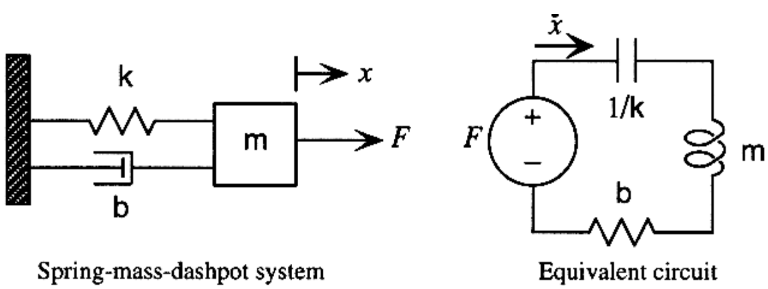
Dynamics

Reference:

- **Senturia's Book: Chapter 7**

Equivalent Circuit for Spring-Mass-Dashpot Systems



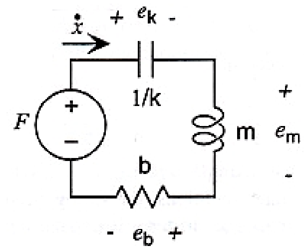
(All 3 elements share the same displacement) → Series Connection

Connection rule for the equivalent circuit for e → V convention:

- **Elements share a common Flow or Displacement → Connected in Series**
- **Elements share a common Effort → Connected in Parallel**

Dynamic Response from Equivalent Circuit

KVL: $-F + e_k + e_m + e_b = 0$



Dynamic Response → Laplace Transform

$$Z_C(s) = \frac{1}{sC}$$
$$Z_L(s) = sL$$
$$Z_b(s) = b$$

$$e(s) = Z(s)f(s) = \left(sL + b + \frac{1}{sC}\right)f(s)$$

Electrical-to-Mechanical Mapping

$$L \to m \qquad b \to b \qquad \frac{1}{C} \to k$$

$$e(s) \to F(s) \qquad f(s) \to \dot{x}(s)$$

$$\frac{\dot{x}(s)}{F(s)} = \frac{1}{sm + b + k/s} = \frac{s}{s^2 m + sb + k}$$

Second-order system

Linear Systems

Consider mass-spring-dashpot system:

$$F = k \cdot x + b \cdot \dot{x} + m \cdot \ddot{x}$$

The "state" of the system can be described by the <u>position</u> and <u>velocity</u>.

$$x_1 : Position$$
$$x_2 = \dot{x}_1 : Velocity$$

Define state variables:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \dot{x}_1 \end{pmatrix}$$

State Equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{m}x_1 - \dfrac{b}{m}x_2 + \dfrac{1}{m}F \end{cases}$$

Matrix Formulation for Linear Systems

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{b}{m}x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m}F \end{pmatrix}$$

$$\longrightarrow \quad \dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} F$$

Generalize:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

x: state variable
u: system inputs
y: output vector
A,B,C,D: matrices constitute the system

Numerical Solution (MathCAD Example)

Parameters $m := 1$ $k := 1$ $b := 0.5$ $F := 1$

Initial Conditions $x := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $u := F$

State Equation

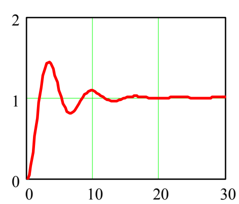
$$D(t, x) := \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-b}{m} \end{pmatrix} \cdot x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} \cdot u$$

Numerical Solu: $Z := \mathrm{rkfixed}(x, 0, 30, 100, D)$

Fourth-order Runge-Kutta method

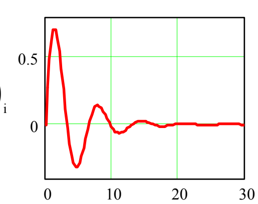
$i := 0 .. \mathrm{rows}(Z) - 1$

Position vs. Time



$\left(Z^{\langle 1 \rangle}\right)_i$ vs. $\left(Z^{\langle 0 \rangle}\right)_i$

