L. Vandenberghe EE236C Spring 2009

Homework assignment #4

Minimum-volume ellipsoid covering a half-ellipsoid. In this problem we derive the updating formulas used in the ellipsoid method (lecture 10, page 17), *i.e.*, we will determine the minimum volume ellipsoid containing the intersection of the ellipsoid

$$\mathcal{E} = \{x \in \mathbf{R}^n \mid (x - x_c)P^{-1}(x - x_c) \leq 1\}$$

and the halfspace

$$\mathcal{H} = \{x \mid g^T(x - x_c) \leq 0\}.$$

We'll assume that $n > 1$, since for $n = 1$ the problem is easy.

1. We first consider a special case: $\mathcal{E}$ is a ball centered at the origin ($P = I$, $x_c = 0$), and $g = -e_1$ (e_1 the first unit vector), so $\mathcal{E} \cap \mathcal{H} = \{x \mid x^T x \leq 1,\ x_1 \geq 0\}$.

   Let
   $$\tilde{\mathcal{E}} = \{x \mid (x - \tilde{x}_c)^T \tilde{P}^{-1}(x - \tilde{x}_c) \leq 1\}$$
   denote the minimum volume ellipsoid containing $\mathcal{E} \cap \mathcal{H}$. Since $\mathcal{E} \cap \mathcal{H}$ is symmetric about the line through the first unit vector e_1, it is clear (and not too hard to show) that $\tilde{\mathcal{E}}$ will have the same symmetry. This means that the matrix $\tilde{A}$ is diagonal, of the form $\tilde{A} = \mathbf{diag}(\alpha, \beta, \beta, \ldots, \beta)$, and that $x_c = \gamma e_1$.

   So now we have only three variables to determine: α, β, and γ. Express the volume of $\tilde{\mathcal{E}}$ in terms of these variables, and also the constraint that $\tilde{\mathcal{E}} \supseteq \mathcal{E} \cap \mathcal{H}$. Then solve the optimization problem directly, to show that
   $$\alpha = \frac{n^2}{(n+1)^2}, \qquad \beta = \frac{n^2}{n^2 - 1}, \qquad \gamma = \frac{1}{n+1}$$
   (which agrees with the formulas we gave, for this special case).

2. Now consider the general case, stated at the beginning of this problem. Show how to reduce the general case to the special case solved in part (a).

   Hint. Find an affine transformation that maps the original ellipsoid to the unit ball, and g to $-e_1$. Explain why minimizing the volume in these transformed coordinates also minimizes the volume in the original coordinates.