

# 17. Primal-dual interior-point methods

- cone programming
- logarithmic barrier function and central path
- symmetrization
- Nesterov-Todd scaling
- path-following algorithm
- quadratic cone program

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## Cone program

### primal problem

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Gx + s = h \\ & && Ax = b \\ & && s \succeq 0 \end{aligned}$$

$s \succeq 0$  is inequality with respect to proper convex cone  $C$

### dual problem

$$\begin{aligned} & \text{maximize} && -h^T z - b^T y \\ & \text{subject to} && A^T y + G^T z + c = 0 \\ & && z \succeq_* 0 \end{aligned}$$

$z \succeq_* 0$  is generalized inequality with respect to dual cone  $C^*$

## Primal-dual path-following methods

closely related to **barrier methods** (EE236B)

- methods follow central path to find approximate solution
- steps are computed by linearizing central path equations

incorporate several **modifications** to improve efficiency and robustness

- symmetric treatment of primal and dual iterates
- barrier parameter is updated after each Newton step; no distinction between outer and inner iterations
- more aggressive step sizes
- infeasible starting points
- higher-order approximation of central path
- (some algorithms) detect infeasibility

## Three standard cones

we will limit the discussion to three types of cones

**nonnegative orthant**

$$\{u \in \mathbf{R}^p \mid u_i \geq 0, i = 1, \dots, p\}$$

**second-order cone**

$$\{(u_0, u_1) \in \mathbf{R} \times \mathbf{R}^{p-1} \mid \|u_1\|_2 \leq u_0\}$$

**semidefinite cone**

$$\{u \in \mathbf{R}^{p(p+1)/2} \mid \mathbf{mat}(u) \succeq 0\}$$

( $\mathbf{mat}(u)$  is symmetric matrix of order  $p$  constructed from  $u$ ; see below)

these cones are self-dual, so we can drop the subscript in  $z \succeq_* 0$

## Notation for symmetric matrices

### symmetric matrix as a vector

if

$$U = \begin{bmatrix} U_{11} & U_{21} & \cdots & U_{p1} \\ U_{21} & U_{22} & \cdots & U_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ U_{p1} & U_{p2} & \cdots & U_{pp} \end{bmatrix} \in \mathbf{S}^p,$$

then

$$\mathbf{vec}(U) = \sqrt{2} \left( \frac{U_{11}}{\sqrt{2}}, U_{21}, \dots, U_{p1}, \frac{U_{22}}{\sqrt{2}}, U_{32}, \dots, U_{p2}, \dots, \frac{U_{pp}}{\sqrt{2}} \right)$$

- $\mathbf{vec}(U) \in \mathbf{R}^{p(p+1)/2}$  are lower triangular entries in column-major order
- off-diagonal entries are scaled by  $\sqrt{2}$  so that

$$\mathbf{tr}(UV) = \mathbf{vec}(U)^T \mathbf{vec}(V)$$

### vector as symmetric matrix

if  $u = (u_1, u_2, \dots, u_{p(p+1)/2}) \in \mathbf{R}^{p(p+1)/2}$ , then

$$\mathbf{mat}(u) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2}u_1 & u_2 & \cdots & u_p \\ u_2 & \sqrt{2}u_{p+1} & \cdots & u_{2p-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_p & u_{2p-1} & \cdots & \sqrt{2}u_{p(p+1)/2} \end{bmatrix}$$

- $\mathbf{mat}(u) \in \mathbf{S}^p$  is the symmetric matrix of order  $p$  constructed from  $u$
- off-diagonal entries are scaled by  $\sqrt{2}$  so that

$$u^T v = \mathbf{tr}(\mathbf{mat}(u) \mathbf{mat}(v))$$

## semidefinite constraints in vector notation

- suppose  $C$  is the cone of vectorized positive semidefinite matrices

$$C = \{u \in \mathbf{R}^{p(p+1)/2} \mid \mathbf{mat}(u) \succeq 0\} = \{\mathbf{vec}(U) \mid U \in \mathbf{S}_+^p\}$$

- for  $G_i \in \mathbf{S}^p$ ,  $h \in \mathbf{S}^p$ , define

$$G = \begin{bmatrix} \mathbf{vec}(G_1) & \mathbf{vec}(G_2) & \cdots & \mathbf{vec}(G_n) \end{bmatrix}, \quad h = \mathbf{vec}(H)$$

