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Lecture 6
Duality

- dual of an LP in inequality form
- variants and examples
- complementary slackness

Dual of linear program in inequality form

we define two LPs with the same parameters $c \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$

- an LP in 'inequality form'

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

- an LP in 'standard form'

$$\begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \geq 0 \end{array}$$

  this problem is called the **dual** of the first LP

in the context of duality, the first LP is called the **primal** problem

Duality theorem

notation

- $p^\star$ is the primal optimal value; $d^\star$ is the dual optimal value
- $p^\star = +\infty$ if primal problem is infeasible; $d^\star = -\infty$ if dual is infeasible
- $p^\star = -\infty$ if primal problem is unbounded; $d^\star = \infty$ if dual is unbounded

duality theorem: if primal or dual problem is feasible, then

$$p^\star = d^\star$$

moreover, if $p^\star = d^\star$ is finite, then primal and dual optima are attained

note: only exception to $p^\star = d^\star$ occurs when primal *and* dual are infeasible

Weak duality

lower bound property: if x is primal feasible and z is dual feasible, then

$$c^T x \geq -b^T z$$

proof: if $Ax \leq b$, $A^T z + c = 0$, and $z \geq 0$, then

$$0 \leq z^T(b - Ax) = b^T z + c^T x$$

$c^T x + b^T z$ is the **duality gap** associated with primal and dual feasible x, z

weak duality: the lower bound property immediately implies that

$$p^\star \geq d^\star$$

(without exception)

Strong duality

if primal and dual problems are feasible, then there exist $x^\star$, $z^\star$ that satisfy

$$c^T x^\star = -b^T z^\star, \qquad A x^\star \leq b, \qquad A^T z^\star + c = 0, \qquad z^\star \geq 0$$

combined with the lower bound property, this implies that

- $x^\star$ is primal optimal and $z^\star$ is dual optimal
- the primal and dual optimal values are finite and equal:

$$p^\star = c^T x^\star = -b^T z^\star = d^\star$$

(proof on next page)

proof: we show that there exist $x^\star$, $z^\star$ that satisfy

$$\begin{bmatrix} A & 0 \\ 0 & -I \\ c^T & b^T \end{bmatrix} \begin{bmatrix} x^\star \\ z^\star \end{bmatrix} \leq \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -A^T \end{bmatrix} \begin{bmatrix} x^\star \\ z^\star \end{bmatrix} = c$$

- the lower-bound property implies that any solution necessarily satisfies

$$c^T x^\star + b^T z^\star = 0$$

- to prove a solution exists we show that the alternative system (p. 5–5)

$$u \geq 0, \quad t \geq 0, \qquad A^T u + tc = 0, \qquad Aw \leq tb, \qquad b^T u + c^T w < 0$$

  has no solution

the alternative system has no solution because:

- if $t > 0$, defining $\tilde{x} = w/t$, $\tilde{z} = u/t$ gives

$$\tilde{z} \geq 0, \qquad A^T\tilde{z} + c = 0, \qquad A\tilde{x} \leq b, \qquad c^T\tilde{x} < -b^T\tilde{z}$$

  this contradicts the lower bound property

- if $t = 0$ and $b^Tu < 0$, u satisfies

$$u \geq 0, \qquad A^Tu = 0, \qquad b^Tu < 0$$

  this contradicts feasibility of $Ax \leq b$ (page 5–2)

- if $t = 0$ and $c^Tw < 0$, w satisfies

$$Aw \leq 0, \qquad c^Tw < 0$$

  this contradicts feasibility of $A^Tz + c = 0$, $z \geq 0$ (page 5–3)

Primal infeasible problems

if $p^\star = +\infty$ then $d^\star = +\infty$ or $d^\star = -\infty$

proof: if primal is infeasible, then from page 5–2, there exists w such that

$$w \geq 0, \qquad A^T w = 0, \qquad b^T w < 0$$

if the dual problem is feasible and z is any dual feasible point, then

$$z + tw \geq 0, \qquad A^T(z + tw) + c = 0 \qquad \text{for all } t \geq 0$$

therefore $z + tw$ is dual feasible for all $t \geq 0$; moreover, as $t \to \infty$,

$$-b^T(z + tw) = -b^T z - t b^T w \to +\infty$$

so the dual problem is unbounded above

Dual infeasible problems

if $d^\star = -\infty$ then $p^\star = -\infty$ or $p^\star = +\infty$

proof: if dual is infeasible, then from page 5–3, there exists y such that

$$A^T y \leq 0, \qquad c^T y < 0$$

if the primal problem is feasible and x is any primal feasible point, then

$$A^T(x + ty) \leq b \quad \text{for all } t \geq 0$$

therefore $x + ty$ is primal feasible for all $t \geq 0$; moreover, as $t \to \infty$,

$$c^T(x + ty) = c^T x + tc^T y \to -\infty$$

so the primal problem is unbounded below

Exception to strong duality

an example that shows that $p^\star = +\infty$, $d^\star = -\infty$ is possible

primal problem (one variable, one inequality)