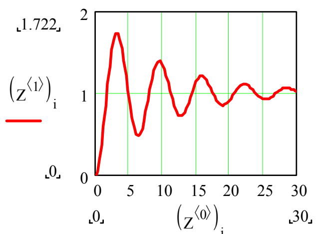
Velocity vs. Time



$\left(Z^{\langle 2 \rangle}\right)_i$ vs. $\left(Z^{\langle 0 \rangle}\right)_i$

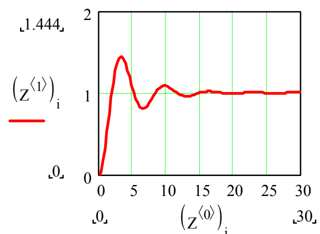
Dynamic Response

b = 0.2 * m



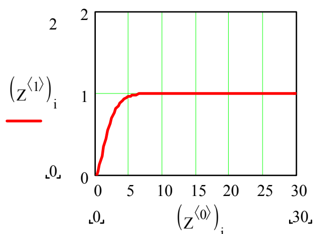
Under-damping

b = 1 * m



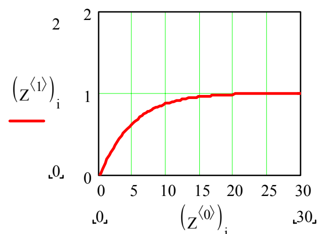
Under-damping

b = 2 * m



Critical-damping

b = 5 * m



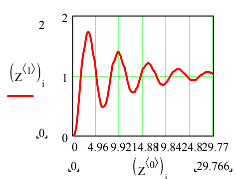
Over-damping

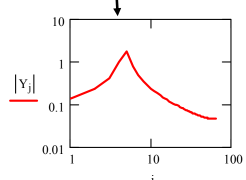
Q and Frequency Response

Quality Factor: $$Q = \frac{\omega_0}{2\alpha} = \frac{m\omega_0}{b}$$

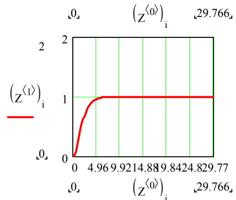
Fourier Transform of Temporal Response

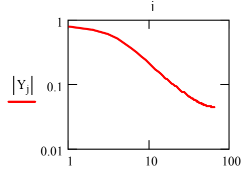
Q = 5
Under-Damped



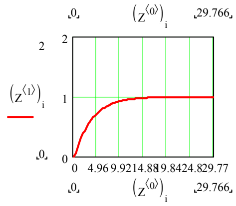


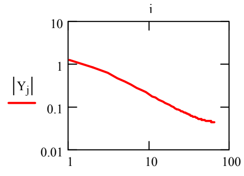
Q = 0.5
Critically Damped





Q = 0.25
Over-Damped





Time

Frequency

Laplace Transform (Frequency Domain)

Single-sided Lapalace transform:

$$X(s) = \int_0^\infty x(t)dt$$

State equation:

$$sX(s) - x(0) = AX(s) + BU(s)$$

Solution:

$$\boxed{X(s) = (s\mathbf{I} - A)^{-1}(x(0) + BU(s))}$$

- **Set U = 0:** $X_{zir}(s) = (s\mathbf{I} - A)^{-1}x(0)$ **Zero-input response**
- **Set x(0) = 0:** $X_{zsr}(s) = (s\mathbf{I} - A)^{-1}BU(s)$ **Zero-state response**

Resonance

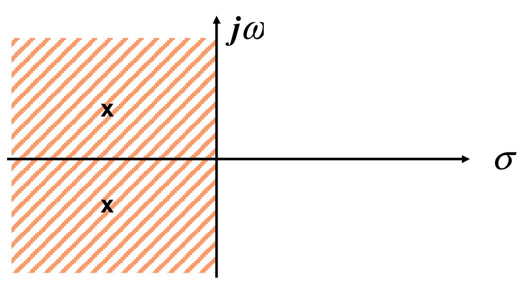
Resonance: $|s\mathbf{I} - A| = 0$

Natural Resonant Frequency: **the s-values at which (poles)** $|s\mathbf{I} - A| = 0$