- $Gx \preceq_C h$  is equivalent to a linear matrix inequality

$$\sum_{i=1}^n x_i G_i \preceq H$$

- $G^T z + c = 0$ ,  $z \succeq_C 0$  is equivalent to

$$\mathbf{tr}(G_i Z) + c_i = 0, \quad i = 1, \dots, n, \quad Z = \mathbf{mat}(z) \succeq 0$$

## Optimality conditions

$$\begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} c \\ b \\ h \end{bmatrix} \quad (1)$$

$$(s, z) \succeq 0, \quad s^T z = 0$$

**primal feasibility:** blocks 2 and 3 of (1) and  $s \succeq 0$

**dual feasibility:** block 1 of (1) and  $z \succeq 0$

**zero duality gap:** inner product with  $(x, y, z)$  on each side of (1) gives

$$z^T s = c^T x + b^T y + h^T z$$

## Barrier function for nonnegative orthant

$$C = \{u \in \mathbf{R}^p \mid u_i \geq 0, i = 1, \dots, p\}$$

**barrier function**

$$\phi(u) = -\sum_{i=1}^p \log u_i$$

**gradient**

$$\nabla\phi(u) = -\left(\frac{1}{u_1}, \frac{1}{u_2}, \dots, \frac{1}{u_p}\right)$$

**Hessian and inverse Hessian**

$$\nabla^2\phi(u) = \mathbf{diag}(u)^{-2}, \quad \nabla^2\phi(u)^{-1} = \mathbf{diag}(u)^2$$

## Barrier function for second-order cone

$$C = \{(u_0, u_1) \in \mathbf{R} \times \mathbf{R}^{p-1} \mid \|u_1\|_2 \leq u_0\}$$

**barrier function**

$$\phi(u) = -\frac{1}{2} \log(u_0^2 - u_1^T u_1)$$

**gradient**

$$\nabla\phi(u) = -\frac{1}{u^T J u} J u, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -I_{p-1} \end{bmatrix}$$

**Hessian and inverse Hessian**

$$\begin{aligned} \nabla^2\phi(u) &= \frac{1}{(u^T J u)^2} (2J u u^T J - (u^T J u) J) \\ \nabla^2\phi(u)^{-1} &= 2u u^T - (u^T J u) J \end{aligned}$$

## Barrier function for positive semidefinite cone

$$C = \left\{ u \in \mathbf{R}^{p(p+1)/2} \mid \mathbf{mat} u \succeq 0 \right\}$$

**barrier function**

$$\phi(u) = -\log \det \mathbf{mat}(u)$$

**gradient**

$$\nabla \phi(u) = -\mathbf{vec}(\mathbf{mat}(u)^{-1})$$

**Hessian**

$$\nabla^2 \phi(u)v = \mathbf{vec}(U^{-1}VU^{-1}), \quad \nabla^2 \phi(u)^{-1}v = \mathbf{vec}(UVU)$$

for all  $v$ , where  $U = \mathbf{mat}(u)$ ,  $V = \mathbf{mat}(v)$

## Barrier function for composite cones

$$C = C_1 \times \cdots \times C_N$$

$C_k$  is one of the three standard cones, with barrier  $\phi_k$

**barrier** at  $u = (u_1, \dots, u_N)$

$$\phi(u) = \sum_{k=1}^N \phi_k(u_k)$$

**gradient**

$$\nabla \phi(u) = (\nabla \phi_1(u_1), \dots, \nabla \phi_N(u_N))$$

**Hessian**

$$\nabla^2 \phi(u) = \begin{bmatrix} \nabla^2 \phi_1(u_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nabla^2 \phi_N(u_N) \end{bmatrix}$$

## Degree of barrier function

the barrier functions satisfy

$$\phi(tu) = \phi(u) - m \log t \quad \text{for } t > 0$$

$m$  is called the logarithmic *degree*

- $m = p$  for nonnegative orthant of dimension  $p$
- $m = 1$  for second-order cone
- $m = p$  for semidefinite cone of order  $p$
- for composite cone,  $m$  is sum of degrees of the components

property of gradient

$$\nabla\phi(u) \prec 0, \quad u^T \nabla\phi(u) = -m$$

## Central path

$$\begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} c \\ b \\ h \end{bmatrix}$$

$$(s, z) \succ 0, \quad z = -\mu \nabla\phi(s)$$

- central path is set of solutions  $s, x, y, z$  for  $\mu > 0$
- points  $s, z$  on the central path satisfy

$$\mu = \frac{s^T z}{m}$$

## Symmetrization

we will write the condition  $z = -\mu \nabla \phi(s)$  as a product  $s \circ z = \mu \mathbf{e}$

