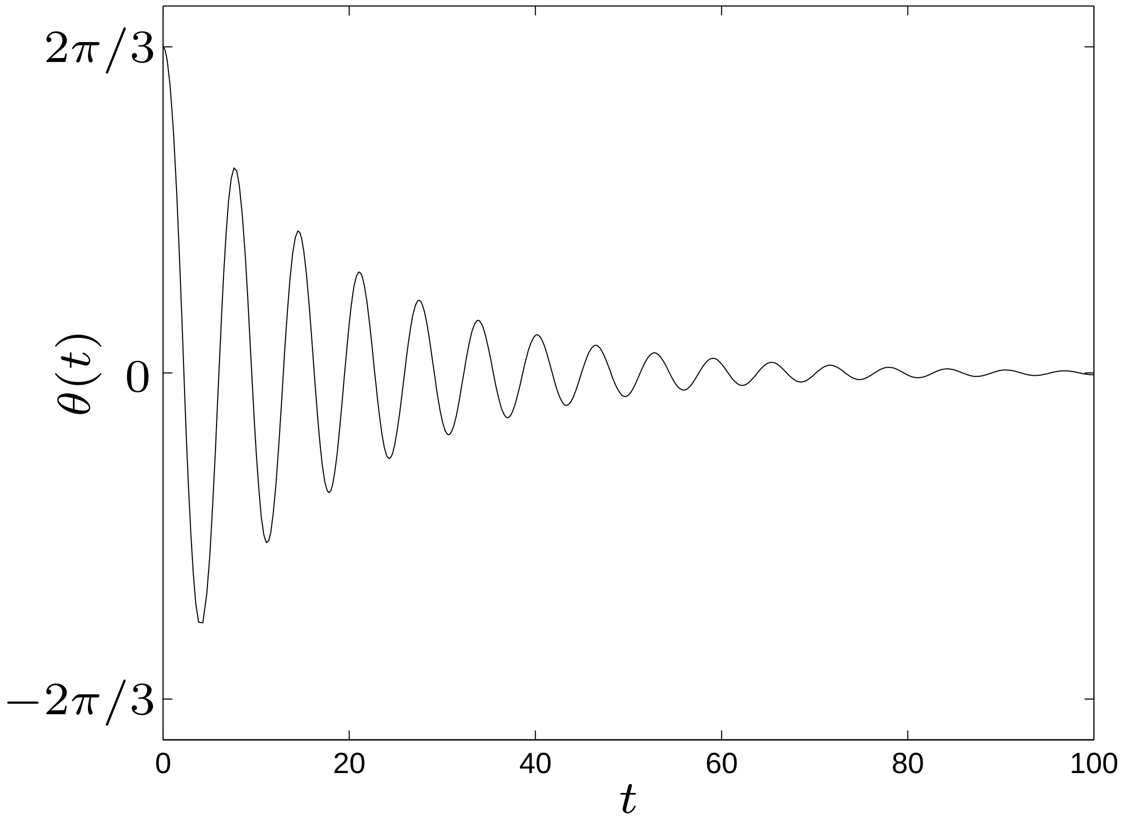
- mass m attached to a pendulum of mass zero and length L
- c is friction coefficient; g is acceleration due to gravity

equivalent to a 1st-order ODE with $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$:

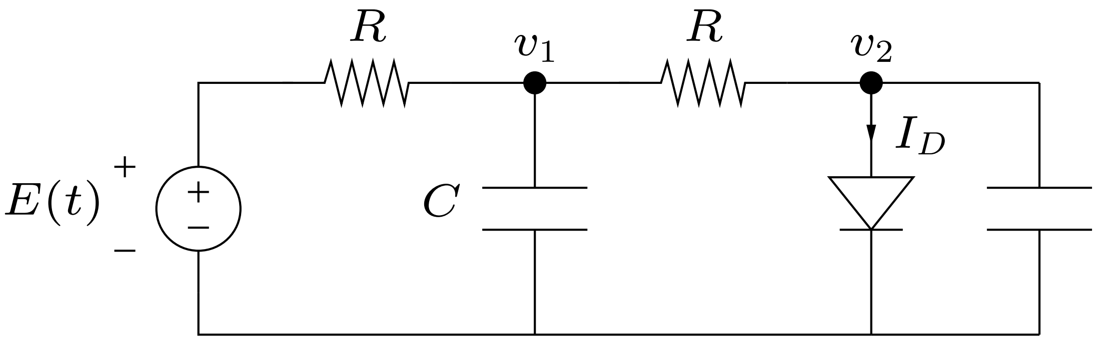
$$\frac{d}{dt}\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] = \left[\begin{array}{c} x_2(t) \\ -(c/m)x_2(t) - (g/L)\sin x_1(t) \end{array}\right]$$

