L. Vandenberghe
T.L. Jensen

Ph.D. Course
Algorithms for Large-Scale Convex Optimization
DTU, August 23-27, 2010

Project Assignment

In the project we apply the accelerated proximal gradient method (FISTA) of page 3-26 to the problem

$$\text{minimize} \quad \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|Wx\|_1. \tag{1}$$

The vector b is a blurred $m \times n$ image, represented as an mn-vector. The variable x is the reconstructed image, also represented as an mn-vector. The matrix A represents a linear blurring operation. The matrix W is orthogonal and represents a wavelet transform. The parameter λ is a positive weight.

Notation

If b is an mn-vector, we use $\mathbf{mat}(b)$ to denote the $m \times n$-matrix constructed from the entries of b as follows:

$$\mathbf{mat}(b) = \begin{bmatrix} b_1 & b_{m+1} & \cdots & b_{(n-1)m+1} \\ b_2 & b_{m+2} & \cdots & b_{(n-1)m+2} \\ \vdots & \vdots & & \vdots \\ b_m & b_{2m} & \cdots & b_{mn} \end{bmatrix}.$$

The inverse operation is denote $\mathbf{vec}$: if B is an $m \times n$-matrix, then $\mathbf{vec}(B)$ denotes the entries of B stored in column-major order as an mn-vector

$$\mathbf{vec}(B) = (B_{11}, \ldots, B_{m1}, B_{12}, \ldots, B_{m2}, \ldots, B_{1n}, \ldots, B_{mn}).$$

In Matlab these operations can be implemented as `b = B(:)` and `B = reshape(b,m,n)`.

Code fragments

The exercise requires the following Matlab toolboxes.

- The Matlab Image Processing Toolbox.
- The HNO Image Deblurring toolbox by P.C. Hansen, J.H. Nagy, and D.P. O'Leary, available at

  `www2.imm.dtu.dk/~pch/HNO/HNO.zip`

- The Matlab Wavelet toolbox of G. Peyre, available at

  `www.ceremade.dauphine.fr/~peyre/matlab/wavelets/content.html`

The problem parameters A and b are specified in the mat-file `project_data.mat` on the course website. Loading the mat-file will create two matrices, B and S. The matrix B contains grayscale levels of a blurred 256×256-image and is equal to $B = \mathbf{mat}(b)$. Use the following code to view the image.

```
load project_data;    % defines two matrices B and S
figure(1);  clf;  imshow(B);  title('Observed Image')
```

The blurring operator A is defined as $A = C^T \Lambda C$ where C is a 2-dimensional discrete cosine transform (DCT) matrix and Λ is the diagonal matrix

$$\Lambda = \mathbf{diag}(\mathbf{vec}(S))$$

with S an $m \times n$ matrix defined in the file `project_data`. Note that C is orthogonal ($C^T C = I$) and A symmetric.

We will use the HNO toolbox to evaluate C, C^T, and A. The following code defines four Matlab functions `A()`, `C()`, `CT()`.

```
addpath HNO  % path to HNO Toolbox
C = @(X) dcts2(X);
CT = @(Y) idcts2(Y);
A = @(X) CT(S.*C(X));
```

After definining these functions you can use function calls `C(X)`, `CT(Y)`, `A(X)` to evaluate

$$\mathbf{mat}(C(\mathbf{vec}(X))), \qquad \mathbf{mat}(C^T(\mathbf{vec}(Y))), \qquad \mathbf{mat}(A(\mathbf{vec}(X))).$$

Similarly, we use the Wavelets toolbox to evaluate W and its transpose. The following code defines two functions `W()` and `WT()` that implement a Symmlet wavelet transform with 6 levels.

```
addpath toolbox_wavelets/toolbox_wavelets
addpath toolbox_wavelets/toolbox_wavelets/toolbox
options.wavelet_type = 'symmlet';
W = @(X) perform_wavelet_transform(X, 6, 1, options);
WT = @(bX) perform_wavelet_transform(bX, 6, -1, options);
```

With these definitions the function calls `W(X)` and `WT(Y)` evaluate

$$\mathbf{mat}(W(\mathbf{vec}(X))), \qquad \mathbf{mat}(W^T(\mathbf{vec}(Y))).$$

Note that W is orthogonal ($W^T W = I$).

Assignment

1. Implement an accelerated proximal gradient method (FISTA) for the problem (1). Terminate the iteration when $\|G_t(x)\|_2/(mn) \leq 10^{-7}$.

2. Explain how can you exploit the structure of A (specifically, the fact that it is diagonalizable by an orthogonal matrix C) to speed up the algorithm.

3. Consider another problem where we set $W = C$, *i.e.*,

$$\text{minimize} \quad \frac{1}{2}\|Ax - b\|_2^2 + \tilde{\lambda}\|Cx\|_1. \tag{2}$$

   (As before, we take $A = C^T \Lambda C$.) Derive a closed form solution of this problem.

4. Compare the solutions of problem (1), with $\lambda = 0.01$, 0.001, 0.0001, and problem (2), with $\tilde{\lambda} = 0.05$, 0.005, 0.0005.

Submit your answers by email to `vandenbe@ee.ucla.edu` and `tlj@es.aau.dk` in the form of an m-file and a short report in pdf.