Lecture 8
Linear-fractional optimization

- linear-fractional program
- generalized linear-fractional program
- examples

Linear-fractional program

$$
\begin{array}{ll}
\text{minimize} & \dfrac{c^T x + d}{g^T x + h} \\
\text{subject to} & Ax \preceq b \\
& g^T x + h \geq 0
\end{array}
$$

- if needed, we interpret $a/0$ as $a/0 = +\infty$ if $a > 0$, $a/0 = -\infty$ if $a \leq 0$
- however, in most applications, $Ax \preceq b$ implies $g^T x + h > 0$

equivalent form (with added variable α)

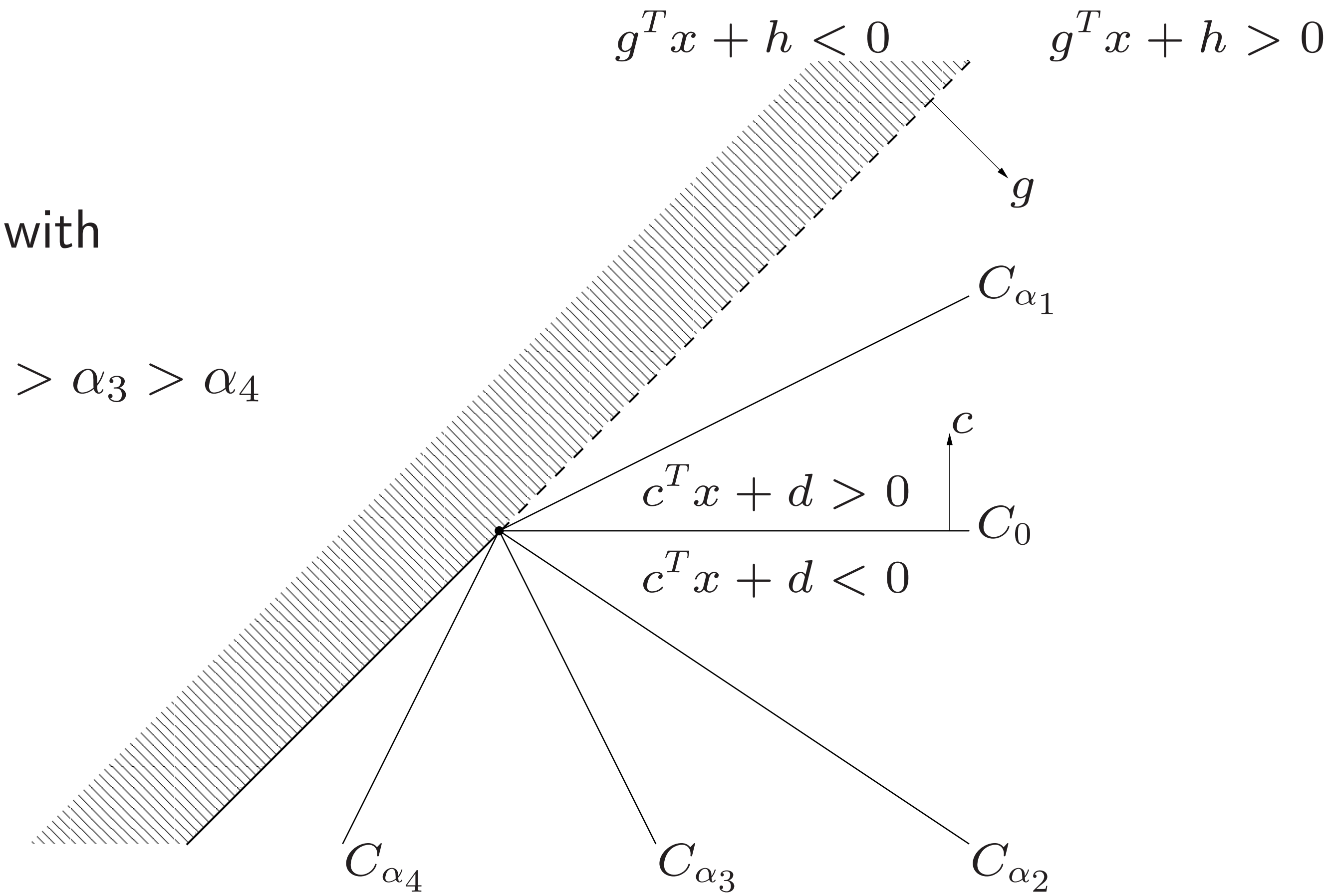
$$
\begin{array}{ll}
\text{minimize} & \alpha \\
\text{subject to} & c^T x + d \leq \alpha(f^T x + g) \\
& Ax \preceq b \\
& f^T x + g \geq 0
\end{array}
$$

