

10. Ellipsoid method

- ellipsoid method
- convergence proof
- inequality constraints

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Ellipsoid method

history

- developed by Shor, Nemirovski, Yudin in 1970s
- used in 1979 by Khachian to show polynomial solvability of LPs

properties

- each step requires cutting-plane or subgradient evaluation
- modest storage ($O(n^2)$)
- modest computation per step ($O(n^2)$), via analytical formula
- extremely simple to implement
- efficient in theory
- slow but steady in practice; rarely used

Motivation

drawbacks of cutting-plane methods

- serious computation needed to find next query point
typically, $O(n^2m)$ for analytic centering in ACCPM, with m inequalities
- localization polyhedron grows in complexity as algorithm progresses
(with pruning, can keep m proportional to n , e.g., $m = 4n$)

ellipsoid method addresses both issues, but retains theoretical efficiency

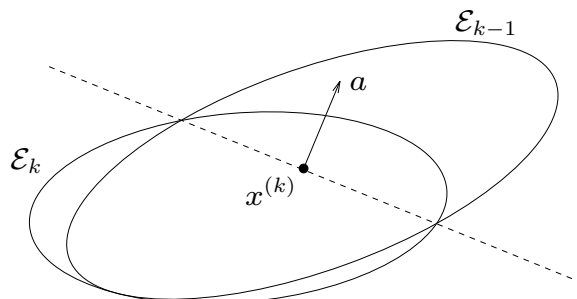
Ellipsoid algorithm for minimizing convex function

idea: localize x^* in an *ellipsoid* instead of a *polyhedron*

given an initial ellipsoid \mathcal{E}_0 known to contain X

repeat for $k = 1, 2, \dots$

1. query oracle to get a neutral cut $a^T z \leq b$ at $x^{(k)}$, the center of \mathcal{E}_{k-1}
2. set $\mathcal{E}_k :=$ minimum volume ellipsoid covering $\mathcal{E}_{k-1} \cap \{z \mid a^T z \leq b\}$



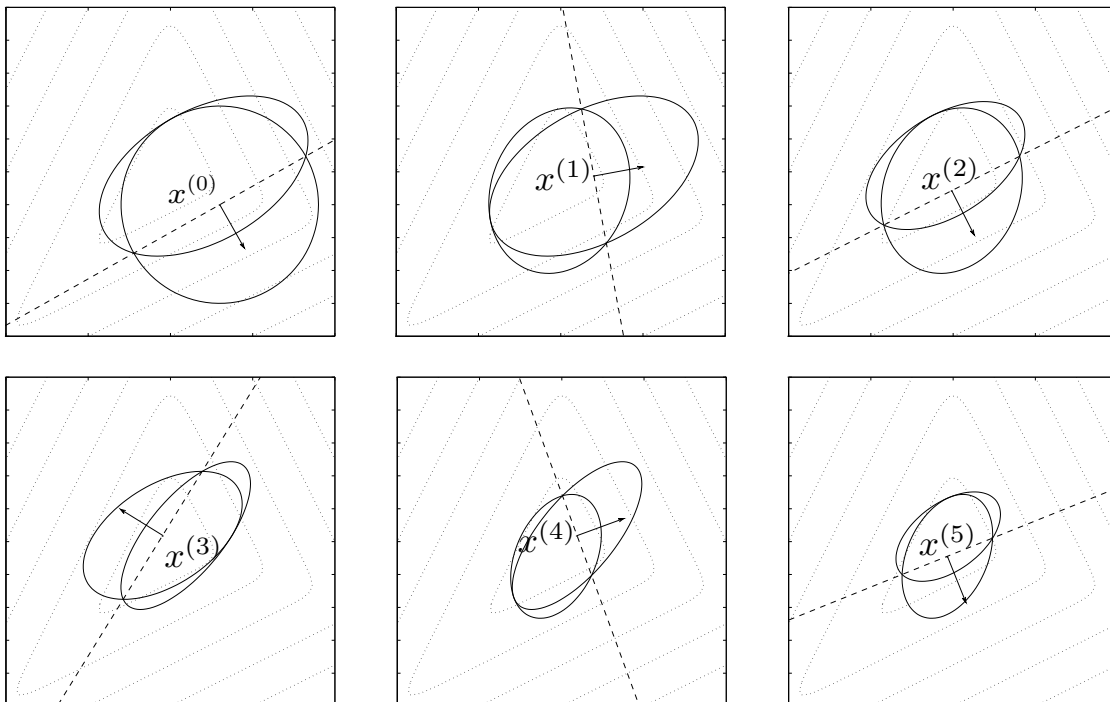
differences with cutting-plane methods

- localization set doesn't grow more complicated
- generating query point is trivial
- but, we add unnecessary points in step 2

interpretation

- reduces to bisection for $n = 1$
- can be viewed as an implementable version of the center-of-gravity cutting-plane method

