3. Convex functions

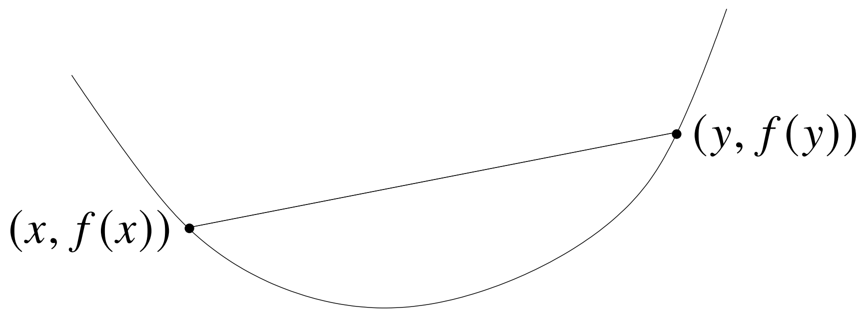
- basic properties and examples
- operations that preserve convexity
- the conjugate function
- quasiconvex functions
- log-concave and log-convex functions

Definition

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \mathbf{dom}\, f$, $x \neq y$, $0 < \theta < 1$

Examples on $\mathbf{R}$

Convex

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- exponential: e^{ax}, for any $a \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

Concave

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$

- affine functions are convex and concave
- all norms are convex

Examples on $\mathbf{R}^n$

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

Examples on $\mathbf{R}^{m\times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \mathrm{tr}(A^T X) + b = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} X_{ij} + b$$

- 2-norm (spectral norm): maximum singular value

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

Extended-value extension

extended-value extension $\tilde{f}$ of f is

$$\tilde{f}(x) = f(x), \quad x \in \mathbf{dom}\, f, \qquad \tilde{f}(x) = \infty, \quad x \notin \mathbf{dom}\, f$$

often simplifies notation; for example, the condition

$$0 \le \theta \le 1 \quad \Longrightarrow \quad \tilde{f}(\theta x + (1-\theta)y) \le \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

(as an inequality in $\mathbf{R} \cup \{\infty\}$), means the same as the two conditions

- $\mathbf{dom}\, f$ is convex
- for $x, y \in \mathbf{dom}\, f$,

$$0 \le \theta \le 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y)$$

Restriction of a convex function to a line

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if the function $g : \mathbf{R} \to \mathbf{R}$,

$$g(t) = f(x + tv), \qquad \mathbf{dom}\, g = \{t \mid x + tv \in \mathbf{dom}\, f\}$$

is convex (in t) for any $x \in \mathbf{dom}\, f$, $v \in \mathbf{R}^n$

can check convexity of f by checking convexity of functions of one variable

Example: $f : \mathbf{S}^n \to \mathbf{R}$ with $f(X) = \log\det X$, $\mathbf{dom}\, f = \mathbf{S}^n_{++}$

$$\begin{aligned} g(t) = \log\det(X + tV) &= \log\det X + \log\det(I + tX^{-1/2}VX^{-1/2}) \\ &= \log\det X + \sum_{i=1}^{n} \log(1 + t\lambda_i) \end{aligned}$$

where λ_i are the eigenvalues of $X^{-1/2}VX^{-1/2}$

g is concave in t (for any choice of $X \succ 0$, V); hence f is concave

First-order condition

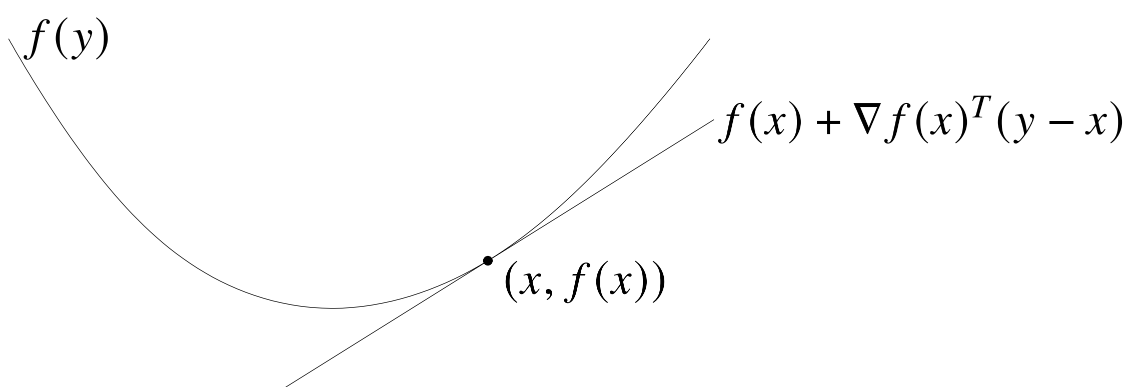
f is **differentiable** if $\mathrm{dom}\, f$ is open and the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n}\right)$$