Level sets

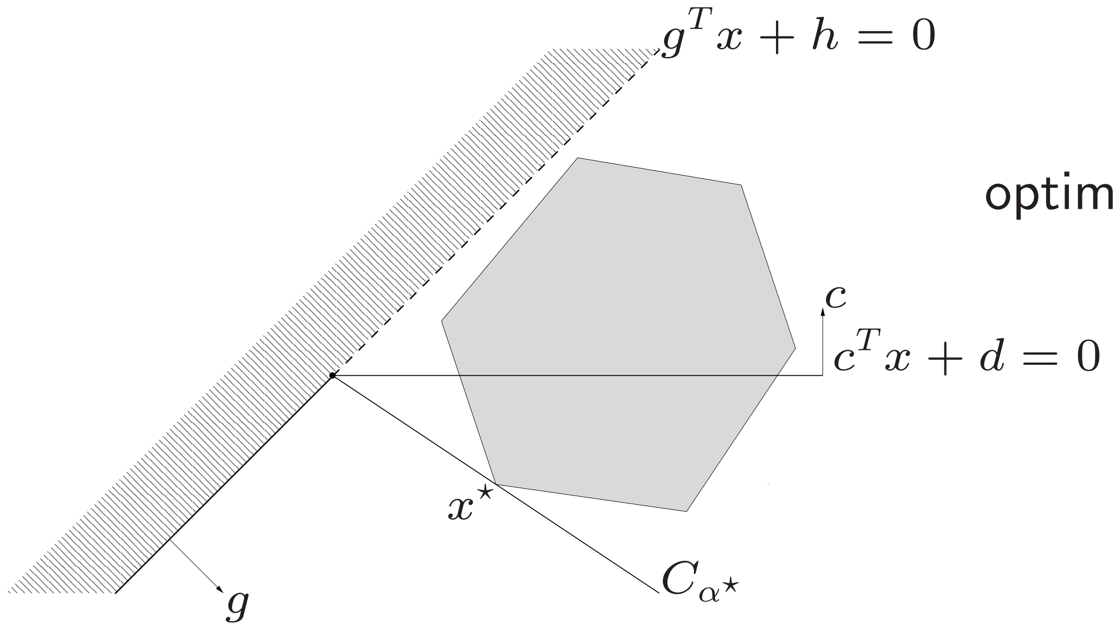
$$\begin{aligned} C_\alpha &= \{x \mid g^T x + h > 0,\ (c^T x + d)/(g^T x + h) = \alpha\} \\ &= \{x \mid g^T x + h > 0,\ (c - \alpha g)^T x = \alpha h - d\} \end{aligned}$$