Example

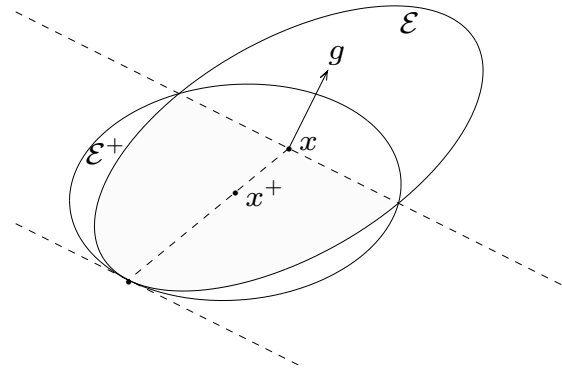


Updating the ellipsoid

$$\mathcal{E} = \{z \mid (z - x)^T P^{-1} (z - x) \leq 1\}$$

\mathcal{E}^+ is min. volume ellipsoid covering

$$\mathcal{E} \cap \{z \mid g^T (z - x) \leq 0\}$$



update formula (for $n > 1$): $\mathcal{E}^+ = \{z \mid (z - x^+)^T (P^+)^{-1} (z - x^+) \leq 1\}$,

$$x^+ = x - \frac{1}{n+1} P \tilde{g}, \quad P^+ = \frac{n^2}{n^2 - 1} \left(P - \frac{2}{n+1} P \tilde{g} \tilde{g}^T P \right)$$

where $\tilde{g} = (1/\sqrt{g^T P g}) g$

Simple stopping criterion

for unconstrained problem

$$\text{minimize } f(x)$$

lower bound on optimal value

$$\begin{aligned} f(x^*) &\geq f(x^{(k)}) + g^{(k)T} (x^* - x^{(k)}) \\ &\geq f(x^{(k)}) + \inf_{z \in \mathcal{E}_{k-1}} g^{(k)T} (z - x^{(k)}) \\ &= f(x^{(k)}) - \sqrt{g^{(k)T} P^{(k-1)} g^{(k)}} \end{aligned}$$

second inequality holds since $x^* \in \mathcal{E}_{k-1}$

simple stopping criterion to guarantee $f(x^{(k)}) - f(x^*) \leq \epsilon$:

$$\sqrt{g^{(k)T} P^{(k-1)} g^{(k)}} \leq \epsilon$$

Basic ellipsoid algorithm

ellipsoid described as $\mathcal{E}(x, P) = \{z \mid (z - x)^T P^{-1} (z - x) \leq 1\}$

given ellipsoid $\mathcal{E}(x, P)$ containing x^* , accuracy $\epsilon > 0$

repeat

1. evaluate $g \in \partial f(x)$
2. if $\sqrt{g^T P g} \leq \epsilon$, return x ; else, update ellipsoid

$$x := x - \frac{1}{n+1} P \tilde{g}, \quad P := \frac{n^2}{n^2 - 1} \left(P - \frac{2}{n+1} P \tilde{g} \tilde{g}^T P \right)$$

where $\tilde{g} = (1/\sqrt{g^T P g})g$

Interpretation

- change coordinates

$$\tilde{z} = P^{-1/2} z$$

so uncertainty is isotropic (same in all directions), *i.e.*, \mathcal{E} is unit ball

- take subgradient step with fixed length $1/(n+1)$

Shor calls ellipsoid method 'gradient method with space dilation in direction of gradient' (which, strangely enough, didn't catch on)

Improvements

- keep track of best upper and lower bounds:

$$f_{\text{best}}^{(k)} = \min_{i=1,\dots,k} f(x^{(i)})$$
$$l_{\text{best}}^{(k)} = \max_{i=1,\dots,k} \left(f(x^{(i)}) - \sqrt{g^{(i)T} P^{(i-1)} g^{(i)}} \right)$$

stop when $f_{\text{best}}^{(k)} - l_{\text{best}}^{(k)} \leq \epsilon$

- propagate Cholesky factor of P (improves numerical stability)

Proof of convergence

assumptions: we consider the unconstrained problem

$$\text{minimize } f(x)$$

- f is Lipschitz: $|f(y) - f(x)| \leq G \|y - x\|_2$
- $\{x \mid f(x) \leq f^* + \epsilon\} \subseteq \mathcal{E}_0$
- \mathcal{E}_0 is ball with radius R

reduction of volume: can show that

$$\text{vol } \mathcal{E}_{k+1} < e^{-\frac{1}{2n}} \text{vol } \mathcal{E}_k$$

(reduction factor degrades rapidly with n , compared to CG or MVE cutting-plane methods)