**nonnegative orthant** (of dimension  $p$ )

$$u \circ v = (u_1 v_1, u_2 v_2, \dots, u_p v_p), \quad \mathbf{e} = (1, 1, \dots, 1)$$

**second-order cone** (of dimension  $p$ ): for  $u = (u_0, u_1)$ ,  $v = (v_0, v_1)$ ,

$$u \circ v = \begin{bmatrix} u^T v \\ u_0 v_1 + v_0 u_1 \end{bmatrix}, \quad \mathbf{e} = (1, 0, \dots, 0)$$

**semidefinite cone** (of order  $p$ ): for  $U = \mathbf{mat}(u)$ ,  $V = \mathbf{mat}(v)$ ,

$$u \circ v = \frac{1}{2} \mathbf{vec}(UV + VU), \quad \mathbf{e} = \mathbf{vec}(I_p)$$

**composite cone:** apply  $\circ$ -product componentwise

## Symmetric parametrization and scaling

$$\begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} c \\ b \\ h \end{bmatrix}$$

$$(s, z) \succ 0, \quad s \circ z = \mu \mathbf{e}$$

can further replace  $s \circ z = \mu \mathbf{e}$  with  $(W^{-T} s) \circ (W z) = \mu \mathbf{e}$ , if

- multiplications with  $W$  and  $W^{-T}$  preserve the cone
- multiplications preserve the central path

$W$  is called a *scaling matrix*

## Nesterov-Todd scaling

a specific scaling, used, for example, in Sedumi and SDPT3

- $W$  is associated with a pair  $(\hat{s}, \hat{z}) \succ 0$  (and defined on the next pages)
- satisfies  $W^T W = \nabla^2 \phi(w)^{-1}$  where  $w$  is the unique point for which

$$\nabla^2 \phi(w) \hat{s} = \hat{z}$$

$w$  is called the *Nesterov-Todd scaling point*

- multiplications by  $W$  and  $W^{-T}$  map  $\hat{s}$  and  $\hat{z}$  to the same point:

$$W^{-T} \hat{s} = W \hat{z} = \lambda$$

this implies that  $\hat{s}^T \hat{z} = \|\lambda\|_2^2$

## NT scaling for nonnegative orthant

$$W = \begin{bmatrix} \sqrt{\hat{s}_1/\hat{z}_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\hat{s}_2/\hat{z}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\hat{s}_p/\hat{z}_p} \end{bmatrix}$$

- a positive diagonal scaling
- NT scaling point is

$$w = \left( \sqrt{\hat{s}_1/\hat{z}_1}, \sqrt{\hat{s}_2/\hat{z}_2}, \dots, \sqrt{\hat{s}_p/\hat{z}_p} \right)$$

- scaled  $\hat{s}, \hat{z}$

$$\lambda = W^{-T} \hat{s} = W \hat{z} = \left( \sqrt{\hat{s}_1 \hat{z}_1}, \sqrt{\hat{s}_2 \hat{z}_2}, \dots, \sqrt{\hat{s}_p \hat{z}_p} \right)$$

## NT scaling for second-order cone

$$W = \left( \frac{\hat{s}^T J \hat{s}}{\hat{z}^T J \hat{z}} \right)^{1/4} \begin{bmatrix} \bar{w}_0 & \bar{w}_1^T \\ \bar{w}_1 & I + (\bar{w}_0 + 1)^{-1} \bar{w}_1 \bar{w}_1^T \end{bmatrix}$$

where

$$\bar{w} = \frac{1}{2\gamma} \left( \frac{\hat{s}}{\sqrt{\hat{s}^T J \hat{s}}} + \frac{J \hat{z}}{\sqrt{\hat{z}^T J \hat{z}}} \right), \quad \gamma = \left( \frac{1}{2} + \frac{\hat{z}^T \hat{s}}{2(\hat{s}^T J \hat{s})^{1/2}(\hat{z}^T J \hat{z})^{1/2}} \right)^{1/2}$$

- $W$  can also be expressed as

$$W = \beta(2vv^T - J), \quad v = \frac{\bar{w} + \mathbf{e}}{(2(\bar{w}_0 + 1))^{1/2}}, \quad \beta = \left( \frac{\hat{s}^T J \hat{s}}{\hat{z}^T J \hat{z}} \right)^{1/4}$$