example

- $c/m = 0.1$, $g/L = 1$
- initial condition $\theta(0) = 2\pi/3$, $\dot{\theta}(0) = 0$



Nonlinear circuit simulation

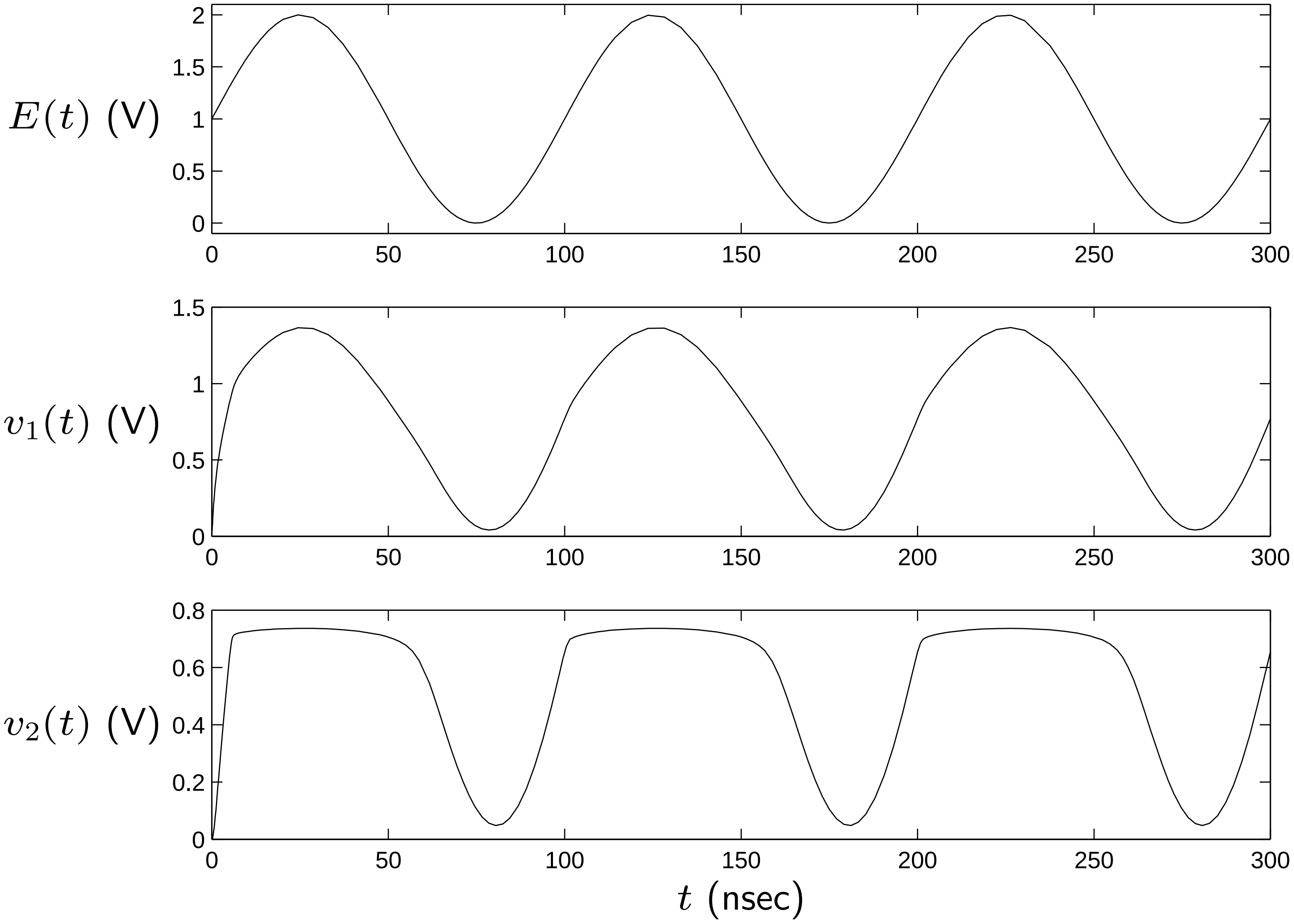


$$\begin{aligned} C\frac{dv_1(t)}{dt} &= -\frac{1}{R}(v_1(t) - E(t)) - \frac{1}{R}(v_1(t) - v_2(t)) \\ C\frac{dv_2(t)}{dt} &= -\frac{1}{R}(v_2(t) - v_1(t)) - I_D(v_2(t)) \end{aligned}$$

where $I_D(v_2(t)) = I_S\ (\exp(\frac{v_2(t)}{V_T}) - 1)$

