Homework assignment #1

Let $f(x) = (1/2)x^T Ax + b^T x + c$ be a convex quadratic function with $0 \prec A \preceq LI$. Show that the 'heavy-ball method'

$$x^{(k)} = x^{(k-1)} - t\nabla f(x^{(k-1)}) + s(x^{(k-1)} - x^{(k-2)}), \qquad k \geq 2,$$

converges from any starting point $x^{(0)}$, $x^{(1)}$ if

$$0 \leq s < 1, \qquad 0 < t < \frac{2(1+s)}{L}.$$

To show this, write the iteration as a linear recursion

$$z(k) = Mz(k-1) + q, \qquad k \geq 2,$$

where

$$z(k) = \begin{bmatrix} x^{(k)} \\ x^{(k-1)} \end{bmatrix}, \qquad M = \begin{bmatrix} (1+s)I - tA & -sI \\ I & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} -tb \\ 0 \end{bmatrix}.$$

Then show that the eigenvalues of M have magnitude less than one. This implies that $z(k)$ converges to the equilibrium point $z^\star = (I - M)^{-1}q = (-A^{-1}b, -A^{-1}b)$.