In Time Domain: $x(t) \sim e^{s_i t}$ **where s_i are pole**

Note s_i is complex: $s_i = \sigma + j\omega$

When Real [s_i] < 0 → Damped sinusoid
When Real [s_i] = 0 → Undamped sinusoid
When Real [s_i] > 0 → Unphysical



Spring-Mass-Dashpot Example

$$sI - A = \begin{pmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{pmatrix}$$

$$|sI - A| = 0$$

$$s^2 + \frac{b}{m}s + \frac{k}{m} = 0$$

$$s_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

Undamped Resonant Frequency $\omega_0 = \sqrt{\frac{k}{m}}$

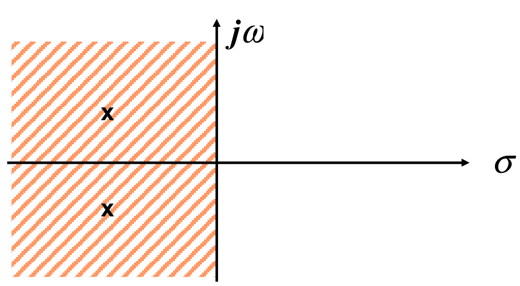
Damping Constant $\alpha = \frac{b}{2m}$

Spring-Mass-Dashpot Example (cont'd)

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$s_{1,2} = -\alpha \pm j\omega_d$$

Damped Resonant Freq. $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$



The poles are always on the left half of the plane for b > 0

Frequency Response of Spring-Mass-Dashpot

Parameters $m := 1$ $k := 1$ $b := 0.2$ $F := 1$

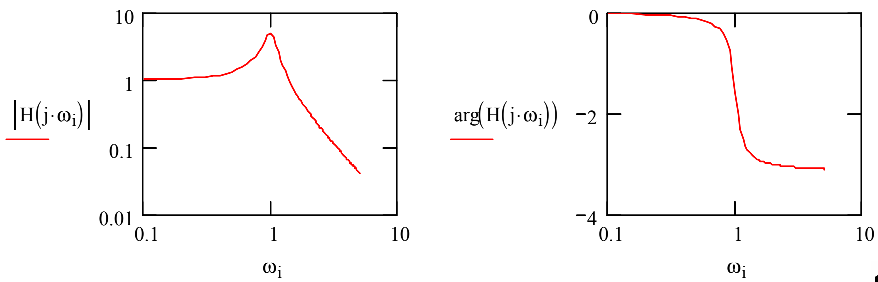
$\omega 0 := \sqrt{\frac{k}{m}}$ $\alpha := \frac{b}{2 \cdot m}$ $\omega d := \sqrt{\omega 0^2 - \alpha^2}$

$s1 := -\alpha + j \cdot \omega d$

$s2 := -\alpha - j \cdot \omega d$

$$H(s) := \frac{1}{m} \cdot \frac{1}{(s - s1) \cdot (s - s2)}$$

$i := 0 .. 100$ $\omega_i := i \cdot \frac{1}{20} + 0.1$

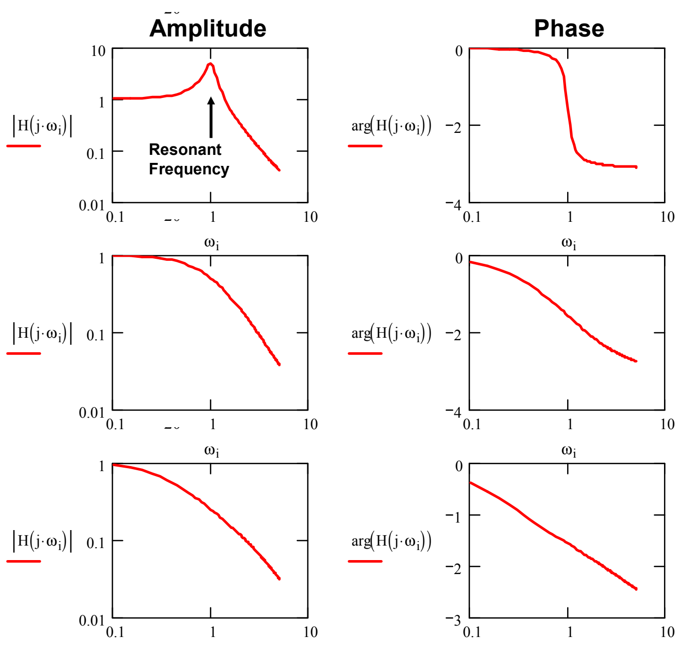


Frequency Response (Bode Plot)