exists at each $x \in \mathrm{dom}\, f$

First-order condition: differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathrm{dom}\, f$$



first-order approximation of f is global underestimator

Second-order conditions

f is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in \mathbf{dom}\, f$

Second-order conditions: for twice differentiable f with convex domain

- f is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

- if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then f is strictly convex

Examples

Quadratic function: $f(x) = (1/2)x^T P x + q^T x + r$ (with $P \in \mathbf{S}^n$)

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

convex if $P \succeq 0$

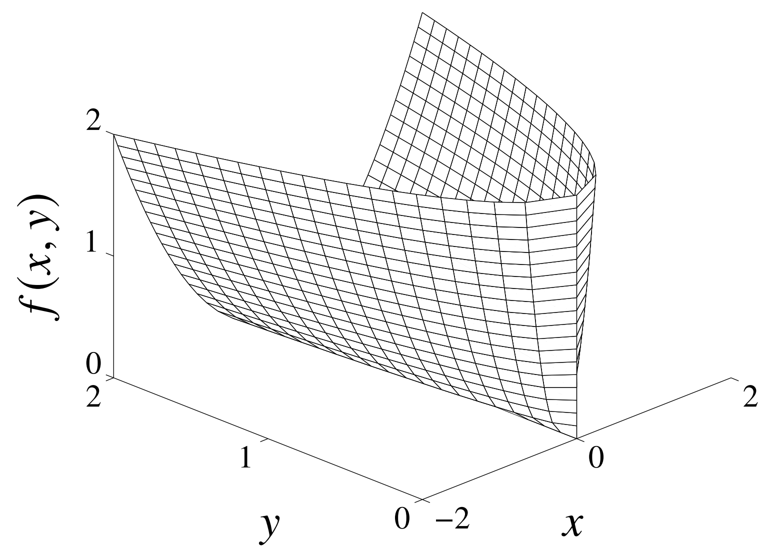
Least squares objective: $f(x) = \|Ax - b\|_2^2$

$$\nabla f(x) = 2A^T(Ax - b), \qquad \nabla^2 f(x) = 2A^T A$$

convex (for any A)

Quadratic-over-linear function: $f(x, y) = x^2/y$

$$\begin{aligned} \nabla^2 f(x,y) &= \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0 \\ \mathbf{dom}\, f &= \{(x,y) \mid y > 0\} \end{aligned}$$



Examples

Log-sum-exp function: $f(x) = \log \sum_{k=1}^{n} \exp x_k$ is convex

$$\nabla^2 f(x) = \frac{1}{\mathbf{1}^T z} \mathbf{diag}(z) - \frac{1}{(\mathbf{1}^T z)^2} z z^T \qquad \text{with } z_k = \exp x_k$$

to show $\nabla^2 f(x) \succeq 0$, we must verify that $v^T \nabla^2 f(x) v \geq 0$ for all v:

$$v^T \nabla^2 f(x) v = \frac{(\sum_{k=1}^{n} z_k v_k^2)(\sum_{k=1}^{n} z_k) - (\sum_{k=1}^{n} v_k z_k)^2}{(\sum_{k=1}^{n} z_k)^2} \geq 0$$

since $(\sum_k v_k z_k)^2 \leq (\sum_k z_k v_k^2)(\sum_k z_k)$ (from Cauchy–Schwarz inequality)

Geometric mean: $f(x) = (\prod_{k=1}^{n} x_k)^{1/n}$ on $\mathbf{R}^n_{++}$ is concave

(similar proof as for log-sum-exp)