( $2vv^T - J$  is called a *hyperbolic Householder matrix*)

- NT scaling point is  $w = \beta \bar{w}$

## NT scaling for positive semidefinite cone

$$Wv = \text{vec}(R^T \text{mat}(v)R)$$

where  $R$  simultaneously diagonalizes  $\text{mat}(\hat{z})$  and  $\text{mat}(\hat{s})^{-1}$

$$R^T \text{mat}(\hat{z})R = R^{-1} \text{mat}(\hat{s})R^{-T} = \Lambda$$

- $W$  is a nonsingular congruence transformation
- NT scaling point is  $w = \text{vec}(RR^T)$
- $R$  can be computed from two Cholesky factorizations and an SVD: if

$$\text{mat}(\hat{s}) = L_1 L_1^T, \quad \text{mat}(\hat{z}) = L_2 L_2^T, \quad L_2^T L_1 = U \Lambda V^T$$

$$\text{then } R = L_1 V \Lambda^{-1/2} = L_2^{-T} U \Lambda^{1/2}$$

## Basic primal-dual update

suppose we are at  $(\hat{s}, \hat{x}, \hat{y}, \hat{z})$  with  $\hat{s} \succ 0, \hat{z} \succ 0$

- define  $\mu = \hat{s}^T \hat{z} / m$  and let  $W$  be the NT scaling matrix for  $\hat{s}, \hat{z}$
- compute  $\Delta s, \Delta x, \Delta y, \Delta z$  by linearizing the central path equation

$$\begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} c \\ b \\ h \end{bmatrix}$$

$$(W^{-T} s) \circ (W z) = \sigma \mu \mathbf{e}$$

around  $\hat{s}, \hat{x}, \hat{y}, \hat{z}$ , for some  $\sigma < 1$

- make an update

$$(\hat{s}, \hat{x}) := (\hat{s}, \hat{x}) + \alpha_p (\Delta x, \Delta s), \quad (\hat{y}, \hat{z}) := (\hat{y}, \hat{z}) + \alpha_d (\Delta y, \Delta z)$$

that preserves positivity of  $\hat{s}, \hat{z}$

## Linearized central path equation

define  $\lambda = W^{-T} \hat{s} = W \hat{z}$  and

$$r = \begin{bmatrix} 0 \\ 0 \\ \hat{s} \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} - \begin{bmatrix} c \\ b \\ h \end{bmatrix}$$

### linearized central path equation

$$\begin{bmatrix} 0 \\ 0 \\ \Delta s \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = -r$$

$$\lambda \circ (W \Delta z + W^{-T} \Delta s) = \sigma \mu \mathbf{e} - \lambda \circ \lambda$$

second equation is linearization of  $(W^{-T}(\hat{s} + \Delta s)) \circ (W(\hat{z} + \Delta z)) = \sigma \mu \mathbf{e}$

## Path-following algorithm

choose starting points  $\hat{s}$ ,  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{z}$  with  $\hat{s} \succ 0$ ,  $\hat{z} \succ 0$

### 1. compute residuals and evaluate stopping criteria

$$r = \begin{bmatrix} 0 \\ 0 \\ \hat{s} \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} - \begin{bmatrix} c \\ b \\ h \end{bmatrix}$$

terminate if  $r$  and  $\hat{s}^T \hat{z}$  are sufficiently small

### 2. compute scaling matrix $W$ associated with $(\hat{s}, \hat{z})$ and set

$$\lambda := W^{-T} \hat{s} = W \hat{z}, \quad \mu := \frac{\lambda^T \lambda}{m} = \frac{\hat{s}^T \hat{z}}{m}$$

( $m$  is logarithmic degree of cone)

### 3. compute affine scaling direction by solving the linear equation

$$\begin{bmatrix} 0 \\ 0 \\ \Delta s_a \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_a \\ \Delta y_a \\ \Delta z_a \end{bmatrix} = -r$$

$$\lambda \circ (W \Delta z_a + W^{-T} \Delta s_a) = -\lambda \circ \lambda$$

### 4. select barrier parameter

$$\sigma = \left( \frac{(\hat{s} + \alpha_p \Delta s_a)^T (\hat{z} + \alpha_d \Delta z_a)}{\hat{s}^T \hat{z}} \right)^\delta$$

where  $\delta$  is an algorithm parameter (typical value is  $\delta = 3$ ) and

$$\alpha_p = \sup\{\alpha \in [0, 1] \mid \hat{s} + \alpha \Delta s_a \succeq 0\}$$

$$\alpha_d = \sup\{\alpha \in [0, 1] \mid \hat{z} + \alpha \Delta z_a \succeq 0\}$$

