

Lumped Element Modeling

Reading: Chapter 5 of Senturia



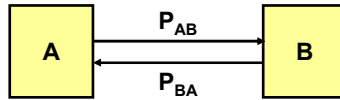
Motivation for Lumped Modeling with Circuit Elements

- Simplified device presentation
- Expressible with equivalent circuit model
 - Can be simulated by SPICE
- Powerful set of tools developed for understanding circuits
- Interface with electronic circuits
 - Modeling of entire system by SPICE



Conjugate Power Variables

Power Flow:



We can express $\begin{cases} P_{AB} = r_1^2 \\ P_{BA} = r_2^2 \end{cases}$ Where r_1 and r_2 are real numbers

$$P_{Net} = P_{AB} - P_{BA} = r_1^2 - r_2^2 = \overbrace{(r_1 + r_2)(r_1 - r_2)}^{\text{Product of 2 real numbers}} = e(t) \cdot f(t)$$

Examples:

$$P = V \cdot I = \text{Voltage} \cdot \text{Current}$$

$$P = F \cdot v = \text{Force} \cdot \text{velocity}$$

Effort \uparrow \uparrow Flow

Conjugate Power Variables



Generalized Displacement and Momentum

In the example $P = F \cdot v = \text{Force} \cdot \text{velocity} = e(t) \cdot f(t)$

We can define
"generalized displacement": $q(t) = \int \text{velocity}(t) dt = \int f(t) dt$

and
"generalized momentum": $p(t) = \int \text{Force}(t) dt = \int e(t) dt$



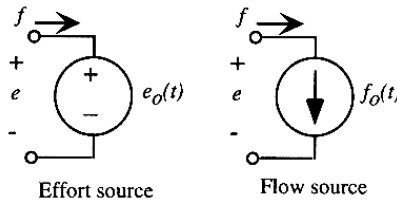
Examples of Conjugate Power Variables

Energy Domain	Effort	Flow	Momentum	Displacement
Mechanical translation	Force F	Velocity \dot{x}, v	Momentum p	Position x
Fixed-axis rotation	Torque τ	Angular velocity ω	Angular momentum J	Angle θ
Electric circuits	Voltage V, v	Current I, i	...	Charge Q
Magnetic circuits	Magnetomotive force MMF	Flux rate $\dot{\phi}$...	Flux ϕ
Incompressible fluid flow	Pressure P	Volumetric flow Q	Pressure momentum Γ	Volume V
Thermal	Temperature T	Entropy flow rate \dot{S}	...	Entropy S

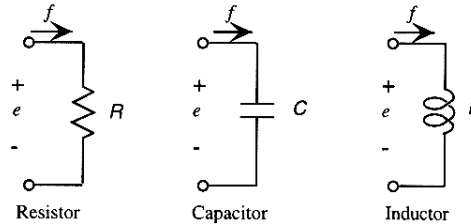


One-Port Circuit Elements

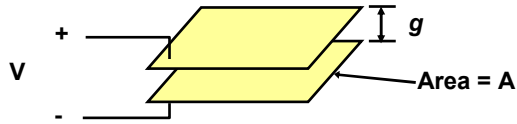
Sources



Circuit Elements

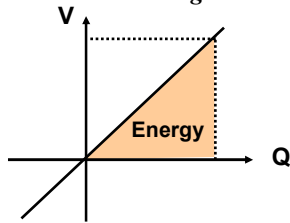


Parallel Plate Capacitor



Linear Capacitor :

$$Q = CV \quad C = \frac{\epsilon A}{g}$$

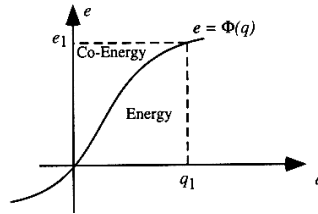


Stored Potential Energy :

$$W(Q) = \frac{Q^2}{2C}$$

Generalized Capacitor:

Well-behaved function that goes through origin of the e-q plane



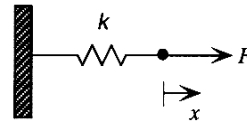
Stored Potential Energy :

$$W(q_1) = \int_0^{q_1} e \cdot dq = \int_0^{q_1} \Phi(q) dq$$



Equivalent Circuit of Spring

	Effort	Flow	Momentum	Displacement
Electrical Circuit	V voltage	I current		Q charge
Mechanical System	F force	v velocity	p = mv	x position



$$V = \frac{1}{C} Q$$

$$W(Q) = \frac{1}{2} \left(\frac{1}{C} \right) \cdot Q^2$$

Electrical

$$F = kx$$

$$W(x) = \frac{1}{2} kx^2$$

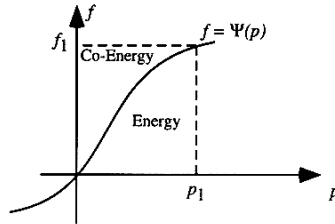
Mechanical

$$\longrightarrow C_{spring} = \frac{1}{k}$$

Equivalent Circuit of a Spring



Generalized Inductors



Stored Kinetic Energy : $W(p_1) = \int_0^{q_1} f \cdot dp = \int_0^{q_1} \Psi(p) dp$