Epigraph and sublevel set

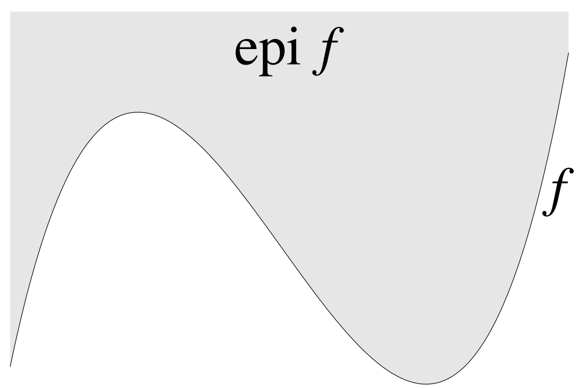
α**-sublevel set** of $f : \mathbf{R}^n \to \mathbf{R}$:

$$C_\alpha = \{x \in \mathrm{dom}\, f \mid f(x) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

Epigraph of $f : \mathbf{R}^n \to \mathbf{R}$:

$$\mathrm{epi}\, f = \{(x, t) \in \mathbf{R}^{n+1} \mid x \in \mathrm{dom}\, f,\ f(x) \leq t\}$$



f is convex if and only if $\mathrm{epi}\, f$ is a convex set

Jensen's inequality

Basic inequality: if f is convex, then for $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

Extension: if f is convex, then

$$f(\mathbf{E}\, z) \leq \mathbf{E}\, f(z)$$

for any random variable z

basic inequality is special case with discrete distribution

$$\mathbf{prob}(z = x) = \theta, \qquad \mathbf{prob}(z = y) = 1 - \theta$$

Operations that preserve convexity

methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that f is obtained from simple convex functions by operations that preserve convexity

    - nonnegative weighted sum
    - composition with affine function
    - pointwise maximum and supremum
    - composition
    - minimization
    - perspective

Positive weighted sum and composition with affine function

Nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

Sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

Composition with affine function: $f(Ax + b)$ is convex if f is convex

Examples

- logarithmic barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i,\ i = 1, \ldots, m\}$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

Pointwise maximum

if $f_1, \ldots, f_m$ are convex, then the function

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

is convex

Examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$ is convex
- sum of r largest components of $x \in \mathbf{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

  is convex ($x_{[i]}$ is ith largest component of x)

  proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

Pointwise supremum

if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then the function

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex

Examples

- *support function* of a set: $S_C(x) = \sup_{y \in C} y^T x$ is convex for any set C
- distance to farthest point in a set C:

$$f(x) = \sup_{y \in C} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$,

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$

Proof.

- suppose $f(x, y)$ is a convex function of x, for any fixed $y \in \mathcal{A}$ (and $\mathcal{A} \neq \emptyset$)
- this means that for all $y \in \mathcal{A}$, x_1, x_2, and $\theta \in [0, 1]$,

$$f(\theta x_1 + (1-\theta)x_2, y) \leq \theta f(x_1, y) + (1-\theta) f(x_2, y)$$

- for simplicity, we use the extended-value convention (where $0 \cdot \infty = 0$)

convexity of g follows from

$$\begin{aligned} g(\theta x_1 + (1-\theta)x_2) &= \sup_{y \in \mathcal{A}} f(\theta x_1 + (1-\theta)x_2, y) \\ &\leq \sup_{y \in \mathcal{A}} (\theta f(x_1, y) + (1-\theta) f(x_2, y)) \\ &\leq \theta \sup_{y \in \mathcal{A}} f(x_1, y) + (1-\theta) \sup_{y \in \mathcal{A}} f(x_2, y) \\ &= \theta g(x_1) + (1-\theta) g(x_2) \end{aligned}$$

Partial minimization

if $f(x, y)$ is convex in (x, y) and C is a convex set, then the function

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex

Examples

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$ with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \qquad C \succ 0$$

  minimizing over y gives $g(x) = \inf_y f(x, y) = x^T (A - BC^{-1}B^T)x$

  g is convex, hence Schur complement $A - BC^{-1}B^T \succeq 0$

- distance to a set: $d(x, S) = \inf_{y \in S} \|x - y\|$ is convex if S is convex

Proof.

