Rayleigh-Ritz Method

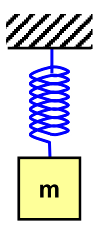
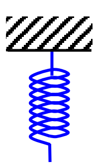
Reference:

- **Senturia's Book: Chapter 10**

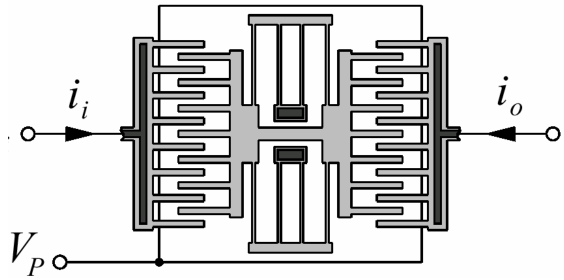
Acknowledgment:

- **Prof. Hiroshi Toshiyoshi (Tokyo University)**

Resonant Frequency

	**Lumped model:** • **Mass is concentrated at one end of the spring** • **Spring has no mass**	$\omega_0 = \sqrt{\frac{k}{m}}$
	**What happen when** • **Mass is distributed over the entire structure?** • **The mass of spring is not negligible?**	$\omega_0 = ?$

Resonant Frequency of Comb Drive Resonator



$$\omega_0 = \sqrt{\frac{k}{m_p + 0.3714 \cdot m_s}}$$

m_p : mass of plate

m_s : mass of spring

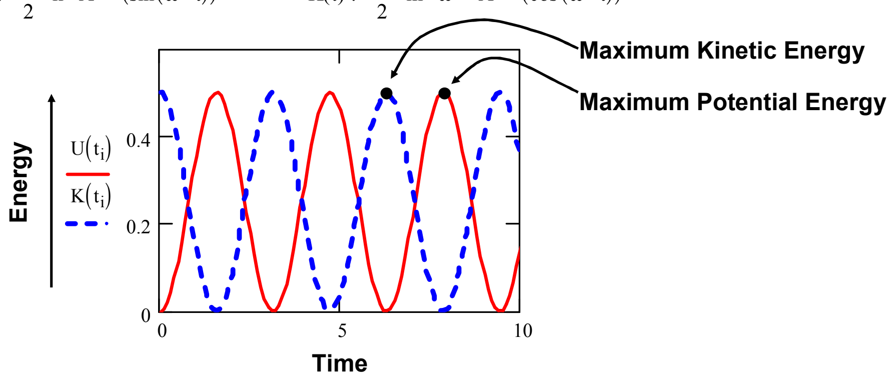
(W. C. Tang, et al.)

How is the weighting factor 0.3714 obtained?

Rayleigh-Ritz Method

$$x(t) := A \cdot \sin(\omega \cdot t)$$

$$U(t) := \frac{1}{2} \cdot k \cdot A^2 \cdot (\sin(\omega \cdot t))^2 \qquad K(t) := \frac{1}{2} \cdot m \cdot \omega^2 \cdot A^2 \cdot (\cos(\omega \cdot t))^2$$

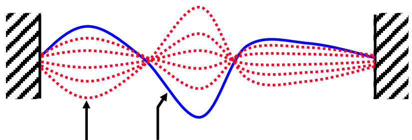


Maximum Kinetic Energy (W_k) = Maximum Potential Energy (W_e)

$$\omega_0 = \sqrt{\frac{W_e}{W_k}}$$

Structure with distributed mass:
→ Masses are weighted by the square of their velocity in the summation

Procedure



Quasistatic displacement function: $\hat{w}(x)$

Assumption: Time-dependent displacement: $\hat{w}(x,t) = \hat{w}(x)\cos(\omega t)$

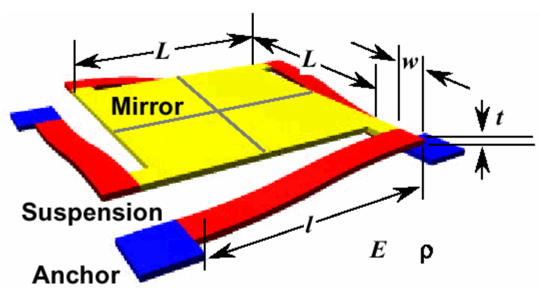
Maximum velocity: $v_{\max}(x) = \left.\frac{\partial \hat{w}(x,t)}{\partial t}\right|_{\omega t=\frac{\pi}{2}} = -\omega \cdot \hat{w}(x)$

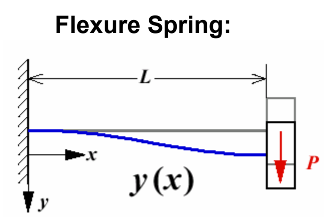
Maximum kinetic energy: $W_k = \int \frac{1}{2}\rho(x)[v_{\max}(x)]^2 dxdydz$

$$= \frac{\omega^2}{2}\int \rho(x)[\hat{w}(x)]^2 dxdydz$$

Resonant Frequency: $$\omega_0^2 = \frac{W_e}{\frac{1}{2}\int \rho(x)[\hat{w}(x)]^2 dxdydz}$$

