15. Saddle-point problems

- definition
- convex-concave games
- primal-dual decomposition

Min-max inequality

the inequalities

$$\inf_{x\in X} f(x,\hat{y}) \leq f(\hat{x},\hat{y}) \leq \sup_{y\in Y} f(\hat{x},y)$$

hold for *any* function f, *any* sets X, Y, *any* point $(\hat{x},\hat{y}) \in X \times Y$

as a consequence, the min-max inequality

$$\sup_{y\in Y} \inf_{x\in X} f(x,y) \leq \inf_{x\in X} \sup_{y\in Y} f(x,y)$$

holds without exception

Saddle-point

$(x^\star, y^\star) \in X \times Y$ is a saddle-point of f on $X \times Y$ if

$$\inf_{x \in X} f(x, y^\star) = f(x^\star, y^\star) = \sup_{y \in Y} f(x^\star, y)$$

min-max equality: if f has a saddle-point $(x^\star, y^\star)$, then

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y)$$

and this quantity is equal to the saddle-point value $f(x^\star, y^\star)$

Lagrangian duality

an important example is the Lagrangian of an optimization problem

$$f(x, y) = f_0(x) + \sum_{i=1}^{m} y_i f_i(x), \qquad X = \bigcap_{i=0,\ldots,m} \mathbf{dom}\, f_i, \qquad Y = \mathbf{R}_+^m$$

- supremum of f over y is the primal objective:

$$\sup_{y \in Y} f(x, y) = \begin{cases} f_0(x) & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ +\infty & \text{otherwise} \end{cases}$$

- infimum of f over x is the dual objective
- min-max inequality corresponds to weak duality

Zero-sum game

f is payoff function of zero-sum two-player game

- player 1 chooses a strategy $x \in X$; player 2 chooses a strategy $y \in Y$
- players pick strategies without knowledge of opponent's choice
- if $f(x,y) > 0$, P1 pays $f(x,y)$ to P2; otherwise, P2 pays $-f(x,y)$ to P1

min-max inequality

- if $y = \hat{y}$ is known to P1, P1 selects $x = \operatorname{argmin}_{x \in X} f(x,y)$
- if $x = \hat{x}$ is known to P2, P2 selects $y = \operatorname{argmax}_{y \in y} f(x,y)$
- the min-max inequality

$$\inf_{x \in X} f(x, \hat{y}) \leq \sup_{y \in Y} f(\hat{x}, y) \qquad \forall (\hat{x}, \hat{y}) \in X \times Y$$

  states the obvious fact that it helps to know your opponent's strategy

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saddle-point

if opponent's choice is unknown, players minimize worst-case loss by taking

$$x = \operatorname*{argmin}_{x \in X} \sup_{y \in Y} f(x,y), \qquad y = \operatorname*{argmax}_{y \in Y} \inf_{x \in X} f(x,y)$$

- if a saddle-point $(x^\star, y^\star)$ exists, we have

$$\inf_{x \in X} f(x, y^\star) = f(x^\star, y^\star) = \sup_{y \in Y} f(x^\star, y)$$

- in this case, there is no advantage in knowing opponent's strategy

Matrix game

$$X = \{1, \ldots, m\}, \qquad Y = \{1, \ldots, n\}, \qquad f(x, y) = A_{xy}$$

$A \in \mathbf{R}^{m \times n}$ is the *payoff matrix*

example

$$A = \begin{bmatrix} -2 & 2 & 0 \\ -3 & 4 & 6 \end{bmatrix}$$

has a saddle-point

$$\max_{y \in Y} \min_{x \in X} f(x, y) = f(1, 2) = \min_{x \in X} \max_{y \in Y} f(x, y)$$

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example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ 0 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

- worst-case choice of P1 is $x = 2$
- worst-case choice of P2 is $y = 1$
- this is not a saddle-point

$$-2 = \max_{y \in Y} \min_{x \in X} f(x, y) < \min_{x \in X} \max_{y \in Y} f(x, y) = 3$$

Convex-concave saddle-point problems

a saddle-point problem is **convex-concave** if

- X and Y are convex sets
- $f(x,y)$ is concave in y for fixed $x \in X$
- $f(x,y)$ is convex in x for fixed $y \in Y$

under weak conditions, saddle-points of convex-concave problems exist

an example of sufficient conditions:

- X and Y are closed and bounded
- $X \times Y \subseteq \mathbf{dom}\, f$ and f is continuous

(see BV exercise 5.25)

Lagrangian duality for convex problems

$$f(x,y) = f_0(x) + \sum_{i=1}^{m} y_i f_i(x), \qquad X = \bigcap_{i=0,\ldots,m} \mathbf{dom}\, f_i, \qquad Y = \mathbf{R}_+^m$$

- f is convex in x for fixed $y \in Y$, concave in y for fixed $x \in X$
- min-max equality corresponds to strong duality
- saddle-point exists if strong duality holds, and primal and dual optimal values are attained