5. **compute search direction** by solving the linear equation

$$\begin{bmatrix} 0 \\ 0 \\ \Delta s \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = -r$$

$$\lambda \circ (W\Delta z + W^{-T}\Delta s) = \sigma\mu\mathbf{e} - \lambda \circ \lambda$$

6. **update iterates**

$$(\hat{x}, \hat{s}) := (\hat{x}, \hat{s}) + \min\{1, 0.99\alpha_p\}(\Delta x, \Delta s)$$

$$(\hat{y}, \hat{z}) := (\hat{y}, \hat{z}) + \min\{1, 0.99\alpha_d\}(\Delta y, \Delta z)$$

where

$$\alpha_p = \sup\{\alpha \geq 0 \mid \hat{s} + \alpha\Delta s \succeq 0\}, \quad \alpha_d = \sup\{\alpha \geq 0 \mid \hat{z} + \alpha\Delta z \succeq 0\}$$

return to step 1

## interpretation and discussion

- step 3: affine scaling direction solves linearized central path equation with  $\sigma = 0$ , *i.e.*, the linearized optimality conditions
- step 4 is a heuristic for choosing  $\sigma$  based on an estimate of the quality of the affine scaling direction

$\sigma$  is small if a step in the affine scaling direction gives a large reduction in  $\hat{s}^T \hat{z}$

- step 5: linear equation has same coefficient matrix as equation in step 3  
if a direct method is used, we can reuse the factorization used in step 3, and solve the two equations at the cost of one

## Mehrotra correction

in step 5, solve

$$\begin{bmatrix} 0 \\ 0 \\ \Delta s \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = -r$$

$$\lambda \circ (W\Delta z + W^{-T}\Delta s) = \sigma\mu\mathbf{e} - \lambda \circ \lambda - (W^{-T}\Delta s_a) \circ (W\Delta z_a)$$

- extra term on the r.h.s. is approximation of the second-order term in

$$(W^{-T}(\hat{s} + \Delta s)) \circ (W(\hat{z} + \Delta z)) = -\sigma\mu\mathbf{e}$$

- adding the correction typically saves a few iterations

## Newton equations

steps 3 and 5 reduce to equations

$$\begin{bmatrix} 0 & A^T & G^T \\ A & 0 & 0 \\ G & 0 & -W^T W \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

usually solved by eliminating  $\Delta z$ :

$$\begin{bmatrix} G^T W^{-1} W^{-T} G & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} d_x + G^T W^{-1} W^{-T} d_z \\ d_y \end{bmatrix}$$

- a KKT system (see §10.4.2 in BV for a discussion of solution methods)
- note (from p.17-17) that

$$G^T W^{-1} W^{-T} G = G^T \nabla^2 \phi(w) G,$$

the Hessian of the barrier function  $\phi(h - Gx)$  at the scaling point  $w$

## Quadratic cone program

$$\begin{aligned} & \text{minimize} && (1/2)x^T P x + q^T x \\ & \text{subject to} && Gx + s = h \\ & && Ax = b \\ & && s \succeq 0 \end{aligned}$$

### optimality conditions

$$\begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} P & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} q \\ b \\ h \end{bmatrix}, \quad (s, z) \succeq 0, \quad s^T z = 0$$

### central path

$$\begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} P & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} q \\ b \\ h \end{bmatrix}, \quad (s, z) \succ 0, \quad s \circ z = \mu \mathbf{e}$$

## Path-following algorithm

algorithm is almost identical to algorithm on page 17-23

- compute search directions from linearized central path equation;  
for example, step 5 becomes

$$\begin{bmatrix} 0 \\ 0 \\ \Delta s \end{bmatrix} - \begin{bmatrix} P & A^T & G^T \\ -A & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = -r$$

$$\lambda \circ (W \Delta z + W^{-T} \Delta s) = \sigma \mu \mathbf{e} - \lambda \circ \lambda$$

- use equal primal and dual step sizes  
for example, in step 6,

$$\alpha_p = \alpha_d = \sup \{ \alpha \geq 0 \mid \hat{s} + \alpha \Delta s \succeq 0, \hat{z} + \alpha \Delta z \succeq 0 \}$$