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

optimal value is $p^\star = +\infty$

dual problem

$$\begin{array}{ll} \text{maximize} & z \\ \text{subject to} & 0 \cdot z + 1 = 0 \\ & z \geq 0 \end{array}$$

optimal value is $d^\star = -\infty$

Summary

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal and attained	
$d^\star = -\infty$	exception		primal unb. dual inf.

- upper-right part of the table is excluded by weak duality
- first column: proved on page 6–8
- bottom row: proved on page 6–9
- center: proved on page 6–5

Outline

- dual of an LP in inequality form
- **variants and examples**
- complementary slackness

Variants

LP with inequality and equality constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T z - d^T y \\ \text{subject to} & A^T z + C^T y + c = 0 \\ & z \geq 0 \end{array}$$

standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

- dual problems can be derived by converting primal to inequality form
- same duality results apply

Piecewise-linear minimization

$$\text{minimize} \quad f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$

LP formulation (variables x, t; optimal value is $\min_x f(x)$)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} A & -\mathbf{1} \end{array}\right] \left[\begin{array}{c} x \\ t \end{array}\right] \leq -b \end{array}$$

dual LP (same optimal value)

$$\begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z = 0 \\ & \mathbf{1}^T z = 1 \\ & z \geq 0 \end{array}$$

Interpretation

- for any $z \geq 0$ with $\sum_i z_i = 1$,

$$f(x) = \max_i (a_i^T x + b_i) \geq z^T(Ax + b) \quad \text{for all } x$$

- this provides a lower bound on the optimal value of the PWL problem

$$\begin{aligned} \min_x f(x) &\geq \min_x z^T(Ax + b) \\ &= \begin{cases} b^T z & \text{if } A^T z = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

- the dual problem is to find the best lower bound of this type

- strong duality tells us that the best lower bound is tight

ℓ_∞-Norm approximation

$$\text{minimize} \quad \|Ax - b\|_\infty$$

LP formulation

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

dual problem

$$\begin{array}{ll} \text{maximize} & -b^T u + b^T v \\ \text{subject to} & A^T u - A^T v = 0 \\ & \mathbf{1}^T u + \mathbf{1}^T v = 1 \\ & u \geq 0, \ v \geq 0 \end{array} \tag{1}$$

simpler equivalent dual

$$\begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z = 0, \quad \|z\|_1 \leq 1 \end{array} \tag{2}$$

proof of equivalence of the dual problems (assume A is $m \times n$)

- if u, v are feasible in (1), then $z = v - u$ is feasible in (2):

$$\|z\|_1 = \sum_{i=1}^{m} |v_i - u_i| \leq \mathbf{1}^T v + \mathbf{1}^T u = 1$$

  moreover the objective values are equal: $b^T z = b^T(v - u)$

- if z is feasible in (2), define vectors u, v by

$$u_i = \max\{z_i, 0\} + \alpha, \quad v_i = \max\{-z_i, 0\} + \alpha, \quad i = 1, \ldots, m$$

  with $\alpha = (1 - \|z\|_1)/(2m)$

  these vectors are feasible in (1) with objective value $b^T(v - u) = b^T z$

Interpretation

- lemma: $u^T v \leq \|u\|_1 \|v\|_\infty$ holds for all u, v
- therefore, for any z with $\|z\|_1 \leq 1$,

$$\|Ax - b\|_\infty \geq z^T(Ax - b)$$

- this provides a bound on the optimal value of the ℓ_∞-norm problem

$$\begin{aligned}\min_x \|Ax - b\|_\infty \quad &\geq \quad \min_x z^T(Ax - b) \\ &= \quad \begin{cases} -b^T z & \text{if } A^T z = 0 \\ -\infty & \text{otherwise} \end{cases}\end{aligned}$$

- the dual problem is to find the best lower bound of this type
- strong duality tells us the best lower bound is tight

Outline

- dual of an LP in inequality form
- variants and examples
- **complementary slackness**