Matrix game with mixed (randomized) strategies

- a mixed strategy for P1 is a probability distribution on $\{1, \ldots, m\}$

$$X = \{x \mid \mathbf{1}^T x = 1,\ x \succeq 0\}$$

- a mixed strategy for P2 is a probability distribution on $\{1, \ldots, n\}$

$$Y = \{y \mid \mathbf{1}^T y = 1,\ y \succeq 0\}$$

- define f as the expected payoff by P1 to P2 is

$$f(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

Optimal mixed strategies

optimal strategy for player 1:

$$\begin{array}{ll} \text{minimize}_x & \sup_{\mathbf{1}^T y = 1,\, y \succeq 0} x^T A y \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

note:

$$\sup_{\mathbf{1}^T y = 1,\, y \succeq 0} x^T A y = \max_{j=1,\ldots,n} (A^T x)_j$$

optimal strategy $x^\star$ can be computed by solving an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \preceq t\mathbf{1} \\ & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array} \qquad (1)$$

(variables x, t)

optimal strategy for player 2:

$$\begin{array}{ll} \text{maximize}_y & \inf_{\mathbf{1}^T x=1,\, x\succeq 0} x^T A y \\ \text{subject to} & \mathbf{1}^T y = 1, \quad y \succeq 0 \end{array}$$

note:

$$\inf_{\mathbf{1}^T x=1,\, x\succeq 0} x^T A y = \min_{i=1,\ldots,m} (Ay)_i$$

optimal strategy $y^\star$ can be computed by solving an LP

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \succeq w\mathbf{1} \\ & \mathbf{1}^T y = 1, \quad y \succeq 0 \end{array} \qquad (2)$$

(variables y, w)

Min-max theorem for matrix games

$$\sup_{y\in Y} x^{\star T} A y = x^{\star T} A y^\star = \inf_{x\in X} x^T A y^\star$$

proof: the LPs (1) and (2) are duals, so they have the same optimal value

example

$$A = \left[\begin{array}{rrrr} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{array}\right]$$

optimal strategies

$$x^\star = (0.37, 0.33, 0.3), \qquad y^\star = (0.4, 0, 0.13, 0.47)$$

expected payoff: $x^{\star T} A y^\star = 0.2$

Primal-dual decomposition

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{r} c_j^T x_j + d^T y \\ \text{subject to} & A_j x_j + B_j y \preceq b_j, \quad j = 1, \ldots, r \\ & \sum_{j=1}^{r} G_j x_j + H y \preceq h \end{array}$$

- y is a complicating variable
- last constraint is a complicating constraint
- without complicating variable and constraint, the problem decomposes into r independent problems
- for simplicity, we consider only linear objective and constraints

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special case: no complicating constraint

- define $f_j(y)$ as optimal value of subproblem (with variable x_j)

$$\begin{array}{ll} \text{minimize} & c_j^T x_j \\ \text{subject to} & A_j x_j + B_j y \preceq b_j \end{array}$$

  this is a convex function of y; in general nondifferentiable

- define master problem

$$\text{minimize} \quad d^T y + \sum_{j=1}^{r} f_j(y)$$

  master problem is equivalent to the original problem

this is known as *primal decomposition*

special case: no complicating variables

- define $g_j(\lambda)$ as optimal value of subproblem (with variable x_j)

$$\begin{array}{ll} \text{minimize} & (c_j + G_j^T \lambda_j)^T x_j \\ \text{subject to} & A_j x_j \preceq b_j \end{array}$$

  this is a concave function of λ; in general nondifferentiable

- define master problem

$$\begin{array}{ll} \text{maximize} & -h^T \lambda + \sum_{j=1}^{r} g_j(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

  master problem is equivalent to the original dual problem

this is known as *dual decomposition*

Lagrangian for problem with complicating variables and constraints

$$L(x, y, \lambda, \mu) = L_0(y, \mu) + \sum_{j=1}^{r} L_j(x_j, y, \lambda_j, \mu)$$

where

$$\begin{array}{rcl} L_0(y, \mu) & = & (d + H^T \mu)^T y - h^T \mu \\ L_j(x_j, y, \lambda_j, \mu) & = & \left(c_j + A_j^T \lambda_j + G_j^T \mu\right)^T x_j + \lambda_j^T B_j y - b_j^T \lambda_j \end{array}$$

master problem

$y^\star$, $\mu^\star$ is a saddle-point of

$$L_0(y,\mu) + \sum_{j=1}^{r} l_j(y,\mu)$$

where

$$\begin{array}{rcl} l_j(y,\mu) & = & \displaystyle\sup_{\lambda_j\succeq 0}\inf_{x_j} L_j(x_j,y,\lambda_j,\mu) \\ & = & \displaystyle\inf_{x_j}\sup_{\lambda_j\succeq 0} L_j(x_j,y,\lambda_j,\mu) \end{array}$$

- follows by repeated application of min-max equality
- $l_j(y,\mu)$ is convex in y for fixed μ, concave in μ for fixed y

subproblems

$l_j(y,\mu)$ is the optimal value of the LP

$$\begin{array}{ll} \text{minimize} & (c_j + G_j^T\mu)^T x_j \\ \text{subject to} & A_j x_j + B_j y \preceq b_j \end{array}$$

conclusion

- decomposition into a master problem and r subproblems
- subproblems are LPs that can be solved independently
- master problem is convex-concave saddle-point problem

References

- S. Boyd, course notes for EE364b, Convex Optimization II

  Notes on convex-concave games and minimax

- D.P. Bertsekas, A. Nedić, A.E. Ozdaglar, *Convex Analysis and Optimization* (2003)

  Section 2.6 for theorems of existence of saddle-points of convex-concave functions