five level sets with

$\alpha_1 > 0 > \alpha_2 > \alpha_3 > \alpha_4$

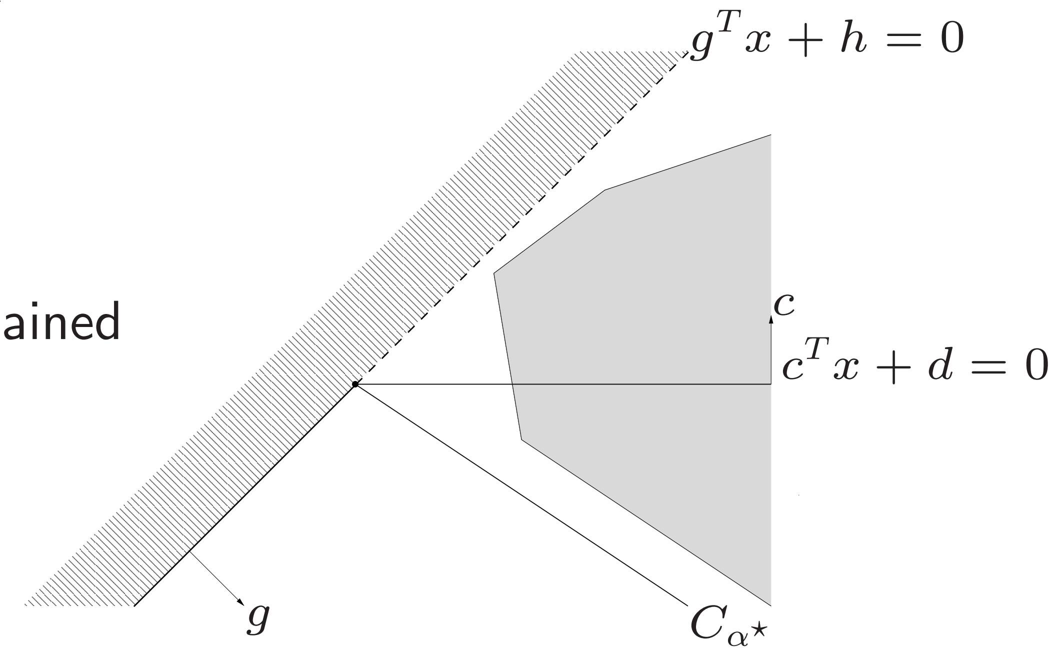


Geometrical interpretation



optimal value $\alpha^\star$ attained at $x^\star$

finite optimal value $\alpha^\star$, not attained



Equivalent linear program

LFP: $\begin{array}{ll} \text{minimize} & \dfrac{c^T x + d}{g^T x + h} \\ \text{subject to} & Ax \leq b \\ & g^T x + h \geq 0 \end{array}$

LP: $\begin{array}{ll} \text{minimize} & c^T y + td \\ \text{subject to} & Ay \leq tb \\ & g^T y + th = 1 \\ & t \geq 0 \end{array}$

we will assume that $g^T x + h > 0$ for all $x \in P = \{x \mid Ax \leq b\}$

- nonlinear change of variables maps $x \in P$ to feasible (y, t) with $t > 0$:

$$y = \frac{1}{g^T x + h} x, \qquad t = \frac{1}{g^T x + h}$$

- inverse transformation $x = y/t$ maps feasible (y, t) with $t > 0$ to $x \in P$
- change of variables and its inverse preserve objective values:

$$(c^T x + d)/(g^T x + h) = c^T y + td$$

Interpretation of $t = 0$

suppose (y, t) is feasible for the LP with $t = 0$ ($i.e.$, $Ay \leq 0$, $g^T y = 1$)

- (y, t) does not correspond to a point $x \in P$ ($x = y/t$ is not defined)
- y can be interpreted as the direction of a half-line based at any $\hat{x} \in P$

$$\{\hat{x} + \lambda y \mid \lambda \geq 0\}$$

- this half-line is in P:

$$A(\hat{x} + \lambda y) \leq b, \quad g^T(\hat{x} + \lambda y) + h \geq 0 \quad \text{for all } \lambda \geq 0$$

- the LFP objective approaches the LP objective $c^T y$ asymptotically:

$$\lim_{\lambda \to \infty} \frac{c^T(\hat{x} + \lambda y) + d}{g^T(\hat{x} + \lambda y) + h} = c^T y$$

Generalized linear-fractional programming

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{i=1,\ldots,m} \frac{c_i^T x + d_i}{f_i^T x + g_i} \\ \text{subject to} & Ax \preceq b \\ & f_i^T x + g_i \geq 0, \quad i = 1, \ldots, m \end{array}$$

equivalent formulation (with auxiliary variable $\alpha \in \mathbf{R}$)

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & Cx + d \preceq \alpha(Fx + g) \\ & Ax \preceq b \\ & Fx + g \succeq 0 \end{array}$$

- C and F are matrices with rows c_i^T, f_i^T
- in contrast to LFP of p. 8–2, generalized LFP is not reducible to an LP
- can be solved efficiently as a sequence of LP feasibility problems

Sublevel sets

definition: α-sublevel set of objective function is

$$\begin{aligned} S_\alpha &= \{x \mid \max_{i=1,\ldots,m} \frac{c_i^T x + d_i}{f_i^T x + g_i} \leq \alpha, \;\; Fx + g \geq 0\} \\ &= \{x \mid Cx + d \leq \alpha(Fx+g), \; Fx + g \geq 0\} \end{aligned}$$