Optimality conditions

primal and dual LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & -b^T z - d^T y \\ \text{subject to} & A^T z + C^T y + c = 0 \\ & z \geq 0 \end{array}$$

optimality conditions: x and (y, z) are primal, dual optimal if and only if

- x is primal feasible: $Ax \leq b$ and $Cx = d$
- y, z are dual feasible: $A^T z + C^T y + c = 0$ and $z \geq 0$
- the duality gap is zero: $c^T x = -b^T z - d^T y$

Complementary slackness

assume A is $m \times n$ with rows a_i^T

- the duality gap at primal feasible x, dual feasible y, z can be written as

$$
\begin{aligned}
c^T x + b^T z + d^T y &= (b - Ax)^T z + (d - Cx)^T y \\
&= (b - Ax)^T z \\
&= \sum_{i=1}^{m} z_i (b_i - a_i^T x)
\end{aligned}
$$

- primal, dual feasible x, y, z are optimal if and only if

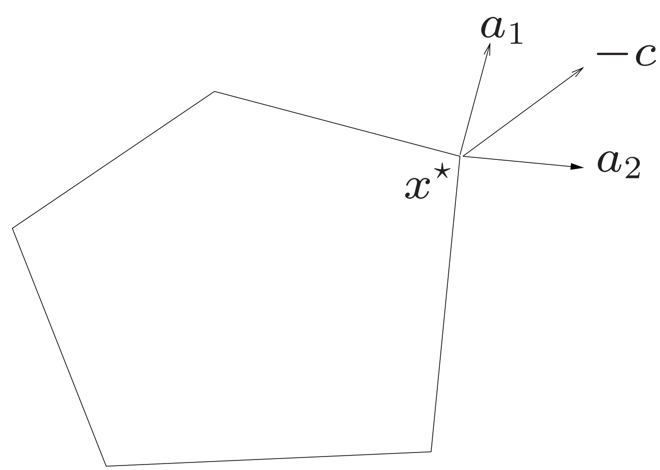
$$z_i (b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

$i.e.$, at optimum, $b - Ax$ and z have a *complementary* sparsity pattern:

$$z_i > 0 \implies a_i^T x = b_i, \qquad a_i^T x < b_i \implies z_i = 0$$

Geometric interpretation

example in $\mathbf{R}^2$



- two active constraints at optimum: $a_1^T x^\star = b_1$, $a_2^T x^\star = b_2$
- optimal dual solution satisfies

$$A^T z + c = 0, \qquad z \geq 0, \qquad z_i = 0 \text{ for } i \notin \{1,2\}$$

  in other words, $-c = a_1 z_1 + a_2 z_2$ with $z_1 \geq 0$, $z_2 \geq 0$

- geometric interpretation: $-c$ lies in the cone generated by a_1 and a_2

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

show that $x = (1, 1)$ is optimal

- second and fourth constraints are active at $(1, 1)$
- therefore any dual optimal z must be of the form $z = (0, z_2, 0, z_4)$ with

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z_2 \\ z_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \qquad z_2 \geq 0, \quad z_4 \geq 0$$

  $z = (0, 1, 0, 2)$ satisfies these conditions

dual feasible z with correct sparsity pattern proves that x is optimal

Optimal set

primal and dual LP (A is $m \times n$ with rows a_i^T)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T z - d^T y \\ \text{subject to} & A^T z + C^T y + c = 0 \\ & z \geq 0 \end{array}$$

assume the optimal value is finite

- let $(y^\star, z^\star)$ be *any* dual optimal solution and define $J = \{i \mid z_i^\star > 0\}$
- x is optimal iff it is feasible and complementary slackness with $z^\star$ holds:

$$a_i^T x = b_i \quad \text{for } i \in J, \qquad a_i^T x \leq b_i \quad \text{for } i \notin J, \qquad Cx = d$$

conclusion: optimal set is a face of the polyhedron $\{x \mid Ax \leq b, Cx = d\}$

Strict complementarity

- primal and dual optimal solutions are not necessarily unique
- any combination of primal and dual optimal points must satisfy

$$z_i(b_i - a_i^T x) = 0, \qquad i = 1, \ldots, m$$

  in other words, for all i,

$$a_i^T x < b_i,\ z_i = 0 \qquad \text{or} \qquad a_i^T x = b_i,\ z_i > 0 \qquad \text{or} \qquad a_i^T x = b_i,\ z_i = 0$$

- primal and dual optimal points are **strictly complementary** if for all i

$$a_i^T x < b_i, \ \ z_i = 0 \qquad \text{or} \qquad a_i^T x = b_i, \ \ z_i > 0$$

it can be shown that strictly complementary solutions exist for any LP with a finite optimal value (exercise 72)