Example: Piston Mirror with Flexure Spring





Quasistatic solution of spring: $\hat{w}(x) = \frac{P}{12EI}(3lx^2 - 2x^3)$

Total kinetic energy: $W_k = W_{k,spring} + W_{k,plate}$

Kinetic energy of flexure spring: $W_{k,spring} = \frac{\omega^2}{2}\int_0^l \rho \cdot \hat{w}(x)^2 wt \cdot dx$

$$= \frac{13\omega^2}{70}\rho wtl^7\left(\frac{P}{12EI}\right)^2$$

Piston Mirror with Flexure Spring (2)

$$\hat{w}(x) = \frac{P}{12EI}(3lx^2 - 2x^3)$$

Kinetic energy of ¼ Mirror Plate:

$$W_{k,plate} = \frac{\omega^2}{2}\rho\left(\frac{L}{2}\right)^2 t \cdot \hat{w}(l)^2$$

$$= \frac{\omega^2}{8}\rho L^2 t l^6 \left(\frac{P}{12EI}\right)^2$$

Total kinetic energy:

$$W_{k,plate} = \frac{13\omega^2}{70}\rho w t l^7 \left(\frac{P}{12EI}\right)^2 + \frac{\omega^2}{8}\rho L^2 t l^6 \left(\frac{P}{12EI}\right)^2$$

$$= \rho t \left[\frac{13}{70} w l^7 + \frac{1}{8} L^2 l^6\right]\left(\frac{P}{12EI}\right)^2 \omega^2$$

Piston Mirror with Flexure Spring (3)

Maximum potential energy stored in the flexure spring

$$W_e = \int_0^l \frac{M(x,t)^2}{2EI} dx$$

Moment

$$M(x,t) = P\left(\frac{l}{2} - x\right)$$

$$W_e = \frac{P^2}{24EI} l^3$$

Moment of inertia

$$I = \frac{wt^3}{12}$$

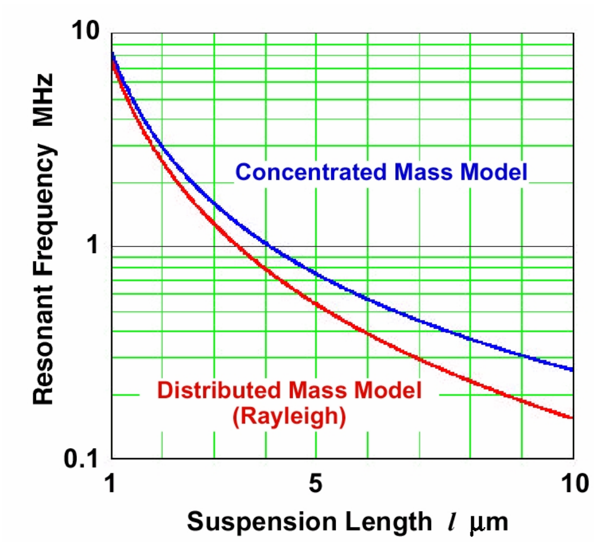
Rayleigh-Ritz Principle

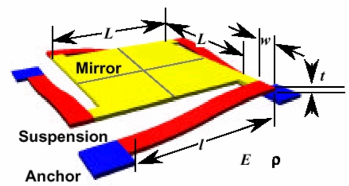
$$W_e = W_{k,total}$$

Resonant Frequency

$$\omega_0 = \sqrt{\frac{Ewt^2}{\rho\left[\frac{13}{35} w l^4 + \frac{1}{4} L^2 l^3\right]}}$$

Piston Mirror with Flexure Spring (4)





$E_{silicon} = 140$ GPa

$\rho_{silicon} = 2.65 \times 10^3$ kg / m^3

$L = 20\ \mu$m

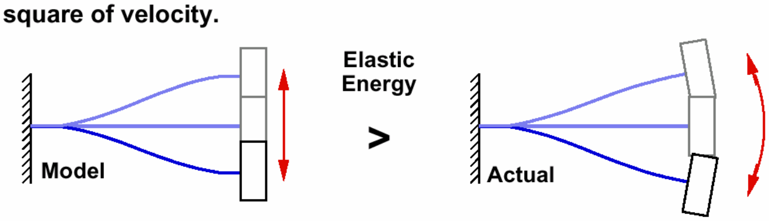
$t = 1\ \mu$m

$w = 5\ \mu$m

Limit of Rayleigh-Ritz Method

Resonant frequencies calculated by Rayleigh's method tend to be higher than those of actual devices due to the following assumptions:

1. **Oscillating beams deflect as if under the static force, neglecting the dynamic inertia of distributed mass (---> Actual beam will deflect with elastic energy smaller than that of theory),**
2. **Viscous loss has been neglected, which is proportional to the square of velocity.**



$$\left[\begin{pmatrix}\text{Contribution}\\ \text{from}\\ \text{Kinetics}\end{pmatrix} + \begin{pmatrix}\text{Contribution}\\ \text{from}\\ \text{Viscosity}\end{pmatrix}\right] \times \omega_0^2 = \begin{pmatrix}\text{Elastic}\\ \text{Energy}\end{pmatrix}$$

Underestimated **Overestimated**