problem: determine $v_1(t)$, $v_2(t)$ given $E(t)$, $v_1(0)$, $v_2(0)$

example $(R = 2\text{k}\Omega,\ C = 1\text{pF},\ I_S = 0.5 \cdot 10^{-13}\text{mA},\ V_T = 0.025\text{V})$



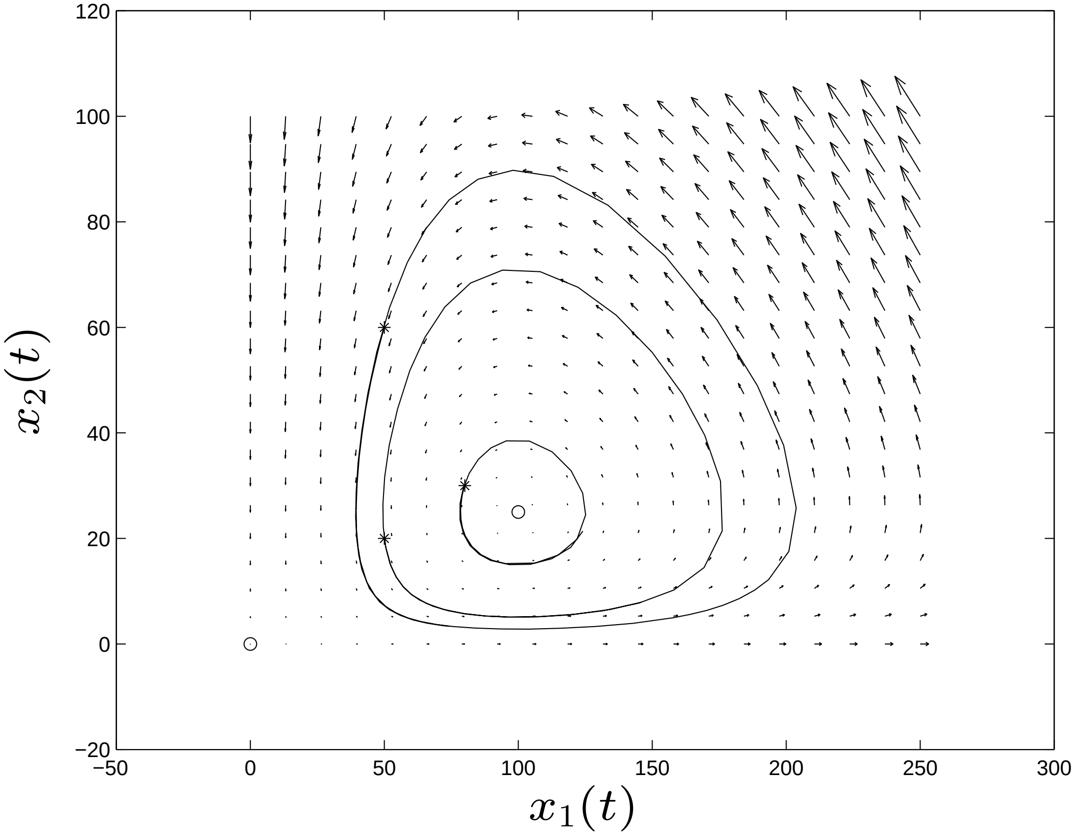
Predator-prey system

$$\begin{aligned}\dot{x}_1(t) &= \alpha x_1(t) - \beta x_1(t)x_2(t) \\ \dot{x}_2(t) &= -\gamma x_2(t) + \delta x_1(t)x_2(t)\end{aligned}$$