(with $a/0$ interpreted as on page 8–2)

properties

- S_α is a polyhedron
- the sublevel sets S_α are nested: if $\alpha < \beta$ then $S_\alpha \subseteq S_\beta$:

$$\left.\begin{array}{c} Cx + d \leq \alpha(Fx+g) \\ Fx + g \geq 0 \end{array}\right\} \quad \Longrightarrow \quad Cx + d \leq \beta(Fx+g)$$

Bisection algorithm

algorithm

given: interval $[l, u]$ of width $\epsilon_0 = u - l$ that contains the optimal α

repeat until $u - l \leq \epsilon$:

- take $\alpha = (u + l)/2$ and solve the feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & Cx + d \leq \alpha(Fx + g) \\ & Ax \leq b \\ & Fx + g \geq 0 \end{array}$$

- if feasible, take $u := \alpha$; if infeasible, take $l := \alpha$

convergence

- after each update, interval $[l, u]$ contains optimal α
- width $u - l$ is halved at each step, so #iterations $= \lceil \log_2(\epsilon_0/\epsilon) \rceil$

Von Neumann economic growth problem

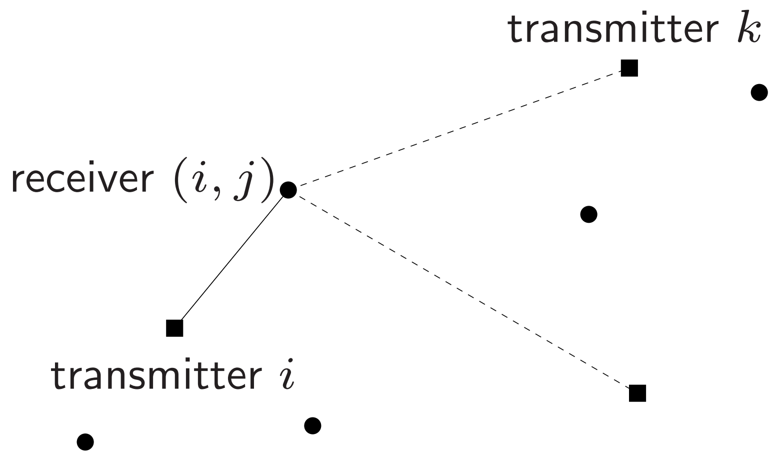
- simple model of an economy with m commodities, n activities (sectors)
- $x_i(t)$ is 'intensity' of activity i in period t
- $a_i^T x(t)$: amount of commodity i consumed in period t
- $b_i^T x(t)$: amount of commodity i produced in period t

maximize growth rate of economy (variables $x(t)$, $x(t+1)$):

$$\begin{array}{ll} \text{maximize} & \min_{i=1,\ldots,n} x_i(t+1)/x_i(t) \\ \text{subject to} & Ax(t+1) \preceq Bx(t) \\ & x(t) \succeq \mathbf{1} \end{array}$$

- cost function is growth rate of sector with slowest growth rate
- a generalized linear-fractional problem

Optimal transmitter power allocation



- m transmitters, mn receivers all at same frequency
- n receivers labeled (i,j), $j = 1,\ldots,n$, listen to transmitter i
- transmitters $k \neq i$ interfere at receivers (i,j)

variables: transmit powers p_i

objective: maximize worst signal to noise-plus-interference ratio

signal to noise-plus-interference ratio at receiver (i, j):

$$\mathrm{SINR}_{ij}(p) = \frac{A_{iji}p_i}{\sum_{k \neq i} A_{ijk}p_k + N_{ij}}$$

- A_{ijk} is path gain from transmitter k to receiver (i, j)
- N_{ij} is (self) noise power of receiver (i, j)

optimization problem

$$\begin{array}{ll} \text{maximize} & \min_{ij} \mathrm{SINR}_{ij}(p) \\ \text{subject to} & 0 \leq p_i \leq p_{\max}, \quad i = 1, \dots, m \end{array}$$

a (generalized) linear-fractional optimization problem in the variables p