proof. suppose $f(x^{(i)}) > f^* + \epsilon$, $i = 1, \dots, k$

- at iteration i we only discard points with $f(z) \geq f(x^{(i)})$; therefore

$$\{z \mid f(z) \leq f^* + \epsilon\} \subseteq \mathcal{E}_k$$

- from Lipschitz condition, $\|z - x^*\|_2 \leq \epsilon/G$ implies $f(z) \leq f^* + \epsilon$; hence

$$B = \{z \mid \|z - x^*\|_2 \leq \epsilon/G\} \subseteq \mathcal{E}_k$$

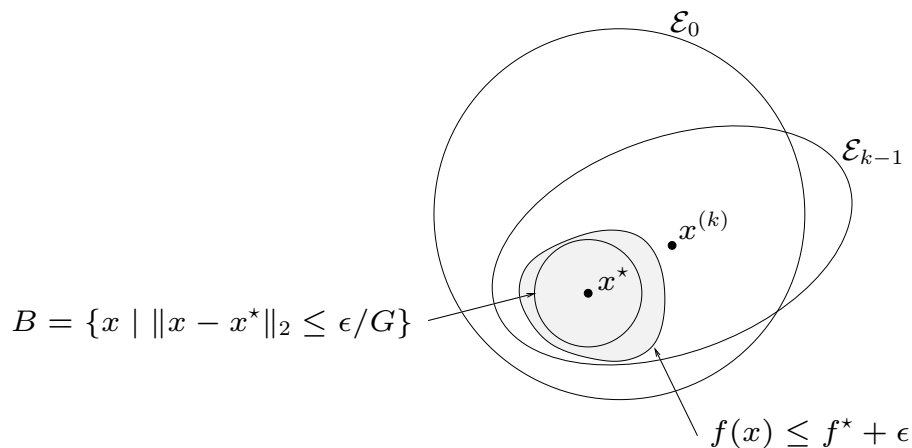
- therefore $\text{vol } B \leq \text{vol } \mathcal{E}_k$, so

$$\alpha_n (\epsilon/G)^n \leq e^{-\frac{k}{2n}} \text{vol } \mathcal{E}_0 = e^{-\frac{k}{2n}} \alpha_n R^n$$

(α_n is volume of unit ball in \mathbf{R}^n); this gives

$$k \leq 2n^2 \log(RG/\epsilon)$$

geometrical illustration



conclusion: for $k > 2n^2 \log(RG/\epsilon)$,

$$f_{\text{best}}^{(k)} \leq f^* + \epsilon$$

Interpretation of complexity

- since $x^* \in \mathcal{E}_0 = \{x \mid \|x - x^{(1)}\|_2 \leq R\}$, our prior knowledge of f^* is

$$f(x^{(1)}) - GR \leq f^* \leq f(x^{(1)})$$

our prior uncertainty in f^* is GR

- after k iterations our knowledge of f^* is

$$f_{\text{best}}^{(k)} - \epsilon \leq f^* \leq f_{\text{best}}^{(k)}$$

posterior uncertainty in f^* is $\leq \epsilon$

- iterations required:

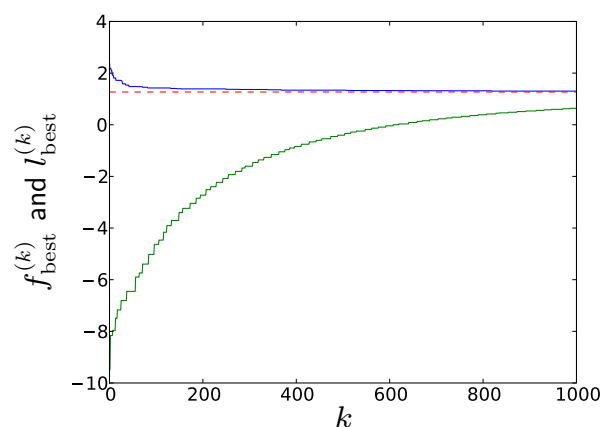
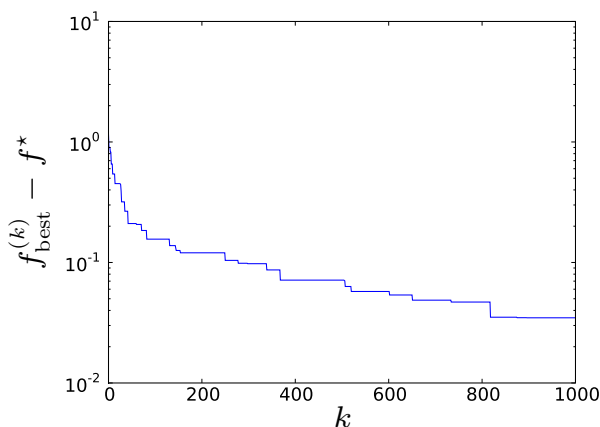
$$2n^2 \log \frac{RG}{\epsilon} = 1.39 n^2 \log_2 \frac{\text{prior uncertainty}}{\text{posterior uncertainty}}$$

efficiency: $0.72/n^2$ bits per gradient evaluation

Example

$$\text{minimize} \quad \max_{i=1, \dots, m} (a_i^T x + b_i)$$

$m = 100$, $n = 20$, $\|x^*\|_2 \approx 1.0$, start with $\mathcal{E} = \{x \mid \|x\|_2 \leq 10\}$