α, β, γ, δ are positive constants

- simple model for population of two species (predator and prey)
- x_1 is population of prey; x_2 is population of predator
- terms $x_1(t)x_2(t)$ model frequency of encounters

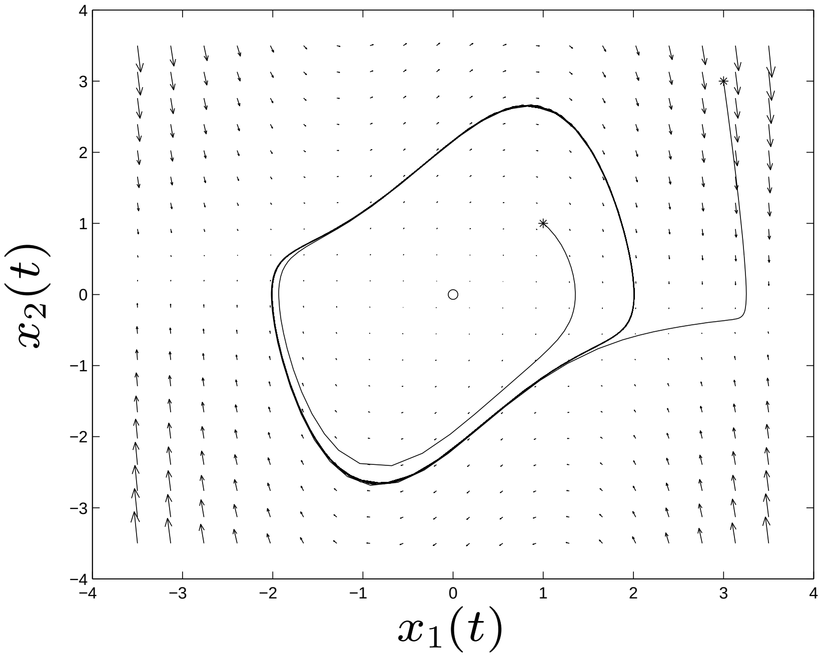
example $(\alpha = 0.25,\ \beta = 0.01,\ \gamma = 1,\ \delta = 0.01)$



- equilibrium states at $x_1 = x_2 = 0$ and at $x_1 = 100$, $x_2 = 25$
- trajectories from other starting points are periodic

Van der Pol oscillator

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_1(t) - \left(x_1(t)^2 - 1\right) x_2(t)\end{aligned}$$



- trajectories converge to a limit cycle
- discovered in 1920s (in electrical circuits with vacuum tubes)

Numerical algorithms for ODEs

$$\frac{dx(t)}{dt} = f(x(t), t), \qquad x(0) = x_0$$

- algorithms compute approximations $\hat{x}_k$ of $x(t_k)$ at discrete times

$$0 = t_0 < t_1 < t_2 < \cdots < t_N$$

- can be interpreted as (approximate) integration methods for

$$x(t_k) = x(t_{k-1}) + \int_{t_{k-1}}^{t_k} f(x(\tau), \tau)\, d\tau$$

we'll discuss two simple methods; methods used in practice are much more complicated

Forward and backward Euler method

forward Euler method

start at $\hat{x}_0 = x_0$; for $k = 1, \ldots, N$,

$$\hat{x}_k := \hat{x}_{k-1} + h_k f(\hat{x}_{k-1}, t_{k-1}) \qquad (h_k = t_k - t_{k-1})$$

backward Euler method

start at $\hat{x}_0 = x_0$; for $k = 1, \ldots, N$, solve $\hat{x}_k$ from the nonlinear equation

$$\hat{x}_k = \hat{x}_{k-1} + h_k f(\hat{x}_k, t_k) \qquad (h_k = t_k - t_{k-1})$$