Linear Inductor: $W(Q) = \frac{1}{2} LI^2 = \frac{1}{2} L \left(\frac{dQ}{dt} \right)^2$

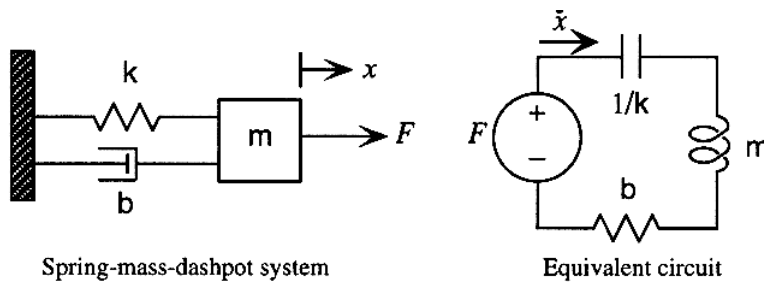
Inertia Mass: $W(p) = \frac{1}{2} mv^2 = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2$

Equivalent Circuit
of a Spring

$$L_{spring} = m$$



Equivalent Circuit for Spring-Mass-Dashpot Systems



(All 3 elements share the same displacement) → Series Connection

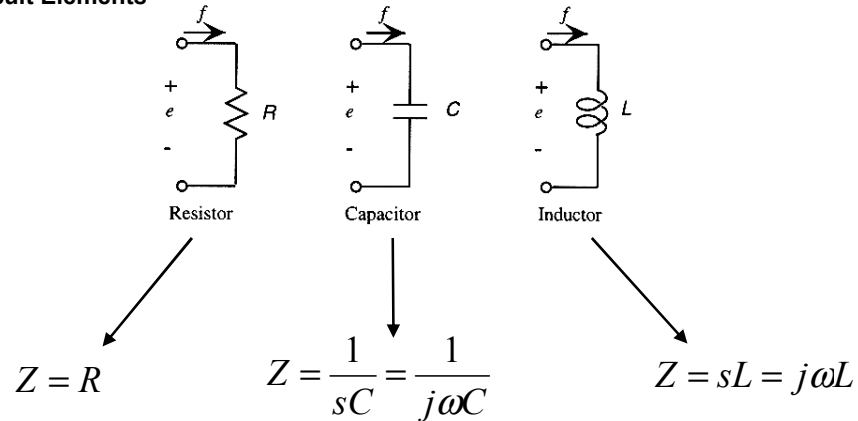
Connection rule for the equivalent circuit for $e \rightarrow V$ convention:

- Elements share a common **Flow or Displacement** → Connected in **Series**
- Elements share a common **Effort** → Connected in **Parallel**



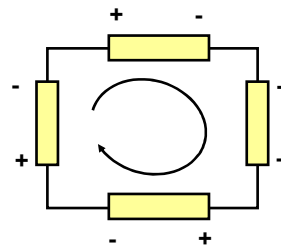
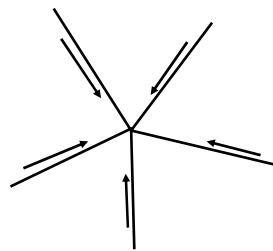
Small Signal (AC) Impedances

Circuit Elements



Kirchhoff's Laws

- Kirchhoff's Current Law (KCL)
 - The sum of all currents (flow) entering a node is zero
- Kirchhoff's Voltage Law (KVL)
 - The oriented sum of all voltages (efforts) around any closed loop is zero



Dynamic Response from Equivalent Circuit

$$\text{KVL: } -F + e_k + e_m + e_b = 0$$

Dynamic Response \rightarrow Laplace Transform

$$Z_C(s) = \frac{1}{sC}$$

$$Z_L(s) = sL$$

$$Z_b(s) = b$$

$$e(s) = Z(s)f(s) = \left(sL + b + \frac{1}{sC} \right) f(s)$$

Electrical-to-Mechanical Mapping

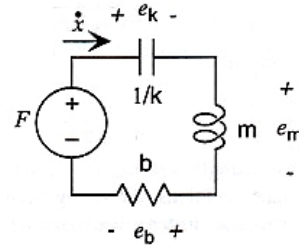
$$L \rightarrow m$$

$$b \rightarrow b$$

$$\frac{1}{C} \rightarrow k$$

$$e(s) \rightarrow F(s)$$

$$f(s) \rightarrow \dot{x}(s)$$



$$\frac{\dot{x}(s)}{F(s)} = \frac{1}{sm + b + k/s} = \frac{s}{s^2m + sb + k}$$

Second-order system