Deep cut ellipsoid method

minimum volume ellipsoid containing ellipsoid intersected with halfspace

$$\mathcal{E} \cap \{z \mid g^T(z - x) + h \leq 0\}$$

with $h \geq 0$, is given by

$$x^+ = x - \frac{1 + \alpha n}{n + 1} P \tilde{g}$$

$$P^+ = \frac{n^2(1 - \alpha^2)}{n^2 - 1} \left(P - \frac{2(1 + \alpha n)}{(n + 1)(1 + \alpha)} P \tilde{g} \tilde{g}^T P \right)$$

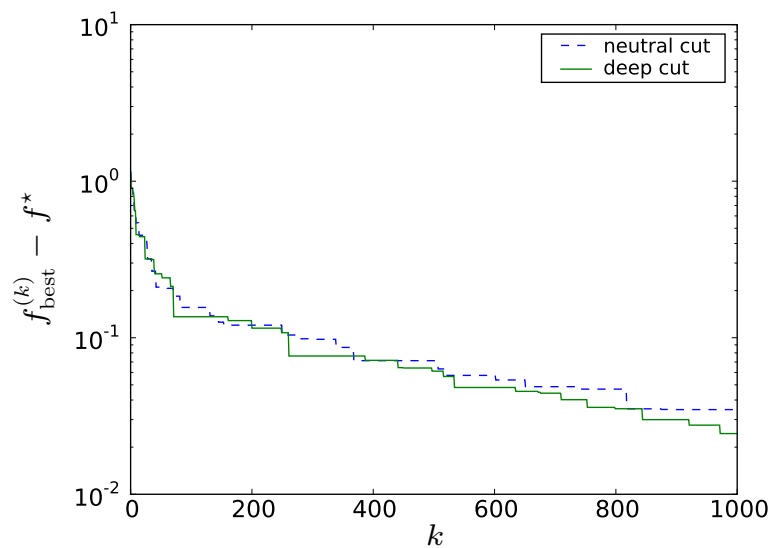
where

$$\tilde{g} = \frac{g}{\sqrt{g^T P g}}, \quad \alpha = \frac{h}{\sqrt{g^T P g}}$$

(if $\alpha > 1$, intersection is empty)

Ellipsoid method with deep objective cuts

same example as on page 10-16



Inequality constrained problems

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- if $x^{(k)}$ feasible, update ellipsoid with objective cut

$$g_0^T(z - x^{(k)}) + f_0(x^{(k)}) - f_{\text{best}}^{(k)} \leq 0, \quad g_0 \in \partial f_0(x^{(k)})$$

$f_{\text{best}}^{(k)}$ is best objective value of feasible iterates so far

- if $x^{(k)}$ infeasible, update ellipsoid with feasibility cut

$$g_j^T(z - x^{(k)}) + f_j(x^{(k)}) \leq 0, \quad g_j \in \partial f_j(x^{(k)})$$

assuming $f_j(x^{(k)}) > 0$

Stopping criterion

- if $x^{(k)}$ is feasible, we have lower bound on p^* as before:

$$p^* \geq f_0(x^{(k)}) - \sqrt{g_0^{(k)T} P^{(k-1)} g_0^{(k)}}$$

- if $x^{(k)}$ is infeasible, we have for all $x \in \mathcal{E}_{k-1}$

$$\begin{aligned} f_j(x) & \geq f_j(x^{(k)}) + g_j^{(k)T}(x - x^{(k)}) \\ & \geq f_j(x^{(k)}) + \inf_{z \in \mathcal{E}_{k-1}} g_j^{(k)T}(z - x^{(k)}) \\ & = f_j(x^{(k)}) - \sqrt{g_j^{(k)T} P^{(k-1)} g_j^{(k)}} \end{aligned}$$

hence, problem is infeasible if for some j ,

$$f_j(x^{(k)}) - \sqrt{g_j^{(k)T} P^{(k-1)} g_j^{(k)}} > 0$$

stopping criteria: terminate algorithm when

- $x^{(k)}$ is known to be ϵ -suboptimal:

$$x^{(k)} \text{ is feasible and } \sqrt{g_0^{(k)T} P^{(k-1)} g_0^{(k)}} \leq \epsilon$$

- or problem is shown to be infeasible:

$$f_j(x^{(k)}) - \sqrt{g_j^{(k)T} P^{(k-1)} g_j^{(k)}} > 0 \text{ for some } j$$