(for example, using Newton's method)

Choice of step size

$$\frac{dx(t)}{dt} = -ax(t), \quad x(0) = 1 \qquad (a > 0)$$

solution: $x(t) = \exp(-at)$

forward Euler with fixed step size $h_k = h$

$$\hat{x}_k = (1 - ah)^k, \quad k = 0, \ldots, N$$

$\hat{x}_k \to 0$ only if $h < 2/a$

backward Euler with fixed step size $h_k = h$

$$\hat{x}_k = \hat{x}_k = (1 + ah)^{-k}, \quad k = 0, \ldots, N$$

$\hat{x}_k \to 0$ for all choices of h; can choose h based on desired accuracy

Example

$$\frac{dx(t)}{dt} = -a(x(t) - \sin t), \qquad x(0) = x_0$$

solution

$$x(t) = (x_0 + \frac{a}{1+a^2})e^{-at} - \frac{a}{1+a^2}(\cos t - a \sin t)$$

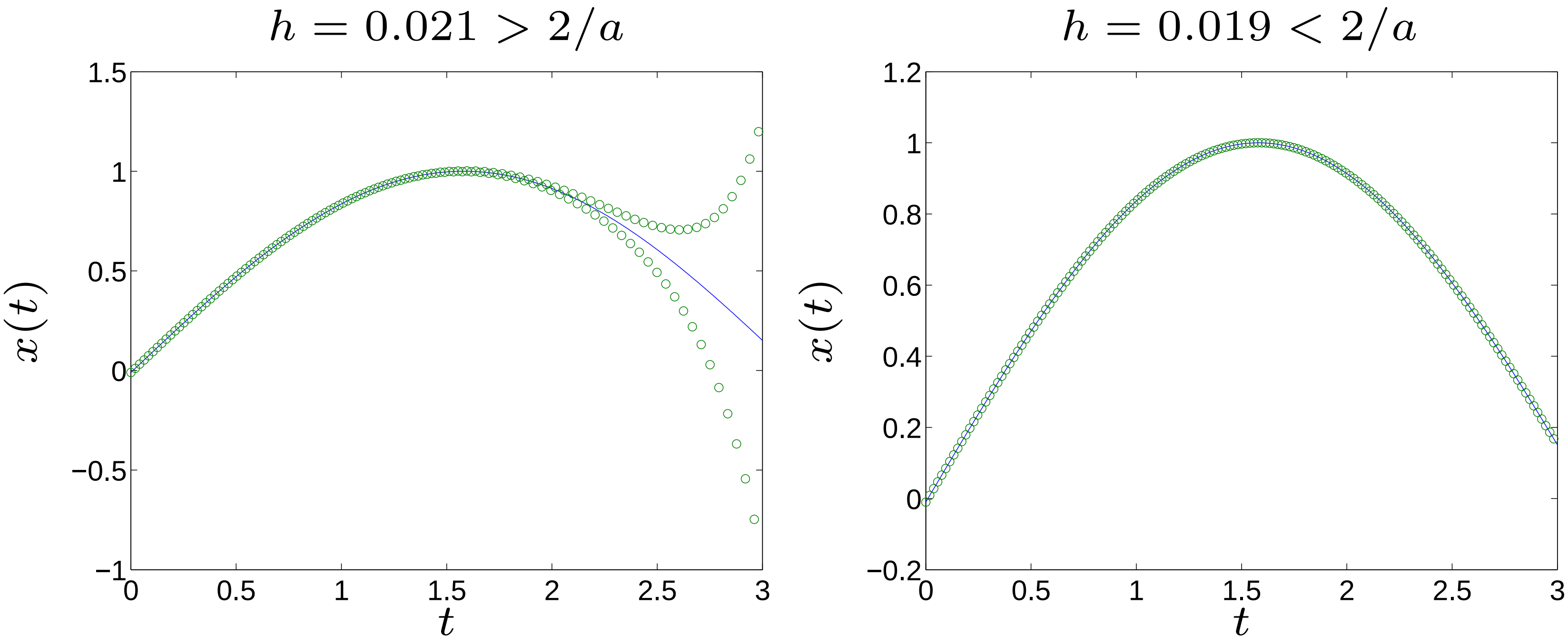
forward Euler with constant step size h