- suppose $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is jointly convex in (x, y)
- without loss of generality, we make C part of the domain of f, *i.e.*, redefine

$$\mathbf{dom}\, f := \mathbf{dom}\, f \cap \{(x, y) \mid y \in C\}, \qquad C := \mathbf{R}^m$$

- convexity of $f(x, y)$ jointly in (x, y) means that for all $x_1, y_1, x_2, y_2, \theta \in [0, 1]$,

$$f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \le \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2)$$

- assume $\inf_y f(x, y) > -\infty$ for all x (we don't allow functions that take value $-\infty$)

convexity of g follows from

$$\begin{aligned}
g(\theta x_1 + (1-\theta)x_2) &= \inf_y f(\theta x_1 + (1-\theta)x_2, y) \\
&= \inf_{y_1, y_2} f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\
&\le \inf_{y_1, y_2} (\theta f(x_1, y_1) + (1-\theta) f(x_2, y_2)) \\
&= \theta \inf_{y_1} f(x_1, y_1) + (1-\theta) \inf_{y_2} f(x_2, y_2) \\
&= \theta g(x_1) + (1-\theta) g(x_2)
\end{aligned}$$

Summary of minimization/maximization rules

if we include the counterparts for concave functions, there are four rules

Maximization

$$g(x) = \sup_{y \in C} f(x, y)$$

- g is convex if f is convex in x for fixed y; C can be any set
- g is concave if f is jointly concave in (x, y) and C is a convex set

Minimization

$$g(x) = \inf_{y \in C} f(x, y)$$

- g is convex if f is jointly convex in (x, y) and C is a convex set
- g is concave if f is concave in x for fixed y; C can be any set

Composition with scalar functions

composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$:

$$f(x) = h(g(x))$$

f is convex if h is convex and one of the following three cases holds

g is convex and $\tilde{h}$ nondecreasing
g is concave and $\tilde{h}$ nonincreasing
g is affine

- monotonicity properties of h must hold for extended-value extension $\tilde{h}$
- quick proof (for $n = 1$, differentiable g, h)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

Examples

- $\exp g(x)$ is convex if g is convex
- $1/g(x)$ is convex if g is concave and positive

Proof (first composition rule)

- suppose g is convex and h is convex
- suppose $\tilde{h}$ is nondecreasing: this means that

$$y \preceq x, \quad x \in \mathbf{dom}\, h \qquad \Longrightarrow \qquad y \in \mathbf{dom}\, h, \quad h(y) \leq h(x)$$

consider convex combination of points $x_1, x_2 \in \mathbf{dom}\, f$

- $x_1, x_2 \in \mathbf{dom}\, f$ means that $x_1, x_2 \in \mathbf{dom}\, g$ and $g(x_1), g(x_2) \in \mathbf{dom}\, h$
- by convexity of g, the convex combination $\theta x_1 + (1-\theta)x_2$ is in $\mathbf{dom}\, g$ and

$$g(\theta x_1 + (1-\theta)x_2) \preceq \theta g(x_1) + (1-\theta)g(x_2)$$

- by monotonicity of $\tilde{h}$ and convexity of h, $g(\theta x_1 + (1-\theta)x_2) \in \mathbf{dom}\, h$ and

$$\begin{aligned} h(g(\theta x_1 + (1-\theta)x_2)) &\leq h(\theta g(x_1) + (1-\theta)g(x_2)) \\ &\leq \theta h(g(x_1)) + (1-\theta)h(g(x_2)) \end{aligned}$$

Vector composition

composition of $g : \mathbf{R}^n \to \mathbf{R}^k$ and $h : \mathbf{R}^k \to \mathbf{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

f is convex if h is convex and for each i, one of the following three cases holds

g_i is convex and $\tilde{h}$ nondecreasing in its ith argument
g_i is concave and $\tilde{h}$ is nonincreasing in its ith argument
g_i is affine

- $\tilde{h}$ is extended-value extension of h
- quick proof (for $n = 1$, differentiable g, h)

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)$$

Examples

- $\sum_{i=1}^{m} \log g_i(x)$ is concave if g_i are concave and positive
- $\log \sum_{i=1}^{m} \exp g_i(x)$ is convex if g_i are convex

Proof (first composition rule)