Q = 5
Under-Damped

Q = 0.5
Critically Damped

Q = 0.25
Over-Damped



Nonlinear Dynamics: Linearization about an Operating Point

State: $\dot{x} = f(x,u)$

Output: $\dot{y} = g(x,u)$

Operating Point: $X_0, \; U_0$ satisfy static equations

Linearization:

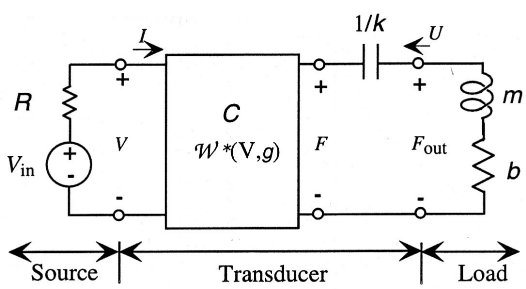
$$x(t) = X_0 + \delta x(t)$$
$$u(t) = U_0 + \delta u(t)$$

Small perturbation around an operating point (DC solution)

$$\Rightarrow \delta\dot{x}(t) = \left.\frac{\partial f}{\partial x}\right|_{X_0,U_0} \cdot \delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{X_0,U_0} \cdot \delta u(t)$$

Matrix   Matrix

Transducer Model for Linearized Actuator



$$V = \frac{Qg}{\varepsilon A}$$

$$F_{out} = \frac{Q^2}{2\varepsilon A} - k(g_0 - g)$$

Find $\delta V, \delta F$

$$F = \frac{\varepsilon A V^2}{2g^2}$$

$$g = g_0 - \frac{\varepsilon A V^2}{2kg^2}$$

Find Operating Point

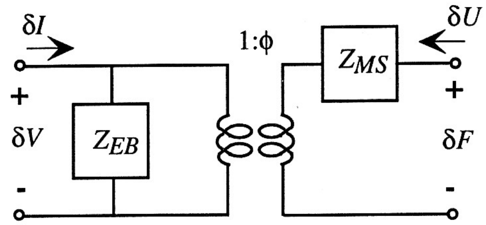
$$Q_0 = \frac{\varepsilon A}{\hat{g}_0} V_{in,0}$$

Linearized Transducer Model

$$\begin{pmatrix}\delta V\\ \delta F\end{pmatrix}=\begin{pmatrix}\frac{\hat{g}_0}{\varepsilon A} & \frac{Q_0}{\varepsilon A}\\ \frac{Q_0}{\varepsilon A} & k\end{pmatrix}\begin{pmatrix}\delta Q\\ \delta g\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\hat{g}_0}{\varepsilon A} & \frac{Q_0}{\varepsilon A}\\ \frac{Q_0}{\varepsilon A} & k\end{pmatrix}\begin{pmatrix}\frac{\delta I}{s}\\ \frac{\delta U}{s}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\hat{g}_0}{s\varepsilon A} & \frac{Q_0}{s\varepsilon A}\\ \frac{Q_0}{s\varepsilon A} & \frac{k}{s}\end{pmatrix}\begin{pmatrix}\delta I\\ \delta U\end{pmatrix}=\begin{pmatrix}Z_{EB} & T_{EM}\\ T_{ME} & Z_{MO}\end{pmatrix}\begin{pmatrix}\delta I\\ \delta U\end{pmatrix}$$



Transformer coupling ratio: $\varphi=\frac{T_{EM}}{Z_{EB}}=\frac{Q_0}{\hat{g}_0}$