$$\hat{x}_k = (1 - ha)\hat{x}_{k-1} + ha \sin t_{k-1}, \quad k = 1, 2, \ldots$$

backward Euler with constant step size h

$$\hat{x}_k = \frac{1}{1+ah}\left(\hat{x}_{k-1} + ha \sin t_k\right), \quad k = 1, 2, \ldots$$

results with forward Euler for $a = 100$, $x_0 = -a/(1+a^2)$

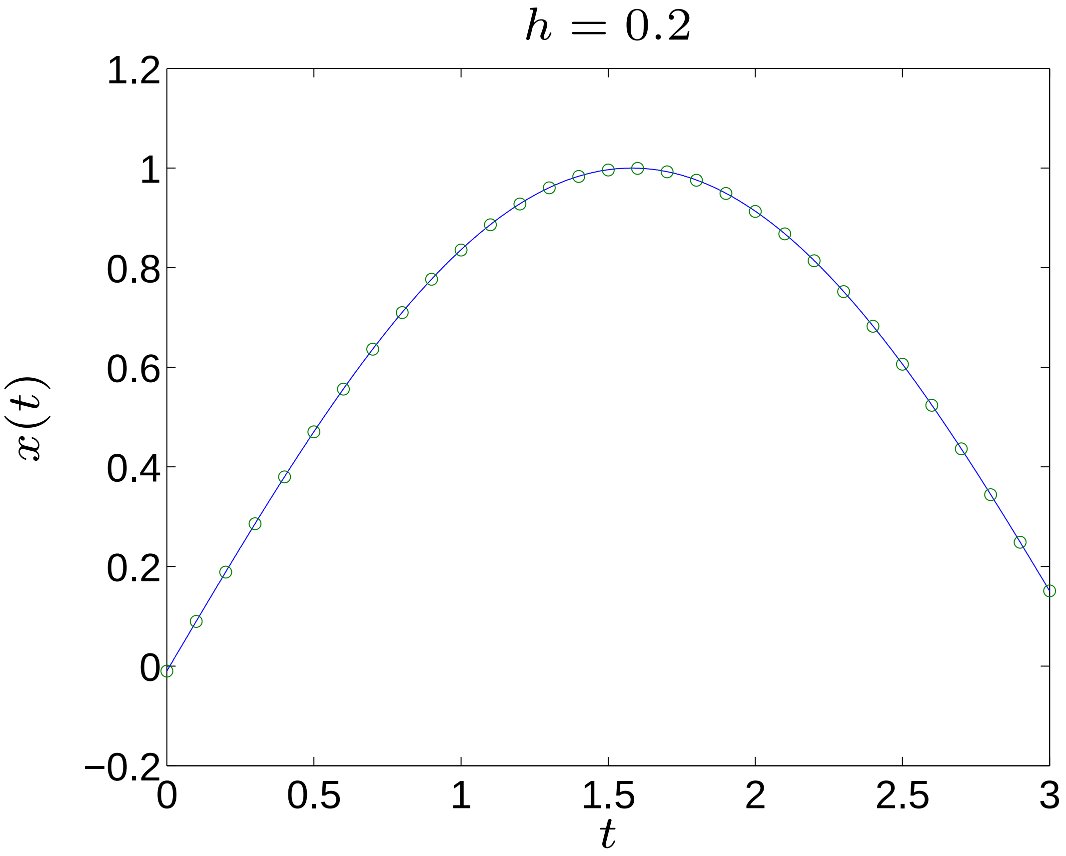


- an example of a 'stiff' ODE: solution contains components on very different time scales

- fast component (e^{-at}) requires small step sizes to maintain stability

- step size is much smaller than needed to track the slow component $(\cos t - a \sin t)$

results with backward Euler for $a = 100$, $x_0 = -a/(1+a^2)$



more efficient, step size determined by desired accuracy