- suppose $g_1, \ldots, g_k$ is convex and h is convex
- Jensen's inequality for $g_1, \ldots, g_k$ can be written as a vector inequality

$$g(\theta x_1 + (1-\theta)x_2) \preceq \theta g(x_1) + (1-\theta)g(x_2)$$

- suppose $\tilde{h}$ is nondecreasing in each argument; this means that

$$y \preceq x, \quad x \in \operatorname{dom} h \qquad \Longrightarrow \qquad y \in \operatorname{dom} h, \quad h(y) \leq h(x)$$

consider a convex combination of points $x_1, x_2 \in \operatorname{dom} f$

- $x_1, x_2 \in \operatorname{dom} f$ means that $x_1, x_2 \in \operatorname{dom} g$ and $g(x_1), g(x_2) \in \operatorname{dom} h$
- by convexity of $g_1, \ldots, g_k$, the convex combination $\theta x_1 + (1-\theta)x_2 \in \operatorname{dom} g$ and

$$g(\theta x_1 + (1-\theta)x_2) \preceq \theta g(x_1) + (1-\theta)g(x_2)$$

- hence, by monotonicity of $\tilde{h}$ and convexity of h, $g(\theta x_1 + (1-\theta)x_2) \in \operatorname{dom} h$ and

$$\begin{aligned} h(g(\theta x_1 + (1-\theta)x_2)) &\leq h(\theta g(x_1) + (1-\theta)g(x_2)) \\ &\leq \theta h(g(x_1)) + (1-\theta)h(g(x_2)) \end{aligned}$$

Perspective

the **perspective** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the function $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$,

$$g(x,t) = tf(x/t), \qquad \mathbf{dom}\, g = \{(x,t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}$$

g is convex if f is convex

Examples

- $f(x) = x^T x$ is convex; hence $g(x,t) = x^T x/t$ is convex for $t > 0$
- negative logarithm $f(x) = -\log x$ is convex; hence relative entropy

$$g(x,t) = t\log t - t\log x$$

  is convex on $\mathbf{R}^2_{++}$
- if f is convex, then

$$g(x) = (c^T x + d) f\left((Ax+b)/(c^T x + d)\right)$$

  is convex on $\{x \mid c^T x + d > 0,\ (Ax+b)/(c^T x + d) \in \mathbf{dom}\, f\}$

Proof.

- consider convex combination of two points $(x_1, t_1), (x_2, t_2) \in \mathbf{dom}\, g$:

$$t_1 > 0, \quad x_1/t_1 \in \mathbf{dom}\, f, \qquad t_2 > 0, \quad x_2/t_2 \in \mathbf{dom}\, f$$

- we verify Jensen's inequality:

$$\begin{array}{ll} g(\theta x_1 + (1-\theta)x_2,\ \theta t_1 + (1-\theta)t_2) & \\ = & (\theta t_1 + (1-\theta)t_2)\, f(\dfrac{\theta x_1 + (1-\theta)x_2}{\theta t_1 + (1-\theta)t_2}) \\ = & (\theta t_1 + (1-\theta)t_2)\, f(\dfrac{\theta t_1}{\theta t_1 + (1-\theta)t_2}(x_1/t_1) + \dfrac{(1-\theta)t_2}{\theta t_1 + (1-\theta)t_2}(x_2/t_2)) \\ \leq & \theta t_1 f(x_1/t_1) + (1-\theta)t_2 f(x_2/t_2) \\ = & \theta g(x_1, t_1) + (1-\theta) g(x_2, t_2) \end{array}$$

the inequality follows from convexity of f:

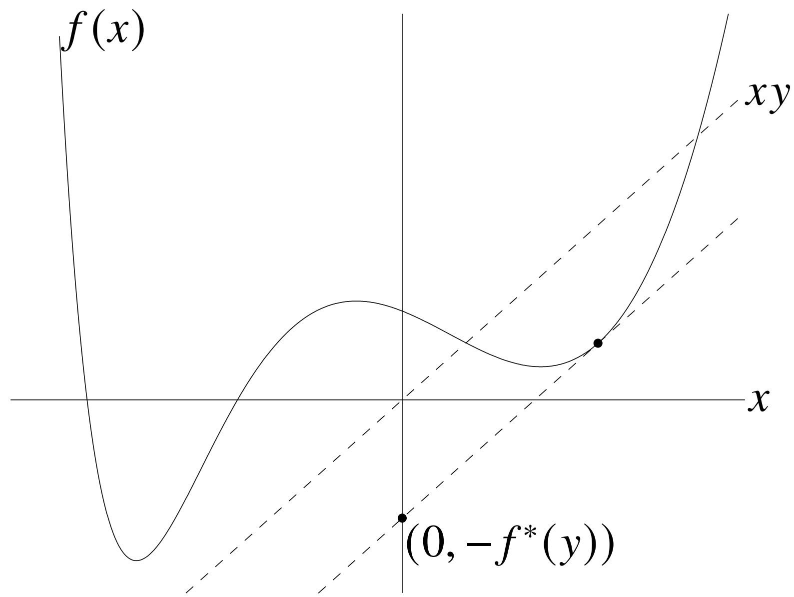
$$f(\mu(x_1/t_1) + (1-\mu)(x_2/t_2)) \leq \mu f(x_1/t_1) + (1-\mu) f(x_2/t_2)$$

where $\mu = \theta t_1/(\theta t_1 + (1-\theta)t_2)$

The conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$



- f^* is convex (even if f is not)
- will be useful when we discuss duality

Examples

- negative logarithm $f(x) = -\log x$

$$\begin{aligned} f^*(y) &= \sup_{x>0} (xy + \log x) \\ &= \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

- strictly convex quadratic $f(x) = (1/2)x^T Qx$ with $Q \in \mathbf{S}^n_{++}$

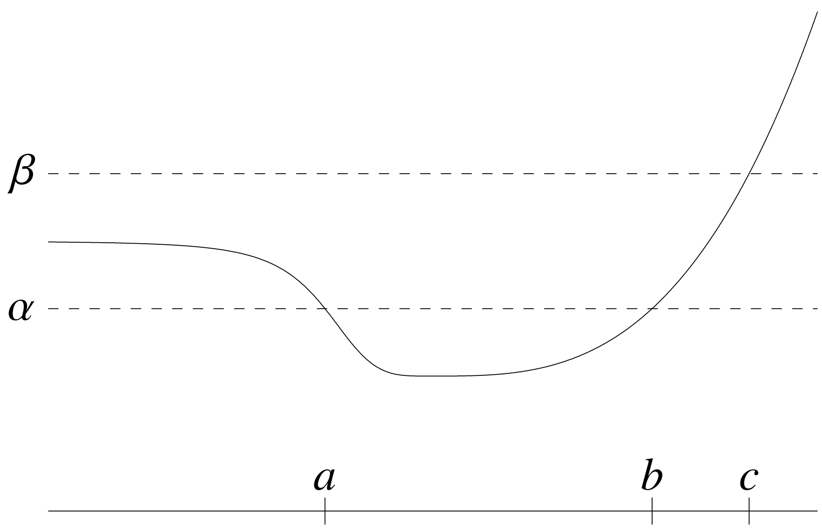
$$\begin{aligned} f^*(y) &= \sup_{x} (y^T x - (1/2)x^T Qx) \\ &= \frac{1}{2} y^T Q^{-1} y \end{aligned}$$

Quasiconvex functions

$f : \mathbf{R}^n \to \mathbf{R}$ is quasiconvex if $\mathrm{dom}\, f$ is convex and the sublevel sets

$$S_\alpha = \{x \in \mathrm{dom}\, f \mid f(x) \leq \alpha\}$$

are convex for all α



$$S_\alpha = [a, b]$$
$$S_\beta = (-\infty, c)$$

- f is quasiconcave if $-f$ is quasiconvex
- f is quasilinear if it is quasiconvex and quasiconcave

Examples

- $\sqrt{|x|}$ is quasiconvex on $\mathbf{R}$
- $\mathrm{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$ is quasilinear
- $\log x$ is quasilinear on $\mathbf{R}_{++}$
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbf{R}^2_{++}$
- linear-fractional function

$$f(x) = \frac{a^T x + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

  is quasilinear

- distance ratio

$$f(x) = \frac{\|x - a\|_2}{\|x - b\|_2}, \qquad \mathbf{dom}\, f = \{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$$

  is quasiconvex

Internal rate of return

- cash flow $x = (x_0, \ldots, x_n)$; x_i is payment in period i (to us if $x_i > 0$)
- we assume $x_0 < 0$ and $x_0 + x_1 + \cdots + x_n > 0$
- *present value* of cash flow x, for interest rate r:

$$\mathrm{PV}(x, r) = \sum_{i=0}^{n} (1+r)^{-i} x_i$$

- *internal rate of return* is smallest interest rate for which $\mathrm{PV}(x, r) = 0$:

$$\mathrm{IRR}(x) = \inf\{r \geq 0 \mid \mathrm{PV}(x, r) = 0\}$$

IRR is quasiconcave: superlevel set is intersection of open halfspaces

$$\mathrm{IRR}(x) \geq R \quad \Longleftrightarrow \quad \sum_{i=0}^{n} (1+r)^{-i} x_i > 0 \quad \text{for } 0 \leq r < R$$

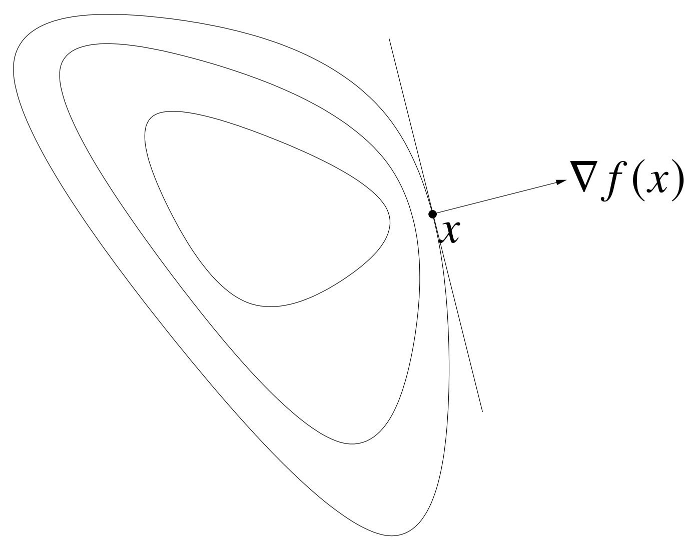
Properties

Modified Jensen inequality: for quasiconvex f

$$0 \leq \theta \leq 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$$

First-order condition: differentiable f with cvx domain is quasiconvex iff

$$f(y) \leq f(x) \quad \Longrightarrow \quad \nabla f(x)^T (y - x) \leq 0$$



Sums: sums of quasiconvex functions are not necessarily quasiconvex

Log-concave and log-convex functions

a positive function f is log-concave if $\log f$ is concave:

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta} \quad \text{for } 0 \leq \theta \leq 1$$

f is log-convex if $\log f$ is convex

- powers: x^a on $\mathbf{R}_{++}$ is log-convex for $a \leq 0$, log-concave for $a \geq 0$
- many common probability densities are log-concave, *e.g.*, normal:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})}$$

- cumulative Gaussian distribution function Φ is log-concave

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$$

Properties of log-concave functions

- twice differentiable f with convex domain is log-concave if and only if

$$f(x)\nabla^2 f(x) \preceq \nabla f(x)\nabla f(x)^T \quad \text{for all } x \in \mathbf{dom}\, f$$

- product of log-concave functions is log-concave

- sum of log-concave functions is not always log-concave

- integration: if $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is log-concave, then

$$g(x) = \int f(x, y)\, dy$$

  is log-concave (not easy to show)

Consequences of integration property

- convolution $f * g$ of log-concave functions f, g is log-concave

$$(f * g)(x) = \int f(x - y)g(y)dy$$

- if $C \subseteq \mathbf{R}^n$ convex and y is a random variable with log-concave p.d.f. then

$$f(x) = \mathbf{prob}(x + y \in C)$$

  is log-concave

  proof: write $f(x)$ as integral of product of log-concave functions

$$f(x) = \int g(x + y)p(y)\, dy, \qquad g(u) = \begin{cases} 1 & u \in C \\ 0 & u \notin C, \end{cases}$$

  p is p.d.f. of y

Example: yield function

$$Y(x) = \mathbf{prob}(x + w \in S)$$

- $x \in \mathbf{R}^n$: nominal parameter values for product
- $w \in \mathbf{R}^n$: random variations of parameters in manufactured product
- S: set of acceptable values

if S is convex and w has a log-concave p.d.f., then

- Y is log-concave
- yield regions $\{x \mid Y(x) \geq \